IV. Symmetries of quantum fields

IV.1 Discrete and continuous symmetries

Symmetries in quantum theory are transformations of states or operators that leave matrix elements invariant

$\Rightarrow$ unitary transformations

$$|\psi\rangle \longrightarrow |\psi'\rangle = U(a)\,|\psi\rangle$$ a - some parameter

(287) $$U^{-1}(a) = U^{+}(a)$$

If a is continuous

$$\hat{U}(a \to 0) \to \mathbb{1}$$

(288) $$\hat{U}(a) = e^{ia\hat{X}} \quad \text{where} \quad \hat{X}^{+} = \hat{X}$$

If holds

(289) $$\boxed{\hat{U}(a)\hat{U}(b) = e^{ia\hat{X}} e^{ib\hat{X}} = e^{i(a+b)\hat{X}} = \hat{U}(a+b) \quad \Rightarrow \text{group!}}$$

group isomorphism

$$a \longleftrightarrow \hat{U}(a)$$

equation of motion of transformed state

(290) $$i\frac{\partial}{\partial t}|\psi'\rangle = i\frac{\partial}{\partial t}\hat{U}|\psi\rangle = \hat{U}\hat{H}|\psi\rangle = \hat{U}\hat{H}\hat{U}^{+}|\psi'\rangle$$

so $\hat{H}' = \hat{U}\hat{H}\hat{U}^{+}$ $\quad \left(\frac{\partial}{\partial t}\hat{U} = 0\right)$ assume

(291) Iff $\boxed{[\hat{H},\hat{U}] = 0 \qquad \hat{H}' = \hat{H}}$

invariance of equation of motion

def.: A group of unitary transformations is called a symmetry group if $\hat{U}(a)$ commutes with $\hat{H}$ for all values of a

If there are two unitary operators $\hat{U}_1$ and $\hat{U}_2$ with

(292) $$[\hat{U}_1,\hat{U}_2] \neq 0 \quad \text{but} \quad [\hat{U}_1,\hat{H}] = [\hat{U}_2,\hat{H}] = 0$$

$\Rightarrow \exists$ linear independent eigenstates of $\hat{H}$ for the same eigenvalue E

$\Rightarrow$ degeneracy

$\boxed{\text{Symmetry groups}}$

$\swarrow \qquad \downarrow \qquad \searrow$

discrete — a: discrete values

continuous (**Lie groups**) — a: continuous

mixed

rem.: We will later see that we can generalize the notion of symmetry groups to operations that leave matrix elements up to complex conjugation invariant
$\Rightarrow$ anti-unitary operations

IV.2 Continuous symmetries and Lie groups

def: A continuous group, whose elements are unitary operators $\hat{U}(a_1, a_2, \dots a_n)$ that are analytic functions of n continuous parameters $a_1, a_2, \dots a_n$ which uniquely determine the element of the group is called a **Lie group**

We set

$$\hat{U}(\underline{a}=0) = \mathbb{1} \tag{293}$$

<u>Def:</u> The n quantities

$$\hat{L}_\mu = i \left.\frac{\partial U}{\partial a_\mu}\right|_{\underline{a}=0} \tag{294}$$

are called the <u>generators of the Lie group</u>

Expansion of $\hat{U}(\underline{a})$ in the vicinity of $\underline{a}=0$ yields

$$\hat{U}(\delta\alpha_\mu) = \hat{U}(0) + \left.\frac{\partial \hat{U}}{\partial \alpha_\mu}\right|_{\underline{\alpha}=0} \delta\alpha_\mu$$

$$= \mathbb{1} - i \hat{L}_\mu \delta\alpha_\mu$$

finite transformations $\quad \Delta\alpha_\mu = N\,\delta\alpha_\mu$

$$\hat{U}(\Delta\alpha_\mu) = \left[\hat{U}(\delta\alpha_\mu)\right]^N = \left[1 - i\frac{\hat{L}_\mu \Delta\alpha_\mu}{N}\right]^N$$

$N \to \infty$

$$\boxed{\hat{U}(\alpha_\mu) = e^{-i\hat{L}_\mu \alpha_\mu}} \tag{295}$$

If the α_μ are chosen real $\Rightarrow$

$$\hat{L}_\mu = \hat{L}_\mu^+ \tag{296}$$

Now consider $\hat{U}(\delta\alpha_\mu)$ in <u>second</u> order in $\{\delta\alpha_\mu\}$

$$\hat{U}(\delta\alpha_\mu) = \mathbb{1} - i\sum_\mu \hat{L}_\mu \delta\alpha_\mu - \frac{1}{2}\sum_{\mu\nu} \hat{L}_\mu \hat{L}_\nu \delta\alpha_\mu \delta\alpha_\nu$$

With this let's look at this sequence of infinitesimal transformations:

$$U^+(\delta\beta_\mu)U^+(\delta\alpha_\mu)U(\delta\beta_\mu)U(\delta\alpha_\mu) = (\neq \mathbb{1}!)$$

$$= \left(1 + i\delta\beta_\mu \hat{L}_\mu - \frac{1}{2}\delta\beta_\nu \delta\beta_\lambda \hat{L}_\nu \hat{L}_\lambda\right)$$
$$\left(1 + i\delta\alpha_\sigma \hat{L}_\sigma - \frac{1}{2}\delta\alpha_\gamma \delta\alpha_\delta \hat{L}_\gamma \hat{L}_\delta\right)$$
$$\left(1 - i\delta\beta_\varepsilon \hat{L}_\varepsilon - \frac{1}{2}\delta\beta_\omega \delta\beta_\xi \hat{L}_\omega \hat{L}_\xi\right)$$
$$\left(1 - i\delta\alpha_\eta \hat{L}_\eta - \frac{1}{2}\delta\alpha_\tau \delta\alpha_\varphi \hat{L}_\tau \hat{L}_\varphi\right) = \dots =$$

$$= 1 + \delta\alpha_\mu \delta\beta_\nu \underbrace{\left(\hat{L}_\mu \hat{L}_\nu - \hat{L}_\nu \hat{L}_\mu\right)}_{\text{this must again be a generator due to group property}}$$

$\Rightarrow$

$$\delta\alpha_\mu \delta\beta_\nu [\hat{L}_\mu, \hat{L}_\nu] \stackrel{!}{=} -i\delta\gamma_\lambda \hat{L}_\lambda$$

we now set $\quad -i\delta\gamma_\lambda = C_{\mu\nu\lambda}\, \delta\alpha_\mu \delta\beta_\nu$

(297) $$\boxed{[\hat{L}_\mu, \hat{L}_\nu] = C_{\mu\nu\lambda} \hat{L}_\lambda}$$ Lie algebra

The coefficients $C_{\mu\nu\lambda}$ are called structure factors.

An important relation for the structure factors follows from Jacobi's identity

$$[[\hat{L}_k, \hat{L}_e], \hat{L}_m] + [[\hat{L}_e, \hat{L}_m], \hat{L}_k] + [[\hat{L}_m, \hat{L}_k], \hat{L}_e] = 0$$

$$\boxed{C_{kej} C_{jmn} + C_{emj} C_{jkn} + C_{mkj} C_{jen} = 0}$$

(298)

def: The number of commuting generators of a Lie group is called the rank of the group.

example:

(i) translational group

(299) $$\hat{U} = e^{-i\vec{a}\cdot\hat{\vec{p}}} \qquad \hat{\vec{p}} = -i\vec{\nabla}$$

generators P_x, P_y, P_z rank: 3

structure factors $C_{abc} \equiv 0$

(ii) rotational group

(300) $$\hat{U} = e^{-i\vec{\theta}\cdot\hat{\vec{L}}}$$

generators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ rank: 1

structure factors $C_{xyz} = i\,\varepsilon_{xyz}$

theorem of Racah

Let $C_{\alpha\beta\gamma}$ be the structure factors of a Lie algebra and

(301) $$g_{\alpha\beta} = g_{\beta\alpha} \equiv C_{\alpha\mu\nu} C_{\beta\nu\mu}$$

If $\det(g_{\alpha\beta}) \neq 0$ and the group has rank ℓ there are ℓ so called

Casimir operators $\hat{C}_\lambda$ $\lambda = 1, 2, \ldots, \ell$.
They are functions of the generators and commute with all of them.

<u>example:</u> rotational group $[\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk}\hat{L}_k$

$$g_{\alpha\beta} = i\varepsilon_{\alpha\mu\nu}\, i\varepsilon_{\alpha\nu\mu} = \varepsilon_{\alpha\mu\nu}\varepsilon_{\beta\mu\nu} = 2\delta_{\alpha\beta}$$

$$\det(g_{\alpha\beta}) = 8 \neq 0 \quad \text{rank} = 1$$

$\Rightarrow \exists$ one Casimir operator $\hat{L}^2$

$$[\hat{L}^2, \hat{L}_i] = 0$$

If a Lie group is a symmetry group of a quantum system then the eigenvalues of the Casimir operator(s) define multipletts. The corresponding eigenstates form an invariant subspace i.e. a subspace that cannot be left by group operations.

<u>example:</u> subspace of rotational group

with total angular momentum (Casimir operator) $\ell = 1$.

$$Y_{11},\ Y_{10}\ Y_{1-1}$$

now $\hat{L}_\pm Y_{\ell m} = (\hat{L}_x \pm i \hat{L}_y) Y_{\ell m}$

$$= \sqrt{\ell(\ell+1) - m(m\pm1)}\ Y_{\ell, m\pm1}$$

i.e. within same subspace

$$\boxed{\text{The Hamiltonian is degenerate on the subspace}}$$

<u>proof:</u> let $|\psi_0\rangle$ be element of multiplett

$\Rightarrow \hat{U}(\alpha)|\psi_0\rangle$ element of multiplett

now $\hat{H}|\psi_0\rangle \overset{!}{=} E|\psi_0\rangle$

$$\hat{H}\, U(\alpha)|\psi_0\rangle = U(\alpha)\hat{H}|\psi_0\rangle = E\, U(\alpha)|\psi_0\rangle$$

<u>remark:</u>

- there is no general construction principle for Casimir operators

An operator $\hat{A}$ that commutes with all generators of a Lie group (and therefore with all elements of the Lie group) is a function of the Casimir operators alone.

$\Rightarrow$ if a Lie group is a symmetry group
$\Rightarrow$ H is a function of the Casimir operators

<u>example:</u> translational invariant system
rank 3 : Casimir operators $\hat{P}_x, \hat{P}_y, \hat{P}_z$

$$\Rightarrow H = \alpha + \vec{\beta}\cdot\hat{\vec{P}} + \gamma\,\hat{\vec{P}}^2 + \dots$$

some groups

group	generators	rank	Casimir op.
translation	$\hat{P}_x, \hat{P}_y, \hat{P}_z$	3	$\hat{P}_x, \hat{P}_y, \hat{P}_z$
rotation	$\hat{L}_x, \hat{L}_y, \hat{L}_z$	1	$\hat{L}^2$
translation + rotation	$\hat{L}_x, \hat{L}_y, \hat{L}_z$ $\hat{P}_x, \hat{P}_y, \hat{P}_z$	3	$\hat{\vec{P}}^2, \hat{\vec{P}}\cdot\hat{\vec{L}}, \hat{L}^2$