$$\{\hat{b}_{n,s}, \hat{b}_{n',s'}\} = \{\hat{d}_{n,s}, \hat{d}_{n',s'}\} = \{\hat{b}_{n,s}, \hat{d}_{n',s'}\} = 0$$

$$\{\hat{b}_{n,s}, \hat{b}^{+}_{n',s'}\} = \delta_{nn'}\delta_{ss'} \tag{227}$$

$$\{\hat{d}_{n,s}, \hat{d}^{+}_{n',s'}\} = \delta_{nn'}\delta_{ss'}$$

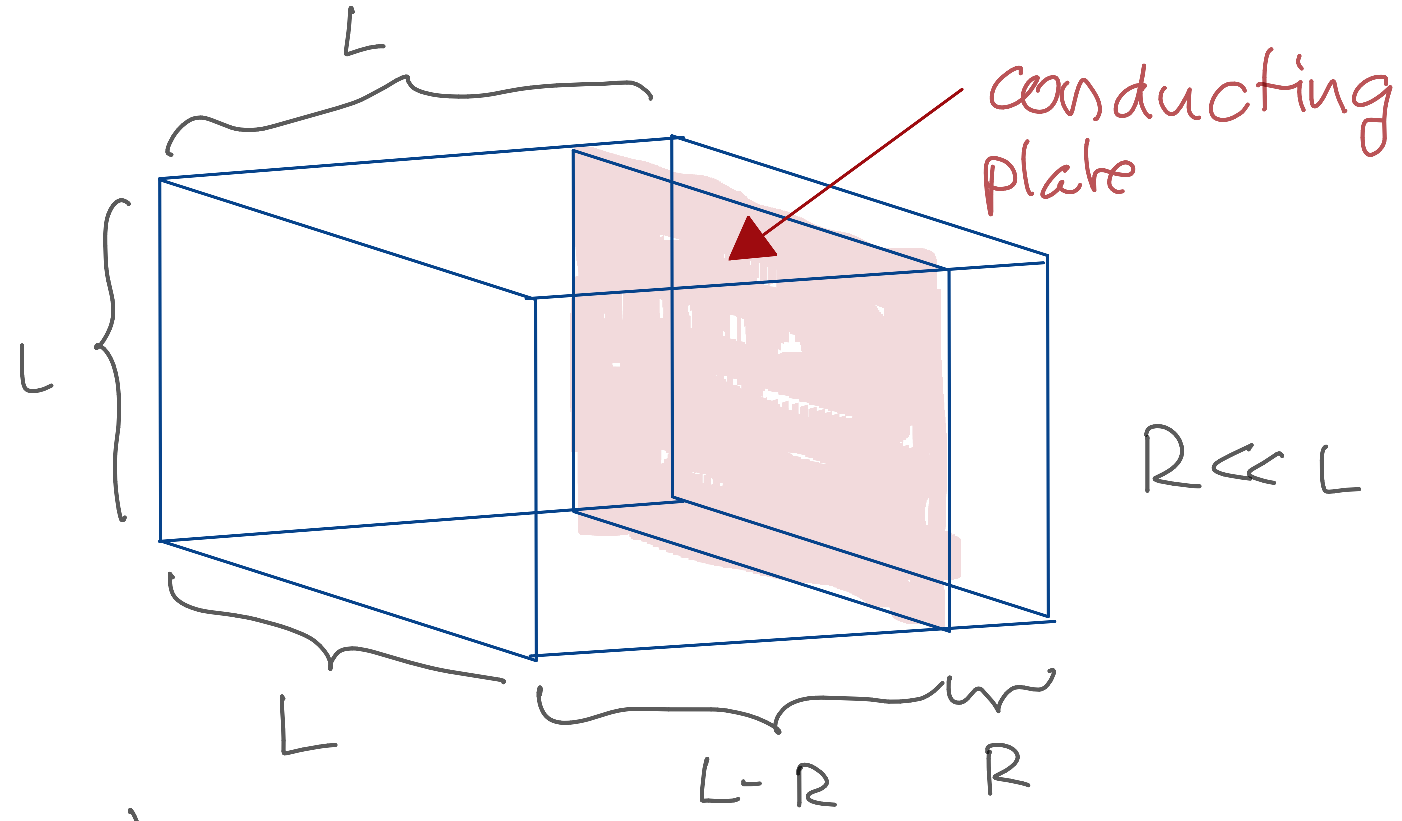
The Dirac field describes spin 1/2 particles and anti-particles of opposite charge and must be quantized as a fermionic field.

III. Simple effects of field quantization

III.1 Vacuum energy of electromagnetic field & Casimir force

In the quantization of the elm. field we have encountered a vacuum contribution to the energy. Although this vacuum contribution per se has no measurable consequences, changes of it by modified boundary conditions can lead to measureable forces.

consider quantization of elm. field in finite box with open boundary conditions



$R \ll L$

Open BC:

$k_n = \frac{\pi}{L} n$

$n = 0, 1, 2 \dots$

(exept $n_x = n_y = n_z = 0$)

What is the energy difference in the elm. vacuum with and without the conducting plate?

(228) $$\Delta E = \langle 0 | \hat{H}_R | 0 \rangle + \langle 0 | \hat{H}_{L-R} | 0 \rangle - \langle 0 | \hat{H}_L | 0 \rangle$$

$$\langle 0 | \hat{H} | 0 \rangle = \sum_{n_x, n_y, n_z} \omega_n = \frac{\pi}{L} \sum_{n_x n_y n_z = 0}^{\infty} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

$$\simeq \frac{\pi}{L} 4\pi \int_0^\infty dn\, n^2 n = \frac{L^3}{2\pi^2} \int_0^\infty dk\, k^3 \xrightarrow[L \to \infty]{} \infty$$

The contribution is divergent. We thus introduce an ultra-violet cut off with a function

$$1 \xleftarrow[k \ll k_c]{} f\left(\frac{k}{k_c}\right) \xrightarrow[k \gg k_c]{} 0$$

This gives

$$(229) \quad \langle 0|\hat{H}_L|0\rangle = \frac{L^3}{2\pi^2}\int_0^\infty dk\, f\left(\frac{k}{k_c}\right)k^3$$

$$(230) \quad \langle 0|\hat{H}_{L-R}|0\rangle = \frac{L^2(L-R)}{2\pi^2}\int_0^\infty dk\, f\left(\frac{k}{k_c}\right)k^3$$

To calculate $\langle 0|\hat{H}_R|0\rangle$ in the small volume we have to be a bit more careful as the limit $L\to\infty$ does not yield continuum of modes in one direction.

$$(231) \quad \langle 0|\hat{H}_R|0\rangle = \sum_{n_x=0}^{\infty}\left(1-\frac{1}{2}\delta_{n0}\right)\frac{L^2}{\pi^2}\, g\left(\frac{n\pi}{R}\right)$$

where the summations in y and z directions was replaced by an integral

$$g(k_x) = \int_0^\infty\!\!\int_0^\infty dk_y\, dk_z\; k\, f\left(\frac{k}{k_c}\right)$$

where $k = \sqrt{k_x^2+k_y^2+k_z^2}$

The term $\left(1-\frac{1}{2}\delta_{n0}\right)$ takes care of the fact that for $n_x = 0$ there is only one polarization direction as $\vec{E}_{\parallel} = 0$. Since

$$\int_0^\infty dk_x \, g(k_x) = \frac{4\pi}{g^?} \int_0^\infty dk \, k^2 k f\left(\frac{k}{k_c}\right) \tag{232}$$

we can also rewrite (229) and (230) in terms of $g(k_x)$. Thus

$$\Delta E = \frac{L^2}{\pi^2} \left\{ \sum_{n=0}^{\infty} \left(1 + \frac{1}{2}\delta_{n0}\right) g\left(\frac{n\pi}{R}\right) - \frac{R}{\pi} \int_0^\infty dk_x \, g(k_x) \right\} \tag{233}$$

If we would approximate the sum in the first term by an integral, we would get

$$\Delta E \to 0$$

but we have to be more careful here!

So let's see: $\quad k_\perp \equiv \sqrt{k_y^2 + k_z^2}$

$$g(k_x) = \frac{2\pi}{4} \int_0^\infty dk_\perp \, k_\perp \sqrt{k_x^2 + k_\perp^2} \, f\left(\frac{\sqrt{k_x^2 + k_\perp^2}}{k_c}\right)$$

set $k_\perp = \alpha \frac{\pi}{R} \qquad k_x = n \frac{\pi}{R}$

$$g(k_x) = \frac{\pi}{2}\frac{\pi^3}{R^3}\int_0^\infty d\alpha \sqrt{n^2+\alpha^2}\, f\left(\frac{\pi\sqrt{n^2+\alpha^2}}{R k_c}\right)$$

define: $\omega = n^2+\alpha^2 \qquad d\omega = 2\alpha\, d\alpha$

$\Rightarrow$

$$g(k_x) = \underbrace{\frac{\pi^4}{4R^3}\int_{n^2}^\infty d\omega \sqrt{\omega}\, f\left(\frac{\pi\sqrt{\omega}}{R k_c}\right)}_{\equiv F(n)}$$

then

$$\Delta E = \frac{L^2\pi^3}{4R^3}\left\{\sum_{n=0}^\infty \left(1-\frac{1}{2}\delta_{n0}\right)F(n) - \int_0^\infty dn\, F(n)\right\}$$

Euler McLaurin formula

given $h(x)$

$h(x) \xrightarrow[x\to\infty]{} 0$

$$\sum_{k=0}^\infty h(k) \simeq \frac{1}{2}h(0) + \int_0^\infty dk\, h(k) - \sum_{n=2}^{p}\frac{B_n}{n!}h^{(n-1)}(0) + \text{corrections}(p)$$

$B_2 = \frac{1}{6} \quad B_3 = 0 \quad B_4 = -\frac{1}{30} \quad B_5 = 0, \ldots$

$$\sum_{n=0}^{\infty}\left(1-\frac{1}{2}\delta_{n0}\right)F(n)-\int_0^{\infty}dn\,F(n) = -\frac{1}{6\cdot 2!}F'(0) + \frac{1}{30\cdot 4!}F'''(0) + \cdots$$

Now

$$F'(n) = -2n^2 f\left(\frac{\pi n}{R k_c}\right)$$

$$F''(n) = -4n f\left(\frac{\pi n}{R k_c}\right) - 2n^2 f'\left(\frac{\pi n}{R k_c}\right)\frac{\pi}{R k_c}$$

$$F'''(n) = -4 f\left(\frac{\pi n}{R k_c}\right) - 8n f'\left(\frac{\pi n}{R k_c}\right)\frac{\pi}{R k_c} - 2n^2 f''\left(\frac{\pi n}{R k_c}\right)\left(\frac{\pi}{R k_c}\right)^2$$

$$\Rightarrow \quad F'(0) = F''(0) = 0 \qquad F'''(0) = -4 f(0) = -4$$

independent on cut off !

(234) $$\boxed{\Delta E = -\frac{L^2 \pi^2}{720 R^3}}$$

Casimir energy of two perfectly conducting plates of size L^2 and distance R

remarks:

- $-\frac{d}{dR}(\Delta E) < 0$ attractive force

area density of force

$$f = \frac{F}{L^2} = -\frac{1}{L^2}\frac{\partial \Delta E}{\partial R} = -\frac{\pi^2}{240 R^4} \tag{235}$$

- f is independent on cut-off $f\left(\frac{k}{k_c}\right)$
- f in SI units

$$f = -\frac{\hbar c \pi^2}{240 R^4} \qquad \text{very small!}$$

- first measurement:

M.J. Sparnaay Physica 24 (1958) 751–764

III.2 Decay and Scattering: general remarks

A typical problem of QFTh:

transition probability between asymptotic states $|\Psi_\alpha\rangle$, $|\Psi_\beta\rangle$ which are eigenstates of free Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_I$$

free hamiltonian — interaction

in the interaction picture

(236) $$|\psi(t)\rangle = T\exp\left\{-i\int_0^t d\tau\, \hat{H}_I(\tau)\right\} |\psi(0)\rangle$$

where

(237) $$\hat{O}(\tau) = e^{i\hat{H}_0\tau}\,\hat{O}\,e^{-i\hat{H}_0\tau}$$

Let $|\psi_\alpha\rangle, |\psi_\beta\rangle$ asymptotic states, i.e. eigenstates of $\hat{H}_0$. Then

(238) $$S_{\beta\alpha} = \frac{1}{w}\langle\psi_\beta|\,\hat{U}_I(\infty,-\infty)\,|\psi_\alpha\rangle$$

gives the transition matrix element for the process

$$|\psi(t=-\infty)\rangle = |\psi_\alpha\rangle \longrightarrow |\psi(t=+\infty)\rangle = |\psi_\beta\rangle$$

where

$$\hat{U}_I(t_2,t_1) \equiv T\exp\left\{-i\int_{t_1}^{t_2} d\tau\, \hat{H}_I(\tau)\right\}$$

w is an appropriately chosen phase factor, which we choose as follows:

We assume without loss of generality

$$\hat{H}_I(\tau) \xrightarrow[|\tau|\to\infty]{} 0 \tag{239}$$

Then it should hold that the ground state $|gs_0\rangle$ of $\hat{H}$ (which is identical to the ground state of H_0 at $t = \pm\infty$) and which we assume to be non-degenerate, should go over into itself apart from a phase. I.e.

$$\langle gs_0 | \hat{U}_I(\infty, -\infty) | gs_0 \rangle \overset{!}{=} e^{i\phi}$$

We use this as the phase normalization i.e.

$$\boxed{S_{\beta\alpha} = \frac{\langle \Psi_\beta | \hat{U}_I(\infty, -\infty) | \Psi_\alpha \rangle}{\langle gs_0 | \hat{U}_I(\infty, -\infty) | gs_0 \rangle}} \tag{240}$$

Scattering matrix

Calculating this scattering matrix will be one of the central problems we are going to deal with!

Relation between scattering matrix and physical quantities

- transition probability per time:

(241)
$$\boxed{\Delta W_{\beta\alpha} \equiv \lim_{T\to\infty} \frac{|S_{\beta\alpha}(T/2, -T/2)|^2}{T}}$$

We will later see that for scattering and decay processes we can write in the limit of large T

(242)
$$S_{\beta\alpha}\left(\frac{T}{2}, -\frac{T}{2}\right) = -2\pi i \frac{T}{\pi} \frac{\sin(\Delta E T/2)}{\Delta E \cdot T} \left(\frac{1}{\sqrt{L^3}}\right)^m f_{\beta\alpha}$$

where ΔE is the energy difference between the asymptotic states and for

decay processes $\quad m = p - 1$

with p being the number of particles in the final state and for

scattering processes $\quad m = q$

where the incoming states are plane waves with q being the number of scattering particles.

The sinc-Term in (242) resulhet from an integral of the form

(243) $$\boxed{d_1(T) \equiv \frac{T}{\pi} \frac{\sin\left(\Delta E \frac{T}{2}\right)}{\Delta E \cdot T} = \frac{1}{2\pi} \int_{-T/2}^{T/2} dt\, e^{-i(E_\alpha - E_\beta)t}}$$

With this we obtain

(244) $$\Delta W_{\beta\alpha} = \lim_{T\to\infty} \frac{d_1^2(T)}{T} (2\pi)^2 \left(\frac{1}{L^3}\right)^m |f_{\beta\alpha}|^2$$

which contains

(245) $$\boxed{d_2(T) \equiv (2\pi)^2 \frac{d_1^2(T)}{T} = 4T \frac{\sin^2\left(\Delta E \frac{T}{2}\right)}{(\Delta E \cdot T)^2}}$$

We note that

(246) $$\boxed{\lim_{T\to\infty} d_1(T) = \delta(E_\alpha - E_\beta)}$$

but also

(247) $$\boxed{\lim_{T\to\infty} d_2(T) = 2\pi\, \delta(E_\alpha - E_\beta)}$$

We note that it is important to keep track of the order of limits. In the above sense of order of limits we have

(248)
$$\boxed{\text{"} \lim_{T\to\infty} \frac{|2\pi\delta(E_\alpha - E_\beta)|^2}{T} = 2\pi\delta(E_\alpha - E_\beta) \text{"}}$$

In summary we find

(250)
$$\boxed{S_{\beta\alpha} = -2\pi i\,\delta(E_\alpha - E_\beta)\left(\frac{1}{\sqrt{L^3}}\right)^m f_{\beta\alpha}}$$

S-matrix elements

and

(251)
$$\boxed{\Delta W_{\beta\alpha} = 2\pi\,\delta(E_\alpha - E_\beta)\left(\frac{1}{\sqrt{L^3}}\right)^m f_{\beta\alpha}}$$

Fermi's golden rule for the transition probability per time (rate)

(252)
$$m = \begin{cases} p-1 & \text{decay} \\ p & \text{scattering} \end{cases}$$

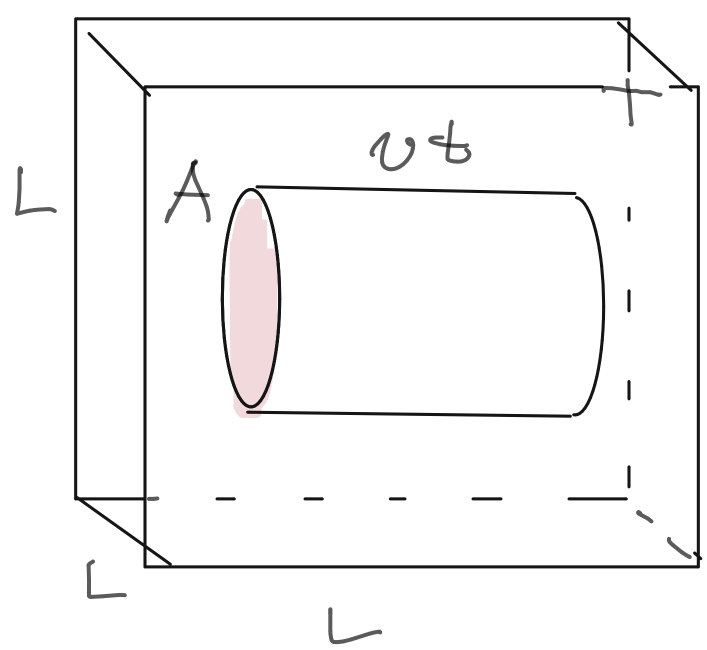
differential scattering cross section

An important experimental quantity is the differential cross section $\Delta\sigma_{\beta\alpha}$ in a spherical angle $\Delta\Omega$

$$\sum_{\beta \in \Delta\Omega} \Delta W_{\beta\alpha}^{\text{scatter}} = j\, \Delta\sigma_{\beta\alpha}$$

j : number of incident particles per area an time

$$j = \frac{n_v}{At} = \frac{1}{L^3} A v t \frac{1}{At} = \frac{v}{L^3}$$



$$\boxed{\Delta\sigma_{\beta\alpha} = \frac{2\pi}{v} \sum_{\beta \in \Delta\Omega} \delta(E_\alpha - E_\beta) \left(\frac{1}{L^3}\right)^m |f_{\beta\alpha}|^2}$$

(253)

III.3 Spontaneous emission

Consider an atom with two relevant states $|a\rangle$ and $|b\rangle$ which we describe as a non relativistic electron field in the fixed Coulomb potential of the atomic nucleus in the presence of the quantized elm. field in Coulomb gauge

$$(254)\quad \hat{H} = \hat{H}_{elm} + \hat{H}_{atom} + \int d^3r\, \hat{\psi}^+(\vec{r}) \frac{1}{2m} \left(\hat{\vec{p}} - e\hat{\vec{A}}\right)^2 \hat{\psi}(\vec{r})$$

Hamiltonian of free elm field

Coulomb potential of nucleus

$\hat{\vec{p}} = -i\vec{\nabla}$

interaction Hamiltonian

$$(255)\quad \hat{H}_I = \int d^3r\, \hat{\psi}^+(\vec{r}) \left(-\frac{e}{m}\hat{\vec{p}}\hat{\vec{A}} + \frac{e^2}{2m}\hat{\vec{A}}^2\right) \hat{\psi}(\vec{r})$$

in Coulomb gauge $\quad [\hat{\vec{p}}, \hat{\vec{A}}] = 0$

Let: $|\alpha\rangle = |a\rangle\,|0\rangle$

photon vacuum

$|\beta\rangle = |b\rangle\,|1_{n\lambda}\rangle$

1 photon state in mode n with polarization λ

$\Delta W_{\beta\alpha}$ in 1st-order perturbation theory

$$\langle\beta|\hat{U}_I(\infty,-\infty)|\alpha\rangle = \langle\beta|T e^{-i\int_{-\infty}^{\infty} d\tau\, \hat{H}_I(\tau)}|\alpha\rangle$$

$$= \underbrace{\langle\beta|\alpha\rangle}_{=0} - i\int_{-\infty}^{\infty} d\tau\, \langle\beta|\hat{H}_I(\tau)|\alpha\rangle$$

One finds that the $\hat{\vec{A}}^2$ term does not contribute and after some calculation (see problem course) one finds

$$\Delta W_{ba} = \underbrace{\sum_{\{n_{\lambda}\}}}_{\substack{\text{sum over all}\\ \text{possible final}\\ \text{photon states}}} \Delta W_{\beta\alpha} = \frac{e^2}{4\pi}\,\frac{4\omega_{ab}}{3m^2}\left|\langle b|\hat{\vec{p}}|a\rangle\right|^2 \tag{256}$$

$$\omega_{ab} = E_b - E_a$$

Making use of

$$\hat{\vec{p}} = -im\left[\hat{\vec{r}},\hat{H}_0\right] = -im\left(\hat{\vec{r}}\hat{H}_0 - \hat{H}_0\hat{\vec{r}}\right)$$

we find

$$\langle b|\hat{\vec{p}}|a\rangle = \frac{m}{i}\langle b|\hat{\vec{r}}|a\rangle\left(E_a - E_b\right) = \frac{m\omega_{ab}}{i}\langle b|\hat{\vec{r}}|a\rangle$$

Thus

(257) $$\boxed{\Delta W_{ba} = \frac{\omega_{ab}^3}{3\pi} \left| \vec{d}_{ab} \right|^2}$$

Einstein A coefficient of spontaneous emission

where

(258) $$\vec{d}_{ab} = e \langle b | \hat{\vec{r}} | a \rangle$$

is the transition dipole matrix element

III. 4 Lamb shift according to Bethe

With the help of microwave and Laser spectroscopy W.E. Lamb and W.E. Retherford found in 1947 that there is an energy splitting between the $2S_{1/2}$ and $2P_{1/2}$ states of Hydrogen despite the fact that these states are precisely degenerate in relativistic quantum mechanics (Dirac theory). This splitting is called Lamb shift and is a direct consequence of the quantization of the elm.