II.5 The Maxwell field

We use Coulomb gauge, i.e.

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad \text{i.e.} \quad \vec{A} = \vec{A}_\perp$$

Lagrange density and Hamiltonian

$$(177) \qquad \mathcal{L} = \frac{1}{2}\left(\vec{E}_\perp^{\,2} - \vec{B}_\perp^{\,2}\right) \qquad \vec{E}_\perp = -\partial_t \vec{A}_\perp$$

$$\vec{B}_\perp = \vec{\nabla} \times \vec{A}_\perp$$

canonical momentum to $\vec{A}_\perp$

$$(178) \qquad \vec{\pi}_\perp = \frac{\partial \mathcal{L}}{\partial(\partial_t \vec{A}_\perp)} = \frac{\partial \vec{A}_\perp}{\partial t} = -\vec{E}_\perp$$

$$(179) \qquad \boxed{H = \int d^3r \, \frac{1}{2}\left(\vec{E}_\perp^{\,2} + \vec{B}_\perp^{\,2}\right)}$$

commutation relation

$$(180) \qquad \boxed{\left[\vec{A}_\perp(\vec{r},t), \vec{E}_\perp(\vec{r}\,',t)\right] = -i \overleftrightarrow{\delta}_\perp(\vec{r}-\vec{r}\,')}$$

remarks:

- one finds (see exercises)

(181) $$\sum_{k=1}^{3} \left[\hat{A}_\perp^k(\vec{r},t), \hat{E}_\perp^k(\vec{r}',t) \right] = -2i\delta(\vec{r}-\vec{r}')$$

- We will see that for the Maxwell field only bosonic quantization makes sense.

Consider quantization in finite volume L^3 with PBC. (Note that quantization in finite volume is not Lorentz invariant. To get Lorentz invariance, one has to consider the limit $L \to \infty$).

To get the normal mode decomposition we consider eigensolutions of the wave eqs:

$$\Box \hat{\vec{A}}(\vec{r},t) = 0 \qquad \text{with } \vec{\nabla}\cdot\vec{A} = 0$$

$\Rightarrow$ only two degrees of freedom

hermitian field $\hat{\vec{A}}_\perp = \hat{\vec{A}}_\perp^\dagger$

(182) $$\boxed{\hat{\vec{A}}(\vec{r},t) = \sum_n \sum_{\alpha=1}^{2} \frac{1}{\sqrt{2\omega_n L^3}} \left(\vec{e}_{n\alpha}\hat{a}_{n\alpha}(t)\, e^{i\vec{k}_n\cdot\vec{r}} + \vec{e}^{\,*}_{n\alpha}\hat{a}^\dagger_{n\alpha}(t)\, e^{-i\vec{k}_n\cdot\vec{r}} \right)}$$

$$\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z) \qquad n_i = 0, \pm 1, \pm 2, \ldots$$
(PBC)

because of Coulomb gauge $\quad \vec{\nabla} \cdot \hat{\vec{A}}_\perp = 0$

(183) $$\boxed{\vec{k}_n \cdot \vec{e}_{n,\alpha} = 0}$$

$\vec{e}_{n,\alpha}$ $\quad \alpha = 1,2 \quad$ transversal polarization (unit) vectors

convention: linear polarization

(184) $$\vec{e}_{n\alpha} = \vec{e}_{n\alpha}^{\,*}$$

(185) $$\vec{e}_{n1} \times \vec{e}_{n2} = \frac{\vec{k}_n}{|\vec{k}_n|} \qquad \text{and cyclic}$$

wave equation implies

(186) $$\boxed{\hat{a}_{n\alpha}(t) = \hat{a}_{n\alpha} e^{-i\omega_n t}} \qquad \boxed{|\vec{k}_n|^2 = \omega_n^2}$$

In the excercises you will prove that the field commutation relations imply

(187) $$\boxed{[\hat{a}_{n\alpha}, \hat{a}_{m\beta}] = [\hat{a}^+_{n\alpha}, \hat{a}^+_{m\beta}] = 0}$$

$$\boxed{\left[\hat{a}_{n\alpha}, \hat{a}^{+}_{m\beta}\right] = \delta_{nm}\,\delta_{\alpha\beta}} \tag{188}$$

and

electric and magnetic field operators

$$\boxed{\begin{aligned}\vec{\hat{E}}_{\perp}(\vec{r},t) &= -\partial_t \vec{\hat{A}}_{\perp} = \sum_{n}\sum_{\alpha=1}^{2} \frac{1}{\sqrt{2\omega_n L^3}}\, i\omega_n \vec{e}_{n\alpha} \\ &\left(\hat{a}_{n\alpha}(t)\, e^{i\vec{k}_n\cdot\vec{r}} - \hat{a}^{+}_{n\alpha}(t)\, e^{-i\vec{k}_n\cdot\vec{r}}\right)\end{aligned}} \tag{189}$$

$$\boxed{\begin{aligned}\vec{\hat{B}}_{\perp}(\vec{r},t) &= \vec{\nabla}\times\vec{\hat{A}}_{\perp} = \sum_{n}\sum_{\alpha=1}^{2} \frac{1}{\sqrt{2\omega_n L^3}}\, i\left(\vec{e}_{n\alpha}\times\vec{k}_n\right) \\ &\left(\hat{a}_{n\alpha}(t)\, e^{i\vec{k}_n\cdot\vec{r}} - \hat{a}^{+}_{n\alpha}(t)\, e^{-i\vec{k}_n\cdot\vec{r}}\right)\end{aligned}} \tag{190}$$

Let us now express the Hamiltonian (179) in terms of the mode operators $\hat{a}_n, \hat{a}_n^+$. So far we will not make use of commutation relations however.

$$\hat{H} = \frac{1}{2}\int_{L^3} d^3r \left(\vec{\hat{E}}_{\perp}^{\,2} + \vec{\hat{B}}_{\perp}^{\,2}\right) = \frac{1}{4L^3}\int_{L^3} d^3r \sum_{\substack{n,m\\ \alpha,\beta}} \frac{1}{\sqrt{\omega_n\omega_m}}$$

*

$$* \Big\{ \left(\omega_n \omega_m \vec{e}_{n\alpha} \cdot \vec{e}_{m\beta} + (\vec{k}_n \times \vec{e}_{n\alpha}) \cdot (\vec{k}_m \times \vec{e}_{m\beta}) \right)$$

$$\left(\hat{a}^+_{n\alpha} \hat{a}_{m\beta} e^{-i(\vec{k}_n - \vec{k}_m)\vec{r}} + \hat{a}_{n\alpha} \hat{a}^+_{m\beta} e^{i(\vec{k}_n - \vec{k}_m)\vec{r}} \right)$$

$$- \left(\omega_n \omega_m \vec{e}_{n\alpha} \cdot \vec{e}_{m\beta} + (\vec{k}_n \times \vec{e}_{n\alpha}) \cdot (\vec{k}_m \times \vec{e}_{m\beta}) \right)$$

$$\left(\hat{a}_{n\alpha} \hat{a}_{m\beta} e^{i(\vec{k}_n + \vec{k}_m)\vec{r}} + \hat{a}^+_{n\alpha} \hat{a}^+_{m\beta} e^{-i(\vec{k}_n + \vec{k}_m)\vec{r}} \right)$$

Making use of (remember $n \to (n_x, n_y, n_z)$)

(191)
$$\frac{1}{L^3} \int_{L^3} d^3r \; e^{\pm i(\vec{k}_n - \vec{k}_m)\vec{r}} = \delta_{nm}$$

$$\frac{1}{L^3} \int_{L^3} d^3r \; e^{\pm i(\vec{k}_n + \vec{k}_m)\cdot\vec{r}} = \delta_{n,-m}$$

and $\vec{k}_{-n} = -\vec{k}_n$ and (185) one finds:

(192)
$$\hat{H} = \frac{1}{4} \sum_{n,\alpha,\beta} \frac{1}{\omega_n} \Big\{ \left(\omega_n^2 \delta_{\alpha\beta} + (\vec{k}_n \times \vec{e}_{n\alpha}) \cdot (\vec{k}_n \times \vec{e}_{n\beta}) \right) \cdot \left(\hat{a}^+_{n\alpha} \hat{a}_{n\beta} + \hat{a}_{n\alpha} \hat{a}^+_{n\beta} \right)$$

$$- \left(\omega_n^2 \vec{e}_{n\alpha} \cdot \vec{e}_{-n\beta} - (\vec{k}_n \times \vec{e}_{n\alpha}) \cdot (\vec{k}_n \times \vec{e}_{-n\beta}) \right) \cdot \left(\hat{a}_{n\alpha} \hat{a}_{-n\beta} + \hat{a}^+_{n\alpha} \hat{a}^+_{-n\beta} \right) \Big\}$$

Now

$$(\vec{k}_n \times \vec{e}_{n\alpha}) \cdot (\vec{k}_n \times \vec{e}_{n\beta}) =$$

$$= k_n^2 \, \vec{e}_{n\alpha} \cdot \vec{e}_{n\beta} - \underbrace{(\vec{k}_n \cdot \vec{e}_{n\alpha})}_{=0} \underbrace{(\vec{k}_n \cdot \vec{e}_{n\beta})}_{=0}$$

$$= \omega_n^2 \, \delta_{\alpha\beta}$$

So the first term in (192) gets a factor 2, and

$$(\vec{k}_n \times \vec{e}_{n\alpha}) \cdot (\vec{k}_n \times \vec{e}_{-n\beta}) =$$

$$= k_n^2 \, \vec{e}_{n\alpha} \cdot \vec{e}_{-n\beta} - \underbrace{(\vec{k}_n \cdot \vec{e}_{n\alpha})}_{=0} \underbrace{(\vec{k}_n \cdot \vec{e}_{-n\beta})}_{=0}$$

I.e. the second term in (192) cancells out.

$\Rightarrow$

$$(193) \quad \hat{H} = \sum_n \sum_{\alpha=1}^{2} \tfrac{1}{2} \omega_n \left[\hat{a}^+_{n\alpha} \hat{a}_{n\alpha} + \hat{a}_{n\alpha} \hat{a}^+_{n\alpha} \right]$$

Now we see that anti-commutator relations for $\hat{a}_n$, and $\hat{a}_n^+$ would yield

$$\hat{H} = 0 \quad \lightning$$

which is obviously nonsense. Thus the

elm. field must be quantized using commutator relations. With this we find for the Hamiltonian of the free elm. field

$$\boxed{\hat{H} = \sum_{n} \sum_{\alpha=1}^{2} \omega_n \left(\hat{a}^{+}_{n\alpha} \hat{a}_{n\alpha} + \frac{1}{2} \right)} \tag{194}$$

which is just a sum of harmonic oscillators.

momentum of the elm. field

It is a short exercise to show that the momentum of the elm. field is

$$\vec{P} = \frac{1}{2} \int d^3r \left(\vec{\hat{E}}_\perp \times \vec{\hat{B}}_\perp - \vec{\hat{B}}_\perp \times \vec{\hat{E}}_\perp \right) = \sum_{n,\alpha} \vec{k}_n \hat{a}^{+}_{n\alpha} \hat{a}_{n\alpha} + \text{const.} \tag{195}$$

$\hat{E}_\perp$ and $\hat{B}_\perp$ do not commute

interpretation: momentum per photon $\hbar \vec{k}_n$

angular momentum of elm field

Even more interesting is the operator of the angular momentum of the elm. field (defined as in classical ED with proper symmetrization) with respect to $\vec{r}_0$

(196) $$\hat{\vec{L}} = \int d^3r \, (\vec{r}-\vec{r}_0) \times \left[\left(\hat{\vec{E}}_\perp \times \hat{\vec{B}}_\perp - \hat{\vec{B}}_\perp \times \hat{\vec{E}}_\perp \right) \right]$$

A little longer calculation shows that $\hat{\vec{L}}$ can be decomposed in a part

- that depends on $\vec{r}_0$ — orbital angular momentum
- that is independent on $\vec{r}_0$ — internal angular momentum or spin

(197) $$\boxed{\hat{\vec{L}} = \hat{\vec{L}}_{orb} + \hat{\vec{S}}}$$

(198) $$\boxed{\hat{\vec{L}}_{orb} \equiv i \int d^3r \; \hat{E}^k_\perp \left(-i(\vec{r}-\vec{r}_0) \times \vec{\nabla} \right) \hat{A}^k_\perp}$$

and

$$\boxed{\begin{aligned}\hat{\vec{S}} &\equiv \int d^3r\ \hat{\vec{E}}_\perp \times \hat{\vec{A}}_\perp \\ &= \int d^3r\ \vec{e}_i\ \varepsilon_{ijk}\ \hat{E}^j_\perp \hat{A}^k_\perp\end{aligned}} \tag{199}$$

<u>proof:</u> We here consider classical variables and drop the index "$\perp$" for simplicity. Also $\vec{x} \equiv \vec{r} - \vec{r}_0$

$$L^i = \int d^3r\ \varepsilon_{ije}\ x^j\ \varepsilon_{ekm} E^k \varepsilon_{mab} \partial_a A^b$$

$$= \int d^3r\ \varepsilon_{ije} x^j \left(\delta_{ea}\delta_{kb} - \delta_{eb}\delta_{ka}\right) E^k \partial_a A^b$$

$$= \int d^3r\ \varepsilon_{ije} x^j \left(E^b \partial_e A^b - (\vec{E}\cdot\vec{\nabla})A^e\right)$$

partial integration of last term and use of $\vec{\nabla}\cdot\vec{E}_\perp = 0$ yields

$$L^i = \int d^3r\ \varepsilon_{ije} \left(\underbrace{x^j E^b \partial_e A^b}_{\text{depends on } \vec{r}_0} - \underbrace{E^j A^e}_{\text{independent on } \vec{r}_0}\right) \tag{200}$$

Lets have a closer look at

$$\hat{\vec{S}} = \int d^3r\ \varepsilon_{ije} \left(-\hat{E}^j \hat{A}^e\right) = \tag{201}$$

$$= \int d^3r\, \varepsilon_{ijk} \sum_{\substack{n,m\\ \alpha,\beta}} \frac{i\omega_n}{\sqrt{\omega_n \omega_m}} (\vec{e}_{n\alpha})^j (\vec{e}_{m\beta})^k$$

$$\Big\{ \hat{a}_{n\alpha} \hat{a}_{m\beta} e^{i(\vec{k}_n + \vec{k}_m)\vec{r}} - \hat{a}^+_{n\alpha} \hat{a}^+_{m\beta} e^{-i(\vec{k}_n+\vec{k}_m)\vec{r}}$$

$$- \hat{a}^+_{n\alpha} \hat{a}_{m\beta} e^{-i(\vec{k}_n - \vec{k}_m)\vec{r}} + \hat{a}_{n\alpha} \hat{a}^+_{m\beta} e^{i(\vec{k}_n - \vec{k}_m)\vec{r}} \Big\}$$

Carrying out the spatial integration and using the corresponding orthogonality relations yields

$$\hat{S}_i = i \sum_{n\alpha\beta} \frac{\varepsilon_{ijk}}{2} \Big\{ (\vec{e}_{n\alpha})^j (\vec{e}_{-n\beta})^k \left[\hat{a}_{n\alpha} \hat{a}_{-n\beta} - \hat{a}^+_{n\alpha} \hat{a}^+_{-n\beta} \right]$$

$$+ (\vec{e}_{n\alpha})^j (\vec{e}_{n\beta})^k \left[\hat{a}_{n\alpha} \hat{a}^+_{n\beta} - \hat{a}^+_{n\alpha} \hat{a}_{n\beta} \right] \Big\}$$

The term in the brackets $\{\cdots\}$ in the first line is symmetric under

$$j \leftrightarrow k \quad \text{and} \quad n \leftrightarrow -n$$

while ε_{ijk} is anti-symmetric. So this term vanishes. The second term is not symmetric if $\alpha \neq \beta$. Thus we find

$$\vec{S} = i \sum_{\substack{n\\ \alpha\neq\beta}} (\vec{e}_{n\alpha} \times \vec{e}_{n\beta})\, \hat{a}^+_{n\beta} \hat{a}_{n\alpha} + \cancel{\text{trans}} \;\to 0 \qquad (202)$$

If we use linear-polarized basis, we find with $\vec{e}_{n1} \times \vec{e}_{n2} = \vec{k}_n / |\vec{k}_n|$

$$\hat{\vec{S}} = i \sum_n \frac{\vec{k}_n}{|\vec{k}_n|} \left(\hat{a}^+_{n2} \hat{a}_{n1} - \hat{a}^+_{n1} \hat{a}_{n2} \right) \tag{203}$$

It is quite instructive to introduce circular polarizations

$$\vec{e}_{n+} \equiv \frac{1}{\sqrt{2}} \left(\vec{e}_{n1} + i \vec{e}_{n2} \right)$$
$$\vec{e}_{n-} \equiv \frac{1}{\sqrt{2}} \left(\vec{e}_{n1} - i \vec{e}_{n2} \right) \tag{204}$$

This yields

$$\hat{\vec{S}} = \sum_n \frac{\vec{k}_n}{|\vec{k}_n|} \left(\hat{a}^+_{n+} \hat{a}_{n+} - \hat{a}^+_{n-} \hat{a}_{n-} \right) \tag{205}$$

spin projection to $\vec{k}_n$ "+1" "−1"

(206) **photons: spin 1 particle**

II.6 The Dirac field

The Lagrange density reads

$\gamma^\mu : 4\times 4$

$\psi : 4\text{-comp.}$

(207) $$\mathcal{L} = \bar{\psi}\left(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right)\psi \xrightarrow[\text{canonical form}]{} \mathcal{L} = \bar{\psi}\left(i\gamma^\mu \overrightarrow{\partial}_\mu - m\right)\psi$$

conjugate momentum (use asymmetric form of $\mathcal{L}$)

(208) $$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^+$$

quantization:

We will see lateron that we have to choose fermionic quantization rules for the Dirac field

(209) $$\boxed{\begin{aligned} \left\{\hat{\psi}_\alpha(\vec{r},t), \hat{\psi}_\beta(\vec{r}\,',t)\right\} &= \left\{\hat{\psi}^+_\alpha(\vec{r},t), \hat{\psi}^+_\beta(\vec{r}\,',t)\right\} = 0 \\ \left\{\hat{\psi}_\alpha(\vec{r},t), \hat{\psi}^+_\beta(\vec{r}\,',t)\right\} &= \delta(\vec{r}-\vec{r}\,')\,\delta_{\alpha\beta} \end{aligned}}$$

$\alpha, \beta \in 1,2,3,4$ are bi-spinor indices

Hamiltonian of free Dirac field

$$(210) \quad \boxed{\hat{H} = \int d^3r \; \hat{\psi}^+(\vec{r}) \left[-i \vec{\alpha} \cdot \vec{\nabla} + \beta m \right] \hat{\psi}(\vec{r})}$$

$$\hat{\psi} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \\ \hat{\psi}_4 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

equation of motion:

$$(211) \quad i \partial_t \hat{\psi} = \left(-i \vec{\alpha} \cdot \vec{\nabla} + \beta m \right) \hat{\psi}$$

We again look for a decomposition of the bi-spinor field in terms of normal modes. This is a bit more involved here

Normal mode decomposition of free Dirac field

In the exercises you will prove that the (classical) free Dirac equation has solutions of the following form

$$(212) \quad \psi^{(\pm)}_{\vec{p}}(\vec{r},t) = N e^{i(\vec{k}_n \cdot \vec{r} \mp E_p t)} \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

with $E_p > 0$ and χ and η are two-component objects and $\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z)$

$\boxed{\psi^{(+)}}$ $\qquad p_\mu x^\mu = E_p t - \vec{k}\cdot\vec{r}$

insert (212) into Dirac eq.

$$E_p \begin{pmatrix} \chi \\ \eta \end{pmatrix} e^{-ip_\mu x^\mu} = \begin{pmatrix} m & -i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} e^{-ip_\mu x^\mu}$$

yields

$$\begin{aligned} E_p \chi &= m\chi + \vec{\sigma}\cdot\vec{p}\,\eta \\ E_p \eta &= \vec{\sigma}\cdot\vec{p}\,\chi - m\eta \end{aligned} \qquad (213) \qquad \vec{p} = \hbar \vec{k}$$

non-trivial solution only if determinant is zero

(214) $\det(\ldots) = 0 \quad \Rightarrow \quad \boxed{E_p^2 = m^2 + p^2}$ relativist. energy-momentum

(215) and $\boxed{\eta = \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\,\chi}$

i.e.

(216)
$$\psi^{(+)}_{\vec{p}}(\vec{r},t) = \mathcal{N} e^{-ip_\mu x^\mu} \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\chi \end{pmatrix}.$$

χ is a two-component spinor that describes the spin degree of freedom of a spin 1/2 particle. Choose the eigenstates of σ_z

$$s=\tfrac{1}{2}: \quad \chi^{(\frac{1}{2})} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad s=-\tfrac{1}{2}: \quad \chi^{(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Normalization condition

$$1 \stackrel{!}{=} \int_{L^3} d^3r \; \psi^\dagger \psi = \mathcal{N}^2 L^3 \left[1 + \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p+m)^2} \right]$$

with $(\vec{\sigma}\cdot\vec{p})^2 = \sigma_i p_i \sigma_j p_j = \sigma_i \sigma_j p_i p_j$

$$= \frac{1}{2} \underbrace{(\sigma_i \sigma_j + \sigma_j \sigma_i)}_{=\delta_{ij}} p_i p_j = \sum_{i=1}^{3} p_i^2 = p^2$$

I.e.

$$1 = \mathcal{N}^2 L^3 \left[1 + \frac{\overbrace{p^2}^{= E_p^2 - m^2 = (E_p+m)(E_p-m)}}{(E_p+m)^2} \right] = \mathcal{N}^2 L^3 \left[1 + \frac{E_p - m}{E_p + m} \right]$$

$$= \mathcal{N}^2 L^3 \frac{2E_p}{E_p+m} \quad \Rightarrow \quad \mathcal{N} = \frac{1}{\sqrt{2E_p L^3}} \sqrt{E_p + m}$$

This gives $(s = \pm\frac{1}{2})$

$$\boxed{\psi^{(+)}_{\vec{p}s}(\vec{r},t) = \frac{1}{\sqrt{2E_p L^3}} \, u(\vec{p},s) \, e^{-ip_\mu x^\mu}}$$

where

$$u(\vec{p},s) = \sqrt{E_p+m}\begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi^{(s)} \end{pmatrix} \xrightarrow[\vec{p}\to 0]{} \sqrt{E_p+m}\begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix} \tag{217}$$

$\boxed{\psi^{(-)}}$ similar but $E_p \to -E_p$

$$-E_p\chi = m\chi + \vec{\sigma}\cdot\vec{p}\,\eta$$

$$-E_p\eta = \vec{\sigma}\cdot\vec{p} - m\eta$$

$E_p^2 = p^2 + m^2$ as before

$$\Rightarrow \quad \boxed{\chi = -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\eta} \tag{218}$$

note that E_p+m large but E_p-m can be small

spin eigenstates

$s = \frac{1}{2} \quad \eta^{(1/2)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad s = -\frac{1}{2} \quad \eta^{(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

"-1" is convenient definition (see later)

This gives

$$\psi^{(-)}_{-\vec{p},-s}(\vec{r},t) = \frac{1}{\sqrt{2E_p L^3}}\, v(\vec{p},s)\, e^{ip_\mu x^\mu}$$

where

$$v(\vec{p},s) = \sqrt{E_p+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\eta^{(-s)} \\ \eta^{(-s)} \end{pmatrix} \xrightarrow[p\to 0]{} \sqrt{E_p+m}\begin{pmatrix} 0 \\ \eta^{(-s)} \end{pmatrix}$$

(219)

<u>Remark:</u> We will see in the following chapter that charge conjugation of the positive-energy solution gives the negative-energy solution

(220) $$\boxed{\left[\psi^{(+)}_{\vec{p},s}\right]^C = \psi^{(-)}_{-\vec{p},-s}}$$

with $-\vec{p}$ and $-s$

Now we can decompose the Dirac field into positive and negative-energy eigensolutions and absorb the time-dependence in the operator valued coefficients (to be verified lateron).

$$\hat{\psi}(\vec{r},t) = \sum_{n,s}\left(\hat{c}_{n,s}\,\psi^{(+)}_{n,s}(\vec{r},t) + \hat{d}^{+}_{n,s}\,\psi^{(-)}_{-n,-s}(\vec{r},t)\right)$$

(221)
$$= \sum_{n,s}\left(\hat{c}_{n,s}\,e^{-iE_p t}\,\psi^{(+)}_{n,s}(\vec{r}) + \hat{d}^{+}_{n,s}\,e^{iE_p t}\,\psi^{(-)}_{-n,-s}(\vec{r})\right)$$

(222)
$$\hat{c}_{n,s}(t) \equiv \hat{c}_{n,s}(0)\,e^{-iE_p t} \qquad E_p = \sqrt{k_n^2 + m^2}$$
$$\hat{d}_{n,s}(t) \equiv \hat{d}_{n,s}(0)\,e^{-iE_p t}$$

We will later show that (222) follows from the Heisenberg eqs. of motion.

Now we want to express the Hamiltonian in terms of the mode operators:

Substituting (221) into the Heisenberg eqs. of motion (Dirac equation) yields with

$$\left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi^{(\pm)} = \pm E_p\,\psi^{(\pm)}$$

$\Rightarrow$

$$\hat{H} = \int d^3r \sum_{n,s}\sum_{n',s'}\left(\hat{c}^{+}_{ns}\,\psi^{(+)+}_{ns}(\vec{r}) + \hat{d}_{ns}\,\psi^{(-)+}_{-n-s}(\vec{r})\right)$$
$$\cdot\, E_{n'}\left(\hat{c}_{n's'}\,\psi^{(+)}_{n's'}(\vec{r}) - \hat{d}^{+}_{n's'}\,\psi^{(-)}_{-n,-s}(\vec{r})\right)$$

Using the orthonormality

$$(223)\quad \int d^3r\ \underset{\substack{\uparrow\\ \text{4-component row vector}}}{\psi^{(\pm)+}_{m,s}}(\vec{r})\ \underset{\substack{\uparrow\\ \text{4 component column vector}}}{\psi^{(\pm)}_{m',s'}}(\vec{r}) = \delta_{mm'}\delta_{ss'}$$

and

$$(224)\quad \int d^3r\ \psi^{(+)+}_{m,s}(\vec{r})\ \psi^{(-)}_{m',s'}(\vec{r}) = 0$$

yields

$$(225)\quad \hat{H} = \sum_{n,s} E_n\left(\hat{c}^+_{n,s}\hat{c}_{n,s} - \hat{d}_{n,s}\hat{d}^+_{n,s}\right)$$

for bosonic quantization this gives

$$\hat{H}_B = \sum_{n,s} E_n\left(\hat{c}^+_{n,s}\hat{c}_{n,s} - \hat{d}^+_{n,s}\hat{d}_{n,s}\right) + \text{const}$$

$\lightning$ not positive definite!

for fermionic quantization holds

$$(226)\quad \boxed{\hat{H}_F = \sum_{n,s} E_n\left(\underset{\text{particle}}{\hat{c}^+_{n,s}\hat{c}_{n,s}} + \underset{\text{anti-particle}}{\hat{d}^+_{n,s}\hat{d}_{n,s}}\right) + \text{const}}$$

positive!

$$\{\hat{b}_{n,s}, \hat{b}_{n',s'}\} = \{\hat{d}_{n,s}, \hat{d}_{n',s'}\} = \{\hat{b}_{n,s}, \hat{d}_{n',s'}\} = 0$$

$$\{\hat{b}_{n,s}, \hat{b}^{+}_{n',s'}\} = \delta_{nn'}\delta_{ss'} \qquad (227)$$

$$\{\hat{d}_{n,s}, \hat{d}^{+}_{n',s'}\} = \delta_{nn'}\delta_{ss'}$$

The Dirac field describes spin 1/2 particles and anti-particles of opposite charge and must be quantized as a fermionic field.

III. Simple effects of field quantization

III.1 Vacuum energy of electromagnetic field & Casimir force

In the quantization of the elm. field we have encountered a vacuum contribution to the energy. Although this vacuum contribution per se has no measurable consequences, changes of