II.4 The charged Klein-Gordon field

The simplest relativistic field is the charged Klein-Gordon field. In order to allow for a non-vanishing current ("charged" KG field) the field must be complex valued

Lagrange density

(153) $$\mathcal{L} = \frac{\partial \phi^*}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu} - m^2 \phi^* \phi$$

conjugate momenta

(154) $$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^* = \dot{\phi}^* \qquad \pi = \dot{\phi}$$

Hamiltonian density & Hamiltonian

$$\mathcal{H} = \pi^* \partial_0 \phi + \partial_0 \phi^* \pi - \mathcal{L}$$

(155) $$= \pi^* \pi + \vec{\nabla} \phi^* \vec{\nabla} \phi + m^2 \phi^* \phi$$

(156) $$\boxed{H = \int d^3 r \; \hat{\pi}^\dagger \hat{\pi} + \vec{\nabla} \hat{\phi}^\dagger \vec{\nabla} \hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi}}$$

We will see now that for the KG field, which is a scalar field, only commutation relations make sense:

Euler-Lagrange eqs. for the classical field:

$$(\Box + m^2)\,\phi = 0$$

From Quantum Mechanics II we know that two types of solutions exist, with positive and negative energies respectively

(166) $$\phi_n^{(\pm)}(\vec{r},t) = \mathcal{N}\, e^{i(\vec{k}_n \vec{r} \mp E_n t)}$$

where

$$E_n = \sqrt{k_n^2 + m^2} \qquad (\hbar = 1)$$

(167) $$\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z) \qquad n_j = 0, \pm 1, \pm 2, \ldots$$

and we have assumed PBC in a volume L^3

We therefore make the ansatz

(168) $$\boxed{\begin{aligned} \hat{\phi}(\vec{r},t) &= \sum_n \left(\hat{a}_n \phi_n^{(+)}(\vec{r},t) + \hat{c}_n^{+} \phi_{-n}^{(-)}(\vec{r},t) \right) \\ \hat{\phi}^{+}(\vec{r},t) &= \sum_n \left(\hat{a}_n^{+} \phi_n^{(-)*}(\vec{r},t) + \hat{c}_n \phi_{-n}^{(-)*}(\vec{r},t) \right) \end{aligned}}$$

ϕ_{-n} has $(-n_x, -n_y, -n_z)$

interpretation

$\hat{a}_n$ - destroys particle with momentum $\vec{p}_n = \hbar \vec{k}_n$ and positive charge

$\hat{c}_n^+$ - creates particle with momentum $-\vec{p}_n = -\hbar \vec{k}_n$ and negative charge

rem.: (i) $\hat{a}_n$ is connected with $\hat{c}_n^+$ such that $\hat{\phi}$ reduces the total charge

(ii) $\hat{c}_n^+$ is associated with $\phi_{-n}^{(-)}$ since states with negative energy and negative momentum corresponds to antiparticles with positive momentum and positive energy

Some properties of normal mode solutions

(166)

$$i \int d^3r \, \phi_n^{(+)*} \overleftrightarrow{\frac{\partial}{\partial t}} \phi_m^{(+)} = 2 \delta_{nm} \mathcal{N}^2 L^3 E_n$$

(169) $\Longrightarrow$ choose $\qquad \mathcal{N} = \frac{1}{\sqrt{2E_n L^3}}$

$$(170) \quad i\int d^3r\, \phi_n^{(\pm)*} \overleftrightarrow{\frac{\partial}{\partial t}} \phi_m^{(\pm)} = \pm \delta_{mn}$$

$$(171) \quad i\int d^3r\, \phi_n^{(\pm)*} \overleftrightarrow{\frac{\partial}{\partial t}} \phi_m^{(\mp)} = 0$$

With this we can re-write the Hamiltonian

$$\hat{H} = \int d^3r \left(\partial_t \hat{\phi}^+ \partial_t \hat{\phi} + \vec{\nabla}\hat{\phi}^+ \vec{\nabla}\hat{\phi} + m^2 \hat{\phi}^+ \hat{\phi} \right)$$

$$\underset{\substack{\text{part.}\\ \text{integr.}}}{=} \int d^3r \left(\partial_t \hat{\phi}^+ \partial_t \hat{\phi} - \hat{\phi}^+ \underbrace{\left(\vec{\nabla}\hat{\phi} - m^2 \hat{\phi} \right)}_{= \partial_t^2 \hat{\phi}} \right)$$

$$= -\int d^3r\, \hat{\phi}^+ \overleftrightarrow{\frac{\partial}{\partial t}} \left(\frac{\partial}{\partial t} \hat{\phi} \right)$$

Lets plug in the mode decomposition (168) $\Rightarrow$

$$\hat{H} = \int d^3r \sum_{m,n} \left(\hat{a}_n^+ \phi_n^{(+)*} + \hat{c}_n \phi_{-n}^{(-)*} \right) i \overleftrightarrow{\frac{\partial}{\partial t}} E_n * \left(\hat{a}_m \phi_m^{(+)} - \hat{c}_m^+ \phi_{-m}^{(-)} \right)$$

with (170), (171) $\Rightarrow$

$$(172) \quad \hat{H} = \sum_n E_n \left(\hat{a}_n^+ \hat{a}_n + \hat{c}_n \hat{c}_n^+ \right)$$

Assuming commutator relations (bosons)

$$(173) \quad \hat{H} = \sum_n E_n \underbrace{\left(\hat{a}_n^+ \hat{a}_n + \hat{c}_n^+ \hat{c}_n \right)}_{> 0} + \sum_n E_n$$

However assuming anti-commutator relations (fermions)

$$\hat{H} = \sum_n E_n \underbrace{\left(\hat{a}_n^{+}\hat{a}_n - \hat{c}_n^{+}\hat{c}_n\right)}_{\text{not definite}} + \sum_n E_n \tag{174}$$

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Thus we need commutator relations!

$$\boxed{\begin{array}{ccc} \text{Spin } 0 & \Longrightarrow & \text{bosons} \\ \text{(no internal dof)} & & \\ \text{but charged} & & \end{array}}$$

fundamental commutation relations

$$\boxed{\begin{aligned} &[\hat{a}_n, \hat{a}_m] = [\hat{a}_n^{+}, \hat{a}_m^{+}] = [\hat{c}_n, \hat{c}_m] = [\hat{c}_n^{+}, \hat{c}_m^{+}] = 0 \\ &[\hat{a}_n, \hat{c}_m^{(+)}] = [\hat{a}_n^{(+)}, \hat{c}_m] = 0 \\ &[\hat{a}_n, \hat{a}_m^{+}] = [\hat{c}_n, \hat{c}_m^{+}] = \delta_{nm} \end{aligned}} \tag{175}$$

From this we find for the fields:

$$[\hat{\phi}(\vec{r},t), \hat{\pi}(\vec{r}',t)] = \sum_{n,m} \Big[\left(\hat{a}_n \phi_n^{(+)}(\vec{r}) + \hat{c}_n^{+} \phi_{-n}^{(-)}(\vec{r})\right), \\ iE_m \left(\hat{a}_m^{+} \phi_m^{(+)*}(\vec{r}') - \hat{c}_m \phi_{-m}^{(-)*}(\vec{r}')\right)\Big]$$

$$= \sum_{n,m} \underbrace{[\hat{a}_n, \hat{a}_m^+]}_{=\delta_{nm}} \, iE_m \, \phi_n^{(+)}(\vec{r}) \, \phi_m^{(+)*}(\vec{r}\,')$$

$$- \sum_{n,m} \underbrace{[\hat{c}_n^+ \hat{c}_m]}_{=-\delta_{nm}} \, iE_m \, \phi_{-n}^{(-)}(\vec{r}) \, \phi_{-m}^{(-)*}(\vec{r}\,')$$

$$= \sum_m iE_m \left(\phi_m^{(+)}(\vec{r}) \phi_m^{(+)*}(\vec{r}\,') + \phi_{-m}^{(-)}(\vec{r}) \phi_{-m}^{(-)*}(\vec{r}\,') \right)$$

From the Klein Gordon theory one knows that

$$\sum_n E_n \, \phi_n^{(+)*}(\vec{r}) \, \phi_n^{(+)}(\vec{r}\,') = \frac{1}{2} \delta(\vec{r}-\vec{r}\,') \quad \text{in } L^3$$

$$\sum_n E_n \, \phi_{-n}^{(-)}(\vec{r}) \, \phi_{-n}^{(-)*}(\vec{r}\,') = \frac{1}{2} \delta(\vec{r}-\vec{r}\,')$$

$$\Rightarrow \quad \boxed{[\hat{\phi}(\vec{r},t), \hat{\pi}(\vec{r}\,',t)] = i\delta(\vec{r}-\vec{r}\,')} \tag{176}$$

With the quantization of the Klein-Gordon field we have seen a first example of an important fundamental theorem of relativistic quantum field theory

Spin-Statistics Theorem

fields with integer spin $\Longleftrightarrow$ bosons

fields with half integer spin $\Longleftrightarrow$ fermions

commutation relations and causality

The commutation relations imply that the fields $\hat{\phi}$ ($\hat{\phi}^+$) and their conjugate at the same time can be measured simultaneously at different points in space; i.e the measurement of the field at

$$(\vec{r}, t)$$

cannot influence the measurement at

$$(\vec{r}' \neq \vec{r}, t)$$

$$(x-x')^2 = \underbrace{(t-t')^2}_{=0} - (\vec{r}-\vec{r}')^2 < 0$$

Since the theory is Lorentz invariant we can generalize this to

measurements of fields or any local observables at spacelike separated points $(x-x')^2 < 0$ cannot influence each other

micro-causality