II. Canonical quantization of fields

II.1 Quantization of the one-dimensional harmonic chain

Let us consider the 1D chain from I.3 again. We can quantize the discrete model and then apply a continuum limit.

$$H = \sum_{j=0}^{N-1} \left(\frac{1}{2m} p_j^2 + \frac{\kappa}{2} \left(q_{j+1} - q_j \right)^2 \right)$$

$$\downarrow$$

(114) $$\hat{H} = \sum_{j=0}^{N-1} \left(\frac{1}{2m} \hat{p}_j^{\,2} + \frac{\kappa}{2} \left(\hat{q}_{j+1} - \hat{q}_j \right)^2 \right)$$

Here we follow a different approach by first considering the classical continuum limit

(115) $$L = \frac{1}{2} \int_0^L dz \left[\frac{1}{v^2} \left(\frac{\partial q(z,t)}{\partial t} \right)^2 - \left(\frac{\partial q(z,t)}{\partial z} \right)^2 \right] \qquad v^2 \equiv \frac{\varkappa}{\mu}$$

The Euler-Lagrange equations read

(116) $$\frac{1}{v^2} \frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial z^2} = 0$$

which can be solved in a finite length L with PBC

$$q_n(z,t) \sim e^{\pm i(k_n z - \omega_n t)}$$

where $\quad k_n = \frac{2\pi}{L} n \qquad n = 0, \pm 1, \pm 2, \dots$

$$\omega_n^2 = k_n^2 v^2 \qquad \omega_n = v|k_n|$$

normalized solutions

(117) $$\phi_n(z,t) = \frac{1}{\sqrt{L}} e^{i(k_n z - \omega_n t)}$$

(118) $$\int_0^L dz\, \phi_n^*(z,t)\, \phi_m(z,t) = \delta_{nm}$$

Now decompose general "field" $q(z,t)$ into normalized solutions $\quad c_n = c_n^*$

$$q(z,t) = \sum_{n=-\infty}^{\infty} c_n \left(a_n(0)\, \phi_n(z,t) + a_n^*(0)\, \phi_n^*(z,t) \right)$$

(119) $$= \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{L}} \left(a_n(t)\, e^{ik_n z} + a_n^*(t)\, e^{-ik_n z} \right)$$

where

(120) $$a_n(t) = a_n(0)\, e^{-i\omega_n t}$$

Obviously

(121) $$\ddot{a}_n + \omega_n^2 a_n = 0 \quad \hat{=} \text{ harmonic oscillator of frequency } \omega_n$$

In a similar way we find for the Hamiltonian

(122) $$H = \sum_{n=-\infty}^{\infty} c_n^2 \frac{2\omega_n^2}{v^2} a_n^*(t) a_n(t)$$

Now we choose the free constants to simplify

(123) $$c_n \equiv \frac{v}{\sqrt{2\omega_n}}$$

which gives

(124) $$\boxed{H = \sum_{n=-\infty}^{\infty} \omega_n a_n^*(t) a_n(t) \qquad \omega_n = \frac{2\pi v}{L}|n|}$$

This is identical to a sum of harmonic oscillators with frequencies ω_n and can be re-written as (introducing the mass $m=1$)

(125) $$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[p_n^2(t) + \omega_n^2 x_n^2(t) \right]$$

where

(126) $$x_n(t) = \frac{1}{\sqrt{2\omega_n}} \left(a_n(t) + a_n^*(t) \right)$$

and

$$p_n(t) = -i\sqrt{\frac{\omega_n}{2}}\left(a_n(t) - a_n^*(t)\right) \tag{127}$$

Each harmonic oscillator can now be quantized as in ordinary quantum mechanics

$$\{x_n, x_m\} = 0 \qquad \{p_n, p_m\} = 0 \qquad \{x_n, p_m\} = \delta_{nm}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$[\hat{x}_n, \hat{x}_m] = 0 \qquad [\hat{p}_n, \hat{p}_m] = 0 \qquad [\hat{x}_n, \hat{p}_m] = i\delta_{nm} \tag{128}$$

equivalent to

$$\boxed{[\hat{a}_n, \hat{a}_m] = [\hat{a}_n^+, \hat{a}_m^+] = 0 \qquad [\hat{a}_n, \hat{a}_m^+] = \delta_{nm}} \tag{129}$$

We will show now, that eq. (129) is equivalent to

$$\boxed{\begin{gathered}[\hat{q}(z,t), \hat{q}(z',t)] = [\hat{\pi}(z,t), \hat{\pi}(z',t)] = 0 \\ [\hat{q}(z,t), \hat{\pi}(z',t)] = i\delta(z-z') \\ \text{on } z, z' \in (0, L]\end{gathered}} \tag{130}$$

proof $\pi(z,t) = \dfrac{\partial \mathcal{L}}{\partial \dot{q}} = \dfrac{1}{v^2}\dot{q}(z,t)$

$$= \frac{1}{v}\sum_n \frac{1}{\sqrt{2\omega_n L}}\left(-i\omega_n a_n(t) e^{ik_n z} + i\omega_n a_n^*(t) e^{-ik_n z}\right)$$

Thus

$$[\hat{q}(z,t), \hat{\pi}(z',t)] = \frac{i}{2L}\sum_{n,m}\sqrt{\frac{\omega_n}{\omega_m}}$$

$$\Big\{ \underbrace{[\hat{a}_m, \hat{a}_n^+]}_{=\delta_{mn}} e^{i(k_m z - k_n z')} e^{-i(\omega_m - \omega_n)t}$$

$$- \underbrace{[\hat{a}_m^+, \hat{a}_n]}_{=-\delta_{mn}} e^{-i(k_m z - k_n z')} e^{i(\omega_m - \omega_n)t} \Big\}$$

$$= \frac{i}{L}\sum_n e^{ik_n(z-z')} = \frac{i}{L}\sum_n e^{i\frac{2\pi}{L}n(z-z')}$$

$$= i\,\delta(z-z') \quad \text{on } (0,L] \qquad \square$$

These results will now be generalized

II.2 Fundamental commutator relations of fields

canonical quantization

Motivated by the relation between classical mechanics and quantum mechanics and the explicit construction of a quantum field theory in II.1 we now postulate

$$\{A(\vec{x},t),\, B(\vec{x}',t)\}_{\text{Poisson}}$$

(131)

bosons	fermions
$-i\left[\hat{A}(\vec{x},t),\, \hat{B}(\vec{x}',t)\right]$	$-i\left\{\hat{A}(\vec{x},t),\, \hat{B}(\vec{x}',t)\right\}$
commutator	anticommutator
scalar fields vector & tensor fields	scalar fields spinor fields

We will see lateron that for certain type of fields we have to use the anti-commutator relations for quantization rules of quantum theory apply

(i) physical state of a field is described by a vector in Hilbert space $\mathcal{H}$ or generalizations thereof

(ii) physical quantities, e.g. the fields, are linear operators in $\mathcal{H}$ or generalizations thereof

(iii) physical observables are in particular self-adjoint linear operators

remark: complex valued fields such as the Schrödinger- or Dirac field do <u>not</u> correspond to self-adjoint operators and are therefore no observables. Only bilinear combinations such as $\hat{\psi}^{+}\hat{\psi}$ (Schrödinger) or $\hat{\bar{\psi}}\hat{\psi}$ (Dirac) are.

(iv) measurement of an observable leads to projection of the state to the eigenspace (eigenvector) corresponding to the measurement outcome; the probability to find a specific measurement result is given by the expectation value of the projection operator to the eigenspace in the quantum state.

(v) time evolution is described by a unitary transformation $\hat{U}(t)$ with the generator being the Hamiltonian $\hat{H}$

$$\hat{U}(t) = e^{-i\hat{H}t} \tag{132}$$

(133) $$|\psi(t)\rangle = \hat{U}(t-t_0)\,|\psi(t_0)\rangle$$ Schrödinger-picture

or

(134) $$\hat{\phi}(\vec{r},t) = \hat{U}^{-1}(t)\,\hat{\phi}(\vec{r},0)\,\hat{U}(t)$$ Heisenberg picture

(135) $$\Rightarrow \quad \boxed{i\,\frac{\partial \hat{\phi}(\vec{r},t)}{\partial t} = \left[\hat{\phi}(\vec{r},t),\hat{H}\right]}$$ always commutator

II.3 Quantization of the non-relativistic Schrödinger-field & many-body quantum mechanics

(A) generals

classical Lagrange density

$$\mathcal{L} = \frac{1}{2}\psi^* i \overrightarrow{\partial_t}\psi - \frac{1}{2}\psi^* i \overleftarrow{\partial_t}\psi - \frac{1}{2m}(\nabla\psi^*)(\nabla\psi)$$

Euler-Lagrange equation = Schrödinger eqs.

(136) $$i\partial_t\psi = -\frac{1}{2m}\Delta\psi$$

canonical momentum

(137) $$\pi^* = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = \frac{i}{2}\psi^* \qquad \pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi^*)} = -\frac{i}{2}\psi$$

problem: fields are their own canonical momentum solution: canonical trafo (add total time-derivative)

$$\mathcal{L} \to \mathcal{L} = i\psi^* \partial_t \psi - \frac{1}{2m} (\nabla\psi^*)(\nabla\psi)$$

only π^* exists

(138) $$\pi^* = i\psi^*$$ (note the "i")

quantization for bosons

(139) $$\boxed{\begin{aligned} \left[\hat\psi(\vec r,t), \hat\psi(\vec r',t)\right] &= \left[\hat\psi^+(\vec r,t), \hat\psi^+(\vec r',t)\right] = 0 \\ \left[\hat\psi(\vec r,t), \hat\psi^+(\vec r',t)\right] &= \delta(\vec r - \vec r') \end{aligned}}$$

quantization for fermions

(140) $$\boxed{\begin{aligned} \left\{\hat\psi(\vec r,t), \hat\psi(\vec r',t)\right\} &= \left\{\hat\psi^+(\vec r,t), \hat\psi^+(\vec r',t)\right\} = 0 \\ \left\{\hat\psi(\vec r,t), \hat\psi^+(\vec r',t)\right\} &= \delta(\vec r - \vec r') \end{aligned}}$$

Hamiltonian (free Schrödinger field)

(141) $$\boxed{\hat H = \int d^3r \; \hat\psi^+(\vec r,t) \left(-\frac{1}{2m}\Delta\right) \hat\psi(\vec r,t)}$$

Heisenberg equation of motion for bosonic quantization

$$[\hat{\psi}(\vec{r},t), \hat{H}] = \int d^3r' \underbrace{[\hat{\psi}(\vec{r},t), \hat{\psi}^+(\vec{r}',t)]}_{=\delta(\vec{r}-\vec{r}')} \left(-\frac{1}{2m}\Delta_{r'}\right)\hat{\psi}(\vec{r}',t)$$

$$= -\frac{1}{2m}\Delta\hat{\psi}(\vec{r},t)$$

i.e.

$$i\frac{\partial}{\partial t}\hat{\psi}(\vec{r},t) = -\frac{1}{2m}\Delta\hat{\psi}(\vec{r},t) \tag{142}$$

formally identical to "classical" Schrödinger eqs!

Interestingly we get the same equation for fermionic quantization:

$$[\hat{\psi}(\vec{r},t), \hat{H}] = \int d^3r' \underline{\hat{\psi}(\vec{r},t)\hat{\psi}^+(\vec{r}',t)} \left(-\frac{1}{2m}\Delta_{r'}\right)\hat{\psi}(\vec{r}',t)$$

$$-\int d^3r' \hat{\psi}^+(\vec{r}',t)\left(-\frac{1}{2m}\Delta_{r'}\right)\underline{\hat{\psi}(\vec{r}',t)\hat{\psi}(\vec{r},t)}$$

with $\underline{\hat{\psi}(\vec{r},t)\hat{\psi}^+(\vec{r}',t)} = \delta(\vec{r}-\vec{r}') - \hat{\psi}^+(\vec{r}',t)\hat{\psi}(\vec{r},t)$

and $\underline{\hat{\psi}(\vec{r}',t)\hat{\psi}(\vec{r},t)} = -\hat{\psi}(\vec{r},t)\hat{\psi}(\vec{r}',t)$

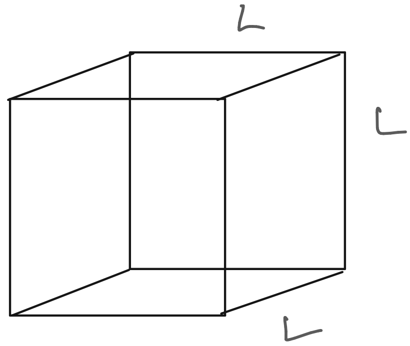
One also finds in the fermionic case

$$[\hat{\psi}(\vec{r},t), \hat{H}] = -\frac{1}{2m}\Delta\hat{\psi}(\vec{r},t) \qquad \square$$

(B) Normal-mode decomposition

Similar to the harmonic chain in II.1 we will search for a decomposition into normal modes

consider cubic box of length L and periodic boundary conditions (PBC)



The "classical" Schrödinger equation has normalized eigensolutions

(143) $$\phi_{\underline{n}}(\vec{r},t) = \frac{1}{\sqrt{L^3}} e^{i(\vec{k}_{\underline{n}}\cdot\vec{r} - \omega_{\underline{n}} t)}$$

where $\vec{k}_{\underline{n}} = \frac{2\pi}{L}(n_x, n_y, n_z)$ $\quad n_x, n_y, n_z \in \mathbb{N}$

(144) $\omega_{\underline{n}} = \frac{|\vec{k}_n|^2}{2m}$ $\quad \underline{n} \equiv (n_x, n_y, n_z)$

from now only simply "n"

decomposition of field operator into $\phi_{\underline{n}}$'s

$$\hat{\psi}(\vec{r},t) = \begin{cases} \sum_n \hat{a}_n \phi_n(\vec{r}) e^{-i\omega_n t} & \text{bosons} \\ \sum_n \hat{c}_n \phi_n(\vec{r}) e^{-i\omega_n t} & \text{fermions} \end{cases}$$

Substitution into eqs. (139), (140) (commutator relations of the fields yields

$$(145)\quad \boxed{[\hat{a}_n, \hat{a}_m] = [\hat{a}_n^+, \hat{a}_m^+] = 0 \qquad [\hat{a}_n, \hat{a}_m^+] = \delta_{nm}}$$

bosonic mode operators

$$(146)\quad \boxed{\{\hat{c}_n, \hat{c}_m\} = \{\hat{c}_n^+, \hat{c}_m^+\} = 0 \qquad \{\hat{c}_n, \hat{c}_m^+\} = \delta_{nm}}$$

fermionic mode operators

The Hamiltonian can be written in terms of the mode operators as (here for bosons)

$$\hat{H} = \int d^3r\, \hat{\Psi}^+(\vec{r},t) \underbrace{\left(-\frac{1}{2m}\vec{\nabla}^2\right)\hat{\Psi}(\vec{r},t)}_{\text{eq. (142)}}$$

$$= \int d^3r\, \hat{\Psi}^+(\vec{r},t)\, i\partial_t \hat{\Psi}(\vec{r},t)$$

$$= \int d^3r \sum_{m,n} \hat{a}_n^+ e^{i\omega_n t} \phi_n^*(\vec{r})\, i\omega_m \hat{a}_m e^{-i\omega_m t} \phi_m(\vec{r})$$

with orthogonality relation

$$\Rightarrow$$

$$(147)\quad \boxed{\hat{H} = \sum_n \omega_n \hat{a}_n^+ \hat{a}_n} \qquad \varepsilon_n = \omega_n$$

similarly one finds for fermions

$$(148)\quad \boxed{\hat{H} = \sum_n \omega_n \hat{c}_n^+ \hat{c}_n}$$

Because of the fermionic commutation relations

$$(\hat{c}_n^+ \hat{c}_n)^2 = \hat{c}_n^+ \hat{c}_n \hat{c}_n^+ c_n$$

$$= \hat{c}_n^+ \hat{c}_n - \underbrace{\hat{c}_n^{+2}}_{=0} \underbrace{\hat{c}_n^2}_{=0} = \hat{c}_n^+ \hat{c}_n$$

Thus the eigenstates of $\hat{H}$ are

(149) $$|\ldots, m_{n-1}, m_n, m_{n+1}, \ldots\rangle$$

where for

bosonic fields $\quad m_k = 0, 1, 2, 3, \ldots$

fermionic fields $\quad m_k = 0, 1$

The integers m_k are eigenvalues of $\hat{a}_k^+ \hat{a}_k$ or $\hat{c}_k^+ \hat{c}_k$ and must be interpreted as

particle number in mode $\phi_k(\vec{r})$

The state

(150) $|0, 0, 0, \ldots, 0\rangle$ is called particle vacuum

$\hat{a}_n^+, \hat{c}_n^+$ are particle creation operators in mode ϕ_n

$\hat{a}_n, \hat{c}_n$ are particle annihilation operators in mode ϕ_n

These properties of the mode operators carry over also to all other field theories $\hat{\psi}^{+}(\vec{r}\,')\left(-\frac{1}{2m}\Delta_{r'}\right)\hat{\psi}(\vec{r}\,')$

The anti-commutation relation for fermionic mode operators

$$\{\hat{c}_n^{+}, \hat{c}_n^{+}\} = 0$$

represents the Pauli-principle

(151) $$\boxed{\hat{c}_n^{+\,2} = 0}$$

$\nexists$ 2 Fermions in the same mode $\phi_n(\vec{r})$

(c) Conservation of particle number in Schrödinger theory

The total number of particles is given by

$$\hat{N} = \int d^3r\, \hat{\psi}^{+}(\vec{r},t)\, \hat{\psi}(\vec{r},t)$$

(152) $$= \int d^3r \sum_{n,m} \hat{a}_n^{+}(t)\, \hat{a}_m(t)\, \underbrace{\phi_n^{*}(\vec{r})\,\phi_m(\vec{r})}_{\text{orthogonality}}$$

$$= \sum_n \hat{a}_n^{+} \hat{a}_n \qquad \square$$

similar in fermion case

Now let us consider the time evolution

$$i\frac{d}{dt}\hat{N} = [\hat{N},\hat{H}] = \int d^3r \int d^3r' \, \hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}^+(\vec{r}')\left(-\frac{1}{2m}\Delta_{r'}\right)\hat{\psi}(\vec{r}')$$

$$-\int d^3r \int d^3r' \, \hat{\psi}^+(\vec{r}')\left(-\frac{1}{2m}\Delta_{r'}\right)\hat{\psi}(\vec{r}')\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r})$$

$$= \int d^3r \int d^3r' \left(\cancel{\hat{\psi}^+(\vec{r})\hat{\psi}^+(\vec{r}')\hat{\psi}(\vec{r})} + \cancel{\delta(\vec{r}-\vec{r}')\hat{\psi}^+(\vec{r}')}\right)\left(-\frac{1}{2m}\Delta_{r'}\right)\hat{\psi}(\vec{r}')$$

$$-\int d^3r \int d^3r' \, \hat{\psi}^+(\vec{r}')\left(-\frac{1}{2m}\Delta_{r'}\right)\left(\cancel{\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r}')\hat{\psi}(\vec{r})} + \cancel{\delta(\vec{r}-\vec{r}')\hat{\psi}(\vec{r}')}\right)$$

$$= 0 \qquad \text{also for fermionic fields}$$

The total number of particles of the non-interacting Schrödinger field is a constant of motion. Particles cannot be created or destroyed

Observation shows that this is also true for interacting Schrödinger fields

(D) Single- and two-particle operators

Looking at eq. (141) one recognizes that the Hamiltonian of the free Schrödinger field has the structure

$$\hat{H} = \int d^3r \, \hat{\psi}^+(\vec{r})\, h_D(\vec{r})\, \hat{\psi}(\vec{r}) \tag{143}$$

where $h_D(\vec{r}) = -\frac{1}{2m}\vec{\nabla}^2$ is the single-particle

Hamilton operator in position representation.

In general one finds for single-particle operators

(144) $$\boxed{\hat{O}_1 = \int d^3r \; \hat{\psi}^+(\vec{r}) \, O_D(\vec{r}) \, \hat{\psi}(\vec{r})}$$

examples:

(145) $$\boxed{\hat{\vec{r}} = \int d^3r \; \hat{\psi}^+(\vec{r}) \, \vec{r} \, \hat{\psi}(\vec{r})}$$

center of mass of Schrödinger field

(146) $$\boxed{\hat{\vec{p}} = \int d^3r \; \hat{\psi}^+(\vec{r}) \, (-i\vec{\nabla}) \, \hat{\psi}(\vec{r})}$$

total momentum

Likewise one can introduce two-particle operators

$$\boxed{\hat{O}_2 = \int d^3r \int d^3r' \; \hat{\psi}^+(\vec{r}) \, \hat{\psi}^+(\vec{r}\,') \, O_D(\vec{r}, \vec{r}\,') \, \hat{\psi}(\vec{r}\,') \, \hat{\psi}(\vec{r})}$$

(147)

All of these operators commute with the total particle number

(148) $$[\hat{O}_1, \hat{N}] = [\hat{O}_2, \hat{N}] = 0$$

(E) Schrödinger field with interactions

The Hamiltonoperator of an interacting Schrödinger-field has the generic structure

$$\hat{H} = \int d^3r\, \hat{\psi}^+(\vec r)\, h_D(\vec r)\, \hat{\psi}(\vec r)$$

$$(149) \quad + \int d^3r \int d^3r'\, \hat{\psi}^+(\vec r)\, \hat{\psi}^+(\vec r')\, V_2(\vec r, \vec r')\, \hat{\psi}(\vec r')\, \hat{\psi}(\vec r)$$

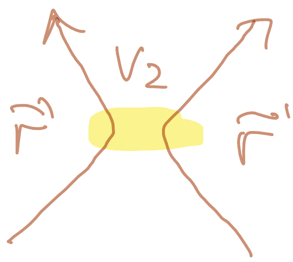
$$+ \int d^3r \int d^3r' \int d^3r''\, \hat{\psi}^+(\vec r)\, \hat{\psi}^+(\vec r')\, \hat{\psi}^+(\vec r'')\, V_3(\vec r, \vec r', \vec r'') \cdot$$

$$\cdot\, \hat{\psi}(\vec r'')\, \hat{\psi}(\vec r')\, \hat{\psi}(\vec r)$$

$$+ \dots$$

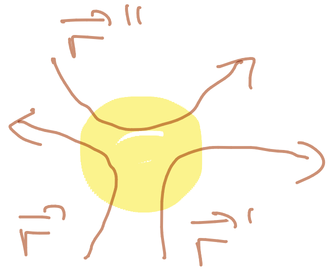
where $V_2(\vec r, \vec r')$ and $V_3(\vec r, \vec r', \vec r'')$ are a two and 3-particle interaction potential

$V_2(\vec r, \vec r')$



e.g. Coulomb
$$\frac{q_1 q_2}{4\pi\varepsilon_0 |\vec r - \vec r'|}$$

$V_3(\vec r, \vec r', \vec r'')$



← we have not encountered this type yet

and

$$(150) \quad h_D(\vec r) = -\frac{1}{2m}\vec\nabla^2 + V_1(\vec r)$$

kinetic   potential energy

A consequence of the particle-number conservation is the

(F) Equivalence between non-relativistic Schrödinger field theory and N-particle quantum mechanics

N-particle wavefunction

(157)
$$\phi_N(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N; t) \equiv \langle 0 | \hat{\psi}(\vec{r}_1, t)\hat{\psi}(\vec{r}_2, t) \cdots \hat{\psi}(\vec{r}_N, t) | \phi \rangle$$

where $|0\rangle$ is the vacuum and $|\phi\rangle$ the N-particle quantum state in Dirac notation

A nice property of the representation in terms of a quantum field ("2nd quantization") is that symmetry requirements of bosonic / fermionic N-particle wavefunctions are automatically taken care of:

$$[\hat{\psi}(\vec{r}_j, t), \hat{\psi}(\vec{r}_k, t)] = 0$$

$$\Longrightarrow$$

$$\phi(\vec{r}_1, \ldots, \vec{r}_j, \ldots, \vec{r}_k, \ldots \vec{r}_N) = \phi(\vec{r}_1, \ldots \vec{r}_k, \ldots, \vec{r}_j, \ldots \vec{r}_N)$$

total symmetry of bosonic N-particle wavefunction

$$\{\hat{\psi}(\vec{r}_j,t), \hat{\psi}(\vec{r}_k,t)\} = 0$$

$$\Rightarrow$$

$$\phi(\vec{r}_1, \ldots, \vec{r}_j, \ldots, \vec{r}_k, \ldots \vec{r}_N) = -\phi(\vec{r}_1, \ldots, \vec{r}_k, \ldots, \vec{r}_j, \ldots \vec{r}_N)$$

Since $\hat{N}$ is a conserved quantity the state vector of the field can always be represented in terms of the N-particle wavefunction if N is the number of particles present at some initial time

$$\boxed{|\phi(t)\rangle = \frac{1}{\sqrt{N!}} \int d^3r_1 \cdots \int d^3r_N \; \phi_N(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N; t) \, \hat{\psi}^+(\vec{r}_1) \hat{\psi}^+(\vec{r}_2) \cdots \hat{\psi}^+(\vec{r}_N) |0\rangle} \quad (152)$$

where $\hat{\psi}^+(\vec{r}_j) = \hat{\psi}^+(\vec{r}_j, t=0)$

This establishes the relation between the quantum field theory of the Schrödinger field and non-relativistic many-body quantum mechanics