the fundamental Poisson brackets for fields

$$\boxed{\begin{gathered}\{\phi(\vec{x},t),\phi(\vec{y},t)\} = \{\pi(\vec{x},t),\pi(\vec{y},t)\} = 0 \\ \{\phi(\vec{x},t),\pi(\vec{y},t)\} = \delta(\vec{x}-\vec{y})\end{gathered}} \tag{68}$$

proof:

$$\{\phi(\vec{x},t),\pi(\vec{y},t)\} = \int d^3z \left(\frac{\delta\phi(\vec{x})}{\delta\phi(\vec{z})}\frac{\delta\pi(\vec{y})}{\delta\pi(\vec{z})} - \underbrace{\frac{\delta\phi(\vec{x})}{\delta\pi(\vec{z})}\frac{\delta\pi(\vec{y})}{\delta\phi(\vec{z})}}_{=0} \right)$$

$$= \int d^3z\, \delta(\vec{x}-\vec{z})\,\delta(\vec{y}-\vec{z})$$

$$= \delta(\vec{x}-\vec{y}) \qquad \square$$

I.4 Lagrange density of some important fields

What are the general properties a Lagrange density should have?

- relativistic covariance $\Rightarrow$ $\mathcal{L}$ is Lorentz-scalar independent on x^μ

- linear field theory (superposition) $\Rightarrow$ $\mathcal{L}$ is at most bilinear in ϕ and $\partial_\mu \phi$

- field equations at most of 2nd order $\Rightarrow$ $\mathcal{L}$ only function of ϕ and $\partial_\mu \phi$

- local field theory $\Rightarrow$ $\mathcal{L} = \mathcal{L}[\phi(x), \partial_\mu \phi(x)]$

- dynamical quantities are real $\Rightarrow$ $\mathcal{L} = \mathcal{L}^*$

(A) non-relativistic Schrödinger field

complex field, only 1st-order derivative in time ($\hbar = 1$), non-relativistic

$$\boxed{\mathcal{L} = -\frac{i}{2}(\partial_t \psi^*)\psi + \frac{i}{2}\psi^*(\partial_t \psi) - \frac{1}{2m}\vec{\nabla}\psi^* \cdot \vec{\nabla}\psi} \tag{69}$$

derivation of field equation and Hamilton-density $\longrightarrow$ problem course note: $\mathcal{L}$ is invariant under the continuous transformation

$$\psi \longrightarrow \psi e^{i\phi} \qquad U(1) \text{ transformation}$$

We will see later that this has an important consequence. There will be a conserved quantity that allows to interprete $|\psi|^2$ as probability density. More on this later.

(B) Klein-Gordon field $(c=1)$

scalar, real field, relativistic theory

$$\phi(x^\mu)$$

the only bilinear Lorentz scalars are

(70) $$\partial_\mu \phi \partial^\mu \phi \quad \text{and} \quad \phi^2$$

(71) $$\boxed{\mathcal{L}_{KG} = \frac{1}{2}\partial_\nu \phi \partial^\nu \phi - \frac{m^2}{2}\phi^2}$$

Euler-Lagrange equations (Note that $\partial_\nu \phi$ and $\partial^\nu \phi$ are dependent)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = -m^2\phi - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}\phi$$

$$\Rightarrow \quad (71) \quad \boxed{(\partial_\nu \partial^\nu + m^2)\phi = (\Box + m^2)\phi = 0}$$

Klein-Gordon equation

canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi = \partial_0 \phi = \dot{\phi}$$

$$\mathcal{H}_{KG} = \dot{\phi}\pi - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\partial_j \partial^j \phi + \frac{m^2}{2}\phi^2$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2$$

$$(72) \quad \boxed{H_{KG} = \frac{1}{2}\int dV \left(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)}$$

Hamiltonian of Klein-Gordon field

(c) Electromagnetic field

6 real fields, Lorentz invariant, field tensor $F^{\mu\nu}$

Lorentzscalar bilinear in $F^{\mu\nu}$: $F^{\mu\nu}F_{\mu\nu}$

$$(73) \quad \boxed{\mathcal{L}_{ED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \vec{E}^2 - \vec{B}^2}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

→ canonical field variable is A^ν or A_ν

Euler-Lagrange

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = 0 + 4 \frac{\partial}{\partial x^\nu} \frac{1}{4} F^{\mu\nu} = 0$$

(74) $$\boxed{\partial_\nu F^{\nu\mu} = 0}$$ inhomogeneous MWE without sources

lets add coupling to external source j^μ

simple Lorentz-scalar $j_\mu A^\mu$

(75) $$\boxed{\mathcal{L}_{ED} \rightarrow \mathcal{L}_{ED} - j_\mu A^\mu = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu}$$

Euler-Lagrange equations then get additional term, since now

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -j^\mu \neq 0$$

Thus

(76) $$\boxed{\partial_\nu F^{\nu\mu} = j^\mu}$$ inhomogeneous MWE with sources

The homogeneous MWE follow from a relation which is automatically fulfilled → problem courses

In order to have a consistent theory with sources, we need to have

$$(77) \quad \partial_\mu j^\mu = \underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} \underbrace{F^{\nu\mu}}_{\text{antisymmetric}} = 0$$

This is of course nothing else than charge conservation, which must hold for the external sources.

<u>gauge transformations</u>

The field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under continuous gauge transformation

$$(78) \quad A^\mu \longrightarrow \tilde{A}^\mu = A^\mu + \partial^\mu \Lambda(x)$$

since $\tilde{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + \underbrace{\partial^\mu \partial^\nu \Lambda(x) - \partial^\nu \partial^\mu \Lambda(x)}_{=0}$

The Lagrange density of the em. field coupled to <u>external sources</u> is <u>not gauge invariant</u>.

But the Lagrangian is gauge invariant up to a canonical trafo (which is irrelevant for the physics)

$$L \to \tilde{L} = \int dV \left(\mathcal{L} - j_\mu \partial^\mu \Lambda \right)$$

$$= \int dV \Big(\mathcal{L} - \underset{\mu = 0,1,2,3}{\partial^\mu (j_\mu \Lambda)} + \Lambda \underbrace{\partial^\mu j_\mu}_{=0} \Big)$$

$$= \int dV \Big(\mathcal{L} - \underset{k = 1,2,3}{\partial^k j_k \Lambda} - \partial^0 j_0 \Lambda \Big)$$

volume integral over
3-divergence $\partial^k O_k$
$\to 0$

i.e.

(79) $$\tilde{L} = L - \frac{d}{dt} \int dV \, j_0 \Lambda(x)$$

However we remember from canonical mechanics that adding a total time derivative to a Lagrangian is just a canonical transformation that does not change the physics.

canonical momenta

fields $\quad A^\mu = (A^0, A^1, A^2, A^3)$

Here we have a problem

(80) $$\boxed{\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = 0 \quad !}$$

i.e. $\nexists$ canonical conjugate field to A^0. Thus the poisson bracket with π^0 always vanishes. This will cause a problem for quantization later!

In order to fix this problem we have to fix the gauge!

e.g. $\partial_\mu A^\mu = 0$ Lorentz gauge

or $\vec{\nabla} \cdot \vec{A} = \partial_j A^j = 0$ Coulomb gauge

(i) elimination of A^0 in Coulomb gauge

This is the conceptually simpler approach but comes at the price of giving up explicit covariance as the Coulomb gauge depends on the inertial frame

(81) $$\vec{\nabla} \cdot \vec{A} = 0 \qquad \text{i.e. } \vec{A} = \vec{A}_\perp$$

interlude: transversal & longitudinal vector fields

- It is always possible to decompose any vector field as

$$\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp}$$

with $\quad \vec{\nabla}\cdot\vec{F}_{\perp} = 0 \quad$ and $\quad \vec{\nabla}\times\vec{F}_{\parallel} = 0$

- How to construct $\vec{F}_{\parallel}$, $\vec{F}_{\perp}$ out of $\vec{F}$?

We know from Edynamics (e.g.)

$$\delta^{(3)}(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi}\Delta\frac{1}{|\vec{r}-\vec{r}\,'|}$$

Thus

$$\vec{F}(\vec{r}) = \int d^3r'\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,\vec{F}(\vec{r}\,')$$

$$= -\frac{1}{4\pi}\int d^3r'\,\vec{F}(\vec{r}\,')\,\Delta\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{4\pi}\int d^3r'\left\{\vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)\right\}$$

note that $\vec{F}$ depends on $\vec{r}\,'$ and <u>not</u> on $\vec{r}$

We see immediately

$$\vec{F}_{\parallel}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) \qquad \vec{\nabla}\times\vec{F}_{\parallel} = 0$$

Define

$$\boxed{\begin{aligned} \overleftrightarrow{\delta}_{\parallel}^{(3)}(\vec{r}-\vec{r}\,') &= \frac{1}{4\pi}\int d^3r' \, \vec{\nabla}\circ\vec{\nabla}\,\frac{1}{|\vec{r}-\vec{r}\,'|} \\ \overleftrightarrow{\delta}_{\perp}^{(3)}(\vec{r}-\vec{r}\,') &= \delta^{(3)}(\vec{r}-\vec{r}\,') - \overleftrightarrow{\delta}_{\parallel}^{(3)}(\vec{r}-\vec{r}\,') \end{aligned}}$$

Fouriertrafo

$$\overleftrightarrow{\delta}_{\parallel}^{(3)}(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int d^3k \, \frac{\vec{k}\circ\vec{k}}{k^2}\, e^{i\vec{k}\cdot\vec{r}}$$

$$\overleftrightarrow{\delta}_{\perp}^{(3)}(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3}\int d^3k \left(\mathbb{1} - \frac{\vec{k}\circ\vec{k}}{k^2}\right) e^{i\vec{k}\cdot\vec{r}}$$

[Notation "$\circ$" dyadic product $\vec{A}\circ\vec{B}$: matrix

$$(\vec{A}\circ\vec{B})_{ij} = A_i B_j$$]

With eq. (87) we can formally eliminate A^0 from the theory (and then don't have to be concerned about π^0)

$$\partial_\mu F^{\mu 0} = j^0 = \rho$$

$$\partial_\mu\left[\partial^\mu A^0 - \partial^0 A^\mu\right] = -\Delta A^0 - \partial_0 \vec{\nabla}\cdot\vec{A} = -\Delta A^0 = \rho$$

($\mu = 0$ term drops out)

The last equation can be solved

(82) $A^0(\vec{r},t) = A_0 = \frac{1}{4\pi}\int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$ Coulomb law

Thus

$A^0 = A_0 \quad A^k = -A_k$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - j_\nu A^\nu$$

$$= -\frac{1}{2}\partial_\mu A^0(\partial^\mu A^0 - \partial^0 A^\mu) + \frac{1}{2}\partial_\mu A^k(\partial^\mu A^k - \partial^k A^\mu)$$

$$-\rho A^0 - j_k A^k$$

$$= -\frac{1}{2}\partial_k A^0(\partial^k A^0 - \partial^0 A^k) + \frac{1}{2}\partial_0 A^k(\partial^0 A^k - \partial^k A^0)$$

$$\underbrace{-\frac{1}{2}\partial_j A^k(\partial_j A^k - \partial_k A^j)}_{\partial^j = -\partial_j} - \rho A^0 - \vec{j}\cdot\vec{A}$$

Let's simplify terms $\quad \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

(i) $\partial_j A^k(\partial_j A^k - \partial_k A^j) = \vec{B}^2 \qquad B_i = \varepsilon_{ijk}\partial_j A^k$

(ii) $\partial_k A^0 \partial^k A^0 = \underbrace{\partial_k(A^0\partial^k A^0)}_{\text{3-divergence will vanish in } L = \int d^3r\,\mathcal{L} \,\to\, \text{disregard}} - A^0 \underbrace{\partial_k \partial^k A^0}_{\Delta A^0 = -\rho} = -A^0\rho$

(iii) $\partial_k A^0 \partial_0 A^k = \underbrace{\partial_0(\partial_k A^0 A^k)}_{\text{total time derivative in } L \text{ can be eliminated by canonical trafo}} - A^k \partial_0 \partial_k A^0$

This leaves

$$\mathcal{L} = \frac{1}{2}\rho A_0 - \frac{1}{2}A^k \partial_0 \partial_k A^0 + \frac{1}{2}\underbrace{\partial_0 A^k \partial^0 A^k}_{= \vec{E}_\perp^{\,2}}$$

$$- \frac{1}{2}A^k \partial_0 \partial_k A^0$$

$$- \frac{1}{2}\vec{B}^2 - \rho A_0 - \vec{j}\cdot\vec{A}$$

Since
$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \vec{\nabla} A^0 = \vec{E}_\perp + \vec{E}_\parallel$$

thus

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}_\perp^{\,2} - \vec{B}^2\right) - \frac{1}{2}\rho A_0 - \underbrace{\left(j^k - \partial_0 \partial_k A^0\right) A_k}_{=?}$$

$$j^k - \partial_0 \partial_k A^0 = \Big(\underbrace{\vec{j} - \partial_t \vec{\nabla}\phi}_{\text{definition } = \vec{j}_\perp}\Big)_k$$

check
$$\vec{\nabla}\cdot\vec{j}_\perp = \vec{\nabla}\cdot\vec{j} - \partial_t \Delta\phi = \vec{\nabla}\cdot\vec{j} + \partial_t \rho = 0 \checkmark$$

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}_\perp^{\,2} - \vec{B}^2\right) - \frac{1}{2}\rho A_0 + \vec{j}_\perp \cdot \vec{A}_\perp$$

Neither $\vec{E}_\perp$ nor $\vec{B}$ depend on A_0 (only $\vec{E}_\parallel$ would). So if we finally eliminate A_0 using Coulombs law (82) we arrive at ($\vec{B} = \vec{B}_\perp$ anyway)

$$L_{ED} = \int d^3r\, \mathcal{L} = \int d^3r \left[\frac{1}{2}\left(\vec{E}_\perp^{\,2} - \vec{B}_\perp^{\,2}\right) + \vec{j}_\perp \cdot \vec{A}_\perp \right]$$

(83) $$- \frac{1}{8\pi} \int d^3r \int d^3r' \, \frac{\rho(\vec{r},t)\,\rho(\vec{r}\,',t)}{|\vec{r} - \vec{r}\,'|}$$

Lagrangian of elm.-field coupled to sources in Coulomb gauge (drawback: not Lorentz invariant) used for quantization of light-matter interaction

Now we can proceed to a

Hamiltonian formulation

(84) • field $\boxed{A_\perp^i(\vec{r},t) = A^i(\vec{r},t)}$

(85) • conjugate field $\boxed{\pi^i(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\perp^i)} = -\frac{\partial A_\perp^i}{\partial t} = -E_\perp^i}$

fundamental Poisson brackets

(86) $$\boxed{\{A_\perp^i(\vec{r},t), A_\perp^j(\vec{r}\,',t)\} = \{E_\perp^i(\vec{r},t), E_\perp^j(\vec{r}\,',t)\} = 0}$$

(87) $$\boxed{\left\{A_\perp^i(\vec r,t),\, \pi^j(\vec r',t)\right\} = \left\{A_\perp^i(\vec r,t),\, -E_\perp^j(\vec r',t)\right\} = \left(\overleftrightarrow{\delta}_\perp(\vec r-\vec r')\right)_{ij}}$$

We have used the transversal Deltafunction instead of $\delta(\vec r-\vec r')$ since

$$\vec\nabla_{\vec r}\cdot\left\{\vec A_\perp(\vec r,t),\, \vec E_\perp(\vec r',t)\right\} = \vec\nabla_{\vec r'}\cdot\left\{\vec A_\perp(\vec r,t),\, \vec E_\perp(\vec r',t)\right\} \overset{!}{=} 0$$

as $\vec\nabla_{\vec r}\cdot\vec A_\perp = 0$ and $\vec\nabla_{r'}\cdot\vec E_\perp(\vec r') = 0$

and $\vec\nabla_{\vec r}\cdot\overleftrightarrow{\delta}_\perp(\vec r-\vec r') = \vec\nabla_{\vec r'}\cdot\overleftrightarrow{\delta}_\perp(\vec r-\vec r') = 0$

The Hamiltonian finally reads in Coulomb gauge

(88) $$\boxed{\begin{aligned} H &= \int d^3r\left(\vec\pi\cdot\frac{\partial\vec A_\perp}{\partial t} - \mathcal{L}\right)\\ &= \int d^3r\left\{\frac{1}{2}\left(\vec E_\perp^2+\vec B^2\right) - \vec j_\perp\cdot\vec A_\perp\right\}\\ &\quad + \frac{1}{8\pi}\int d^3r\int d^3r'\,\frac{\rho(\vec r,t)\,\rho(\vec r',t)}{|\vec r-\vec r'|}\end{aligned}}$$

An alternative to using the Coulomb gauge to circumvent the problem with π^0 is the so-called Gupta-Bleuler approach:

(ii) gauge fixing term in Lagrange density

If one uses the Lorentz gauge, which is Lorentz-invariant and thus does not require to fix the inertial frame,

(89) $$\partial_\mu A^\mu = 0$$

One can add a term to the Lagrange density

(90) $$\mathcal{L}_{GB} = -\frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right)$$

which does not affect the physics as long as the Lorentz gauge is applied later. If we not fix Lorentz gauge apriori and let

(91) $$\mathcal{L}_{ED} \longrightarrow \tilde{\mathcal{L}}_{ED} = \mathcal{L}_{ED} + \mathcal{L}_{GB}$$

Now $\tilde{\mathcal{L}}_{ED}$ contains $\partial_0 A^0$ and π^0 can be defined. With this we can quantize the field with all 4 components. We can then not require the Lorentz gauge to hold on the level of operators but must require that for all physically allowed states holds

(92) $$\partial_\mu \hat{A}^\mu |\psi\rangle = 0 \quad \Leftarrow \quad \cancel{\partial_\mu \hat{A}^\mu = 0}$$

(D) Dirac field

Lagrange density should contain first derivatives only in a linear way

Lorentz scalars that are linear in derivatives are

(93) $$\bar{\psi}\gamma^\mu\partial_\mu\psi \quad \text{or} \quad (\partial_\mu\bar{\psi})\gamma^\mu\psi$$

mass term without derivatives

(94) $$m\bar{\psi}\psi$$

Thus the Lagrange density reads

(95) $$\boxed{\mathcal{L}_D = \bar{\psi}\left[\frac{i}{2}\gamma^\mu\overleftrightarrow{\partial}_\mu - m\right]\psi}$$

Lagrange density of Dirac field

where

(96) $$\overleftrightarrow{\partial}_\mu \equiv \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$$

i.e.

$$\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi$$

Since ψ is a complex valued object, we can either treat $\mathrm{Re}[\psi]$ and $\mathrm{Im}[\psi]$ or ψ and $\bar{\psi}$ as independent fields.

The Euler-Lagrange equations read (see problem course) e.g. for $\bar{\psi}$

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}_D}{\left(\partial\frac{\partial\bar{\psi}}{\partial x^\mu}\right)} - \frac{\partial\mathcal{L}_D}{\partial\bar{\psi}} = -\frac{i}{2}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\gamma^\mu\partial_\mu\psi + m\psi \tag{97}$$

$$= -(i\gamma^\mu\partial_\mu - m)\psi = 0$$

This is of course just the Dirac equation for ψ. In a similar way one could derive the Dirac equation for $\bar{\psi}$

conjugate fields

$$\boxed{\begin{aligned}\pi &= \frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\psi})} = -\frac{i}{2}\gamma^0\psi \\ \bar{\pi} &= \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \frac{i}{2}\bar{\psi}\gamma^0 = \frac{i}{2}\psi^+\end{aligned}} \tag{98}$$

ψ^+ is conjugate to ψ; note $\bar{\pi} = \psi^+$ is a row vector while ψ is a column vector of 4 components.

Hamilton density

$$\mathcal{H}_D = \bar{\pi}\frac{\partial\psi}{\partial t} + \frac{\partial\bar{\psi}}{\partial t}\pi - \mathcal{L}_D = \frac{i}{2}\bar{\psi}\gamma^0\frac{\overleftrightarrow{\partial}}{\partial t}\psi - \mathcal{L}_D$$

row column

$$\mathcal{H}_D = \bar{\psi}\left[-\frac{i}{2}\gamma^k \frac{\overleftrightarrow{\partial}}{\partial x^k} + m\right]\psi = \psi^+\left[-\frac{i}{2}\alpha_k \frac{\overleftrightarrow{\partial}}{\partial x^k} + \beta m\right]\psi \tag{99}$$

α_k, β Dirac matrices

$$H_D = \int d^3r\, \mathcal{H}_D \underset{\text{part. Int.}}{=} \int d^3r\, \psi^+\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right]\psi \tag{100}$$

Hamiltonian of Dirac field

I.5 Noether theorem for fields: continuous symmetries & conservation laws

We know from classical mechanics that continuous symmetries of a Lagrangian imply conservation laws. We will now see that the same is true for fields, albeit in a more general form

Noether theorem

For each continuous transformation of coordinates x^μ and fields ϕ^i which leave the action in any finite volume invariant, there is a function of the fields and their derivatives which is conserved.

let ε^i be a set of infinitesimal parameter that characterizes the transformations ("i" numbers trafo)

$$x^\mu \longrightarrow x'^\mu = x^\mu + \lambda_i^\mu(x)\,\varepsilon^i \qquad \text{coordinates}$$

$$\varphi_\alpha \longrightarrow \varphi'_\alpha(x') = \varphi_\alpha(x) + \Omega_{\alpha i}(x)\,\varepsilon^i \qquad \text{fields}$$

and assume the action is invariant

$$S_V = \int_V d^4x\, \mathcal{L}(x) \overset{!}{=} S'_{V'} = \int_{V'} d^4x'\, \mathcal{L}'(x')$$

(note $V \to V'$)

Noether theorem

(101) $$\boxed{\partial_\mu O_i^\mu = 0}$$

local conservation law

"i" labels symmetry trafo

where

(102) $$\boxed{O_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)} \left[\frac{\partial \varphi_\alpha}{\partial x^\nu} \lambda_i^\nu - \Omega_{\alpha i} \right] - \mathcal{L}\, \lambda_i^\mu}$$

the conservation law implies a conserved Noether charge

(103) $$\boxed{Q_i = \int d^3r\, O_i^0(\vec{r},t) = \text{const}_t}$$

proof:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}'(x')$$

$$S' = \int_{V'} d^4x'\, \mathcal{L}'(x') \stackrel{!}{=} S = \int_V d^4x\, \mathcal{L}(x)$$

$$0 = S' - S = \int_{V'} d^4x' \left(\mathcal{L}'(x') - \mathcal{L}(x')\right) + \int_{V'} d^4x'\, \mathcal{L}(x') - \int_V d^4x\, \mathcal{L}(x)$$

1st order in ε

$$= \int_V d^4x \left(\mathcal{L}'(x) - \mathcal{L}(x)\right) + \int_V d^4x \det\left|\frac{\partial x^{\mu\prime}}{\partial x^\mu}\right| \left(\mathcal{L}(x) + \frac{\partial \mathcal{L}(x)}{\partial x^\mu}\frac{\partial x^\mu}{\partial \varepsilon}\varepsilon\right)$$

$$-\int_V d^4x\, \mathcal{L}(x)$$

expand Jacobi-determinant $\det\left(\frac{\partial x^{\mu\prime}}{\partial x^\mu}\right)$ to first order in ε

- $$\det\left(\frac{\partial x^{\mu\prime}}{\partial x^\mu}\right) = \det\left(\mathbb{1} + \frac{\partial \lambda^\mu}{\partial x^\nu}\varepsilon\right) \simeq 1 + \frac{\partial \lambda^\mu}{\partial x^\mu}\varepsilon$$

- $$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi_\alpha}\delta\varphi_\alpha + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_\alpha}{\partial x^\mu}\right)}\delta\left(\frac{\partial \varphi_\alpha}{\partial x^\mu}\right)$$

$$\delta\varphi_\alpha = \varphi'_\alpha(x) - \varphi_\alpha(x) = \underbrace{\varphi'_\alpha(x) - \varphi'_\alpha(x')}_{-\frac{\partial \varphi_\alpha}{\partial x^\mu}\underbrace{\delta x^\mu}_{=\lambda^\mu\varepsilon}} + \underbrace{\varphi'_\alpha(x') - \varphi_\alpha(x)}_{=\Omega_\alpha(x)\varepsilon}$$

i.e.

$$\delta\varphi_\alpha = \left(-\frac{\partial \varphi_\alpha}{\partial x^\mu}\lambda^\mu + \Omega_\alpha\right)\varepsilon$$

similarly

$$\delta\left(\frac{\partial \psi_\alpha}{\partial x^\mu}\right) = \cdots = \frac{\partial}{\partial x^\mu}\left(-\frac{\partial \psi_\alpha}{\partial x^\nu}\lambda^\nu + \Omega_\alpha\right)\varepsilon$$

this yields

$$0 = \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \psi_\alpha}\left(\Omega_\alpha - \frac{\partial \psi_\alpha}{\partial x^\nu}\lambda^\nu\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\alpha}{\partial x^\mu}\right)} \frac{\partial}{\partial x^\mu}\left(\Omega_\alpha - \frac{\partial \psi_\alpha}{\partial x^\nu}\lambda^\nu\right) + \frac{\partial \mathcal{L}}{\partial x^\mu}\lambda^\mu + \frac{\partial \lambda^\mu}{\partial x^\mu}\mathcal{L} \right]\varepsilon$$

We now eliminate $\partial\mathcal{L}/\partial\psi_\alpha$ using the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \psi_\alpha} = \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\alpha}{\partial x^\mu}\right)}$$

This yields

$$0 = \int_V d^4x \frac{\partial}{\partial x^\mu}\underbrace{\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\alpha}{\partial x^\mu}\right)}\left(\Omega_\alpha - \frac{\partial \psi_\alpha}{\partial x^\nu}\lambda^\nu\right) + \lambda^\mu \mathcal{L}\right]}_{= -\Theta^\mu}\varepsilon$$

So

$$\partial_\mu \Theta^\mu = 0 \qquad \square$$

example **invariance of Dirac theory under phase transformation**

$$x \to x' = x$$

$$\psi(x) \longrightarrow \psi'(x) = e^{-iq\varepsilon}\psi(x) \simeq \psi(x) - iq\varepsilon\psi(x)$$

so $\quad \lambda^\mu \equiv 0 \qquad \Omega = -iq\psi(x)$

conserved 4-current

$$\Theta^\mu = -\frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \psi)}\Omega = iq\frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \psi)}\psi$$

Dirac Lagrangian $\quad \dfrac{\partial \mathcal{L}_D}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu$

(104) $$\boxed{\Theta^\mu = -q\bar{\psi}\gamma^\mu\psi = -j_D^\mu}$$

The continuous phase symmetry of Lagrange densities of complex valued fields, such as the Dirac field implies some sort of charge conservation

(105) $$\boxed{\partial_\mu j_D^\mu = 0}$$

I.6 Invariance under local phase trafo's

We have seen that the Dirac field as well as other field theories for complex fields are invariant under global phase transformations

global $U(1)$ symmetry

$$\psi(x) \longrightarrow \psi'(x) = e^{iq\theta}\psi(x)$$

What happens if the global phase trafo is generalized to a local one? $\theta \to \theta(x^\mu)$

Dirac:
$$\psi(x) \longrightarrow \psi'(x) = e^{iq\theta(x)}\psi(x)$$
$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = e^{-iq\theta(x)}\bar{\psi}(x)$$

$$\mathcal{L}_D \to \mathcal{L}_D' = \frac{1}{2}\bar{\psi}'(i\gamma^\mu \overrightarrow{\partial}_\mu - m)\psi' + \frac{1}{2}\bar{\psi}'(-i\gamma^\mu \overleftarrow{\partial}_\mu - m)\psi'$$

$$= \mathcal{L}_D + q\bar{\psi}\gamma^\mu\psi\,\partial_\mu\theta(x) \neq \mathcal{L}_D \tag{106}$$

$\Rightarrow$ Lagrange density does not remain invariant!

Interestingly invariance can be restored if we add a coupling to a vector field A_μ

(107) $$\mathcal{L}_D \longrightarrow \mathcal{L} = \mathcal{L}_D - j^\mu A_\mu$$

and if this vector field transforms simultaneously like

(108) $$\boxed{A_\mu \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)}$$

which is nothing else than a gauge transformation with gauge $\theta(x)$

(108) together with

(109) $$\boxed{\psi \longrightarrow \psi'(x) = e^{iq\theta(x)}\,\psi(x)}$$

constitutes what is called a joint

local gauge (phase) transformation

Under such a transformation one finds invariance of

(110) $$\boxed{\mathcal{L}' = \mathcal{L}_D' - j'^\mu A'_\mu = \mathcal{L}_D - j^\mu A_\mu = \mathcal{L}}$$

If we assume that the vector field A_μ, which is called a gauge field is a dynamical quantity with a (free) Lagrangian of its own

(111) $$\boxed{\mathcal{L}_F \equiv \alpha F_{\mu\nu} F^{\mu\nu}}$$

we have constructed a minimal version of an interacting field theory, which is invariant under local gauge transformations. These types of theories play an important role in quantum field theory and are called

gauge field theories

If Ψ is the Dirac field and A_μ the gauge potential of electromagnetism local gauge invariance yields the minimum coupling Lagrangian of Quantum Electrodynamics QED

(112) $$\mathcal{L}_{MWD} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{ED} + \underbrace{\frac{1}{2} \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi}_{Dirac} \underbrace{- q \bar{\Psi} \gamma^\mu \Psi A_\mu}_{\text{minimum coupling}}$$

minimum coupling can be generated by

(113) $$\boxed{\overrightarrow{\partial}_\mu \to \overrightarrow{D}_\mu = \partial_\mu + iqA_\mu \qquad \overleftarrow{\partial}_\mu \to \overleftarrow{D}_\mu = \partial_\mu - iqA_\mu}$$