I. Elements of classical field theory

I.1 A little reminder about special theory of relativity (c=1)

special relativity uses Minkowski space

4-vector position

$\mu = 0, \underbrace{1,2,3}$
time space

(1) $x^\mu = (t, \vec{r})$ contravariant

$x_\mu = (t, -\vec{r})$ covariant component

metric tensor

(2) $$g_{\mu\nu} \hat{=} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g^{\mu\nu}$$

sum convention

$$f^\mu g_\mu \equiv \sum_{\mu=0}^{3} f^\mu g_\mu$$

4-gradient

(3a) $$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

for

(3b) $$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

Scalar product

(4) $$x \cdot y \equiv x^\mu y_\mu = x_\mu y^\mu$$
$$x \cdot x = x_0^2 - \vec{x} \cdot \vec{x}$$

relation between covariant & contravariant components

(5) $$a^\mu = g^{\mu\nu} x_\nu = \sum_{\nu=0}^{3} g^{\mu\nu} x_\nu$$
$$a_\mu = g_{\mu\nu} x^\nu$$

transformation between inertial systems

$$x^\mu \longrightarrow x'^\nu$$

(6) $$x'^\mu = L^\mu{}_\nu \, x^\nu$$ Lorentztrafo

Lorentztrafo has to conserve length of a 4-vector, i.e.

$$x'^{\mu} x'_{\mu} = \underbrace{L^{\mu}{}_{\nu} L_{\mu}{}^{\beta}}_{\text{sum over } \mu} x^{\nu} x_{\beta} \overset{!}{=} x^{\beta} x_{\beta}$$

$\Rightarrow$

(7) $$\det(L) = \pm 1$$

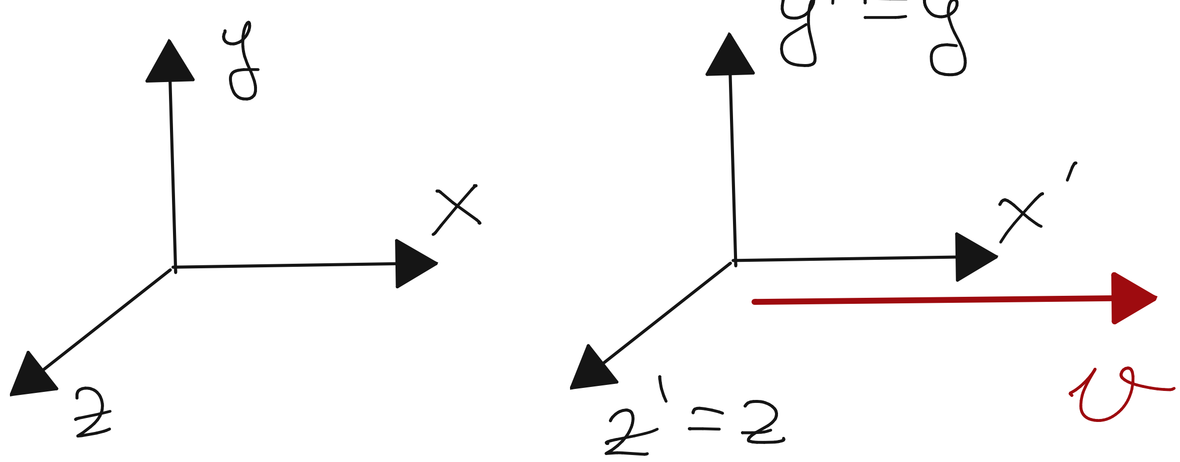
"+1" proper Lorentz trafo

(8) $$L^{0}{}_{0} \geq 1$$

orthochrone

special L-trafo's

- boost (in x)



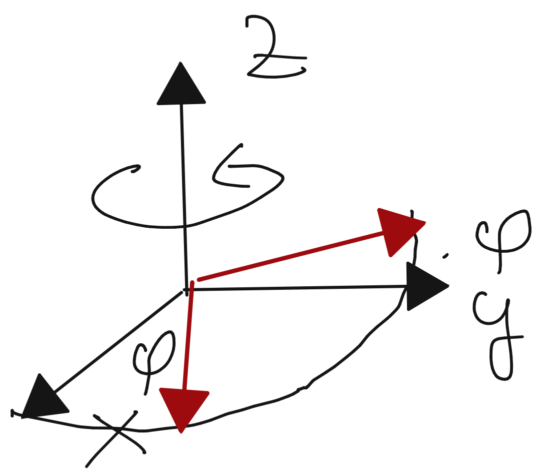
$$\cosh \xi = \frac{1}{\sqrt{1-v^2/c^2}} \qquad \sinh \xi = \frac{v}{\sqrt{1-v^2/c^2}}$$

(9) $$L^{\mu}{}_{\nu}(\xi) = \begin{pmatrix} \cosh \xi & -\sinh \xi & 0 & 0 \\ -\sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ξ rapidity

rotation about z-axis

$$L^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & \cos\varphi & \sin\varphi & \\ & -\sin\varphi & \cos\varphi & \\ & & & 1 \end{pmatrix} \tag{10}$$



time reversal

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{11}$$

inversion

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{12}$$

I.2 Examples of important field theories you may or may not know

(A) Non-relativistic quantum mechanics: the Schrödinger field ($\hbar = 1$)

$\psi(\vec{r}, t)$ single-component complex field

$$(13) \quad \boxed{i\frac{\partial}{\partial t}\psi(\vec{r},t) = -\frac{1}{2m}\Delta\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t)}$$

(B) Maxwell field

(rational Gauß units)
$c=1$ no "4π"

4-potential

$$A^\mu = (\phi, \vec{A}) \qquad (14)$$
$$A_\mu = (\phi, -\vec{A})$$

4-current density

$$j^\mu = (\rho, \vec{j}) \qquad (15)$$
$$j_\mu = (\rho, -\vec{j})$$

charge conservation

$$\partial_\mu j^\mu = \partial^\mu j_\mu = 0 \qquad (16)$$

Lorentz gauge

$$\partial_\mu A^\mu = \partial^\mu A_\mu = 0 \qquad (17)$$

Coulomb gauge (not L-invariant!)

$i=1,2,3$

$$\partial_i A^i = \partial^i A_i = 0 \qquad (18)$$

Wave equation in Lorentz gauge

(19) $$\boxed{\partial_\mu \partial^\mu A^\nu = \Box A^\nu = j^\nu}$$

field tensor

(20) $$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -F^{\nu\mu}$$

antisymmetric 4-dim matrix $\Rightarrow$ 6 components

(21) $$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & B_y \\ E_y & -B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

inhomogeneous Maxwell equations

(22) $$\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$

homogeneous Maxwell equations are automatically fulfilled

(23) gauge freedom $\quad A_\mu \to \tilde{A}_\mu = A_\mu - \partial_\mu \Lambda$

Λ Lorentzscalar

$$\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu =$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu \underbrace{- \partial^\mu \partial^\nu \Lambda + \partial^\nu \partial^\mu \Lambda}_{=0} = F^{\mu\nu}$$

C. Dirac field

$\psi(x)$ bispinor

(23) $$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$ γ^μ 4x4 matrix

(24) $$\gamma^\mu \hat{=} (\beta, \beta\alpha_i)$$ Dirac matrices

(25) $$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \alpha_i \hat{=} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

σ_i - Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

it holds

(26) $$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

furthermore:

(27) $$(\gamma^0)^+ = \gamma^0 \qquad (\gamma^i)^+ = -\gamma^i$$

hermitian $\qquad$ anti-hermitian

and generalization of both

(28) $$\boxed{\gamma^0\gamma^{\mu +}\gamma^0 = \gamma^\mu}$$

$\psi^+\psi$ is not a Lorentz scalar but

(29) $\bar{\psi}\psi$ is Lorentz scalar

where

(30) $\bar{\psi} = \psi^+\gamma^0$ adjoint bispinor

bilinear covariant objects

(31) $\bar{\psi}\psi$ scalar

(32) $\bar{\psi}\gamma^\mu\psi$ 4-vector

(33) $\bar{\psi}\,\sigma^{\mu\nu}\,\psi$ antisymmetric tensor

with

(34) $$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right] = -\sigma^{\nu\mu}$$

commutator of Dirac matrices

(35) $$\sigma^{0j} = i\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \qquad \sigma^{ij} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

ijk cyclic $= 1,2,3$

Lorentz transformation of bispinor

if $$x^\mu \to x'^\mu = L^\mu{}_\nu\, x^\nu$$

than $$\psi(x) \to \psi'(x') = S\,\psi(x)$$

in order for the Dirac equation to have the same form in all inertial frames it needs to hold

(36) $$S^{-1}\gamma^\mu S \overset{!}{=} L^\mu{}_\nu\,\gamma^\nu$$

$\Rightarrow$

(37) $$S = \exp\left\{\frac{1}{2}\vec{\xi}\cdot\vec{\alpha} - \frac{i}{2}\gamma^5\,\vec{n}_\theta\cdot\vec{\alpha}\,\theta\right\}$$

$\vec{\xi} = (\xi_x, \xi_y, \xi_z)$ rapidities in x, y, z directions

$\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ α-matrices

θ = rotation angle $\quad \vec{n}_\theta$: direction of rotation

(38) $$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

I.3 Lagrange-Hamilton formalism for fields

(A) as a reminder: mechanics

$q_j, \dot{q}_j$ coordinates and velocities

(39) $$L(q_j, \dot{q}_j, t) = T - V$$ Lagrangian

the fundamental principle of Lagrangian mechanics was that of minimal action

(40)
$$\boxed{\delta \int_{t_1}^{t_2} dt\, L(q_j, \dot{q}_j, t) = 0 \quad \text{where} \quad \delta q_j(t_{1,2}) = 0}$$

$$\delta L = \sum_j \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right)$$

partial integration in (40) yields

(41)
$$0 \overset{!}{=} \int_{t_1}^{t_2} dt\, \delta L = \sum_j \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right) \delta q_j$$
$$+ \sum_j \frac{\partial L}{\partial \dot{q}_j} \underbrace{\delta q_j \Big|_{t_1}^{t_2}}_{=0}$$

since (41) has to hold for all variations δq_j $\Rightarrow$

(42)
$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0}$$

Euler-Lagrange equations

this suggests to introduce generalized momenta

(43)
$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$

and a Legendre transformation from the set of variables $\{q_j, \dot{q}_j\} \longrightarrow \{q_j, p_j\}$

(44)
$$\boxed{H = \sum_j \dot{q}_j p_j - L} \qquad \boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j} \quad \dot{q}_j = \frac{\partial H}{\partial p_j}}$$

Hamiltonian — Hamilton equations

These equations can be cast in a compact form, introducing Poisson brackets

(45) $$\{f,g\} \equiv \sum_j \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}$$

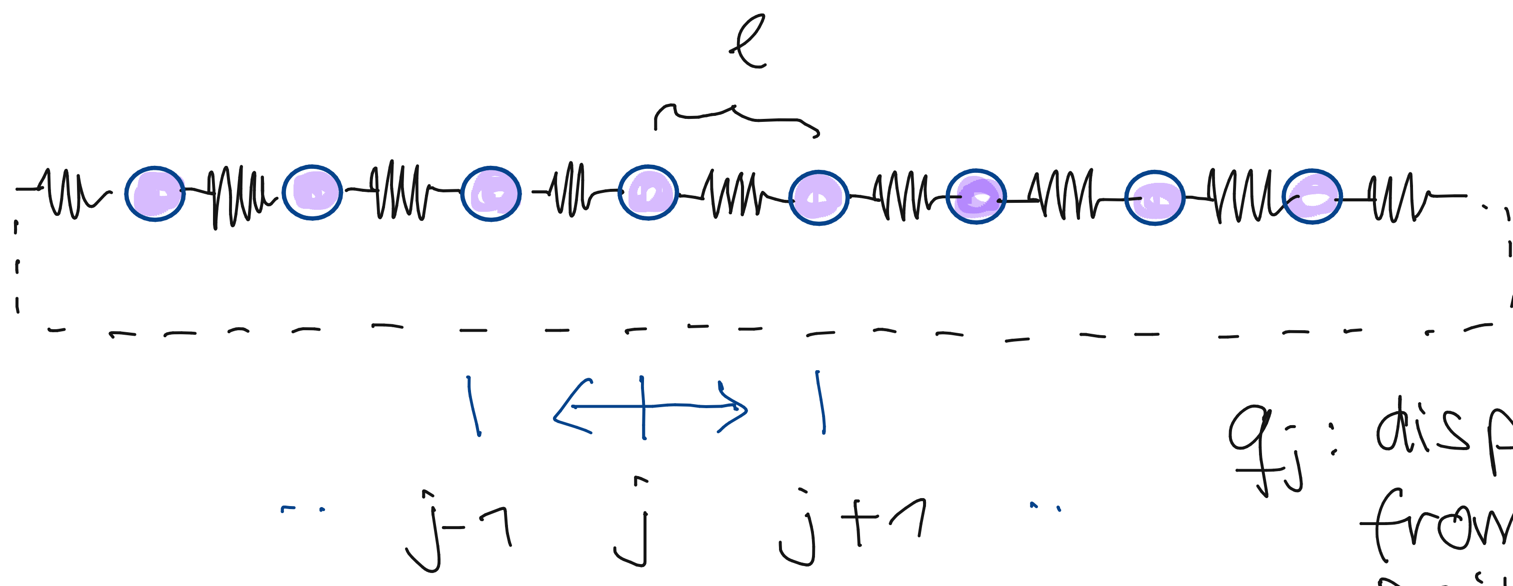
(46) $$\Rightarrow \quad \boxed{\dot{p}_j = \{p_j, H\}} \qquad \boxed{\dot{q}_j = \{q_j, H\}}$$

or for any function $f(q_j, p_j, t)$

(47) $$\boxed{\frac{d}{dt} f = \{f, H\} + \frac{\partial f}{\partial t}}$$

(B) 1D harmonic chain in an external force field

In order to see how the concept of Lagrange-Hamilton formalism can be extended from point particles to fields, we consider a one-dimensional harmonic chain of coupled masses m with springs of spring constant k:



PBC

q_j: displacement from equilibrium position

N coupled harmonic oscillators + external force

$$T = \frac{m}{2} \sum_{j=0}^{N-1} \dot{q}_j^{\,2} \tag{48}$$

$$V = \underbrace{\frac{k}{2} \sum_{j=0}^{N-1} (q_{j+1} - q_j)^2}_{\text{spring}} + \underbrace{\frac{g}{2} \sum_{j=0}^{N-1} q_j^2}_{\text{external harmonic confinement}}$$

Lagrange function

$$L = T - V = \sum_{j=0}^{N-1} \left[\frac{m}{2} \dot{q}_j^{\,2} - \frac{k}{2} (q_{j+1} - q_j)^2 - \frac{g}{2} q_j^2 \right] \tag{49}$$

Now lets make a continuum limit

$\ell \to 0$, $N \to \infty$, $m \to 0$, $k \to \infty$

$$L = N\ell = \text{const} \qquad \mu = \frac{m}{\ell} = \text{const}$$

$$\varkappa = k\ell = \text{const} \qquad \gamma = \frac{g}{\ell} = \text{const} \tag{50}$$

$$q_j(t) \longrightarrow q(x_j, t) \longrightarrow q(x, t)$$

$$\sum_{j=0}^{N-1} \ell \longrightarrow \int_0^L dx$$

This then yields

$$(51)\quad L = \int_0^L dx \left[\frac{\mu}{2}\left(\frac{\partial q(x,t)}{\partial t}\right)^2 - \frac{\varkappa}{2}\left(\frac{\partial q(x,t)}{\partial x}\right)^2 - \frac{\gamma}{2} q(x,t)^2 \right]$$

We notice that the Lagrangian has now the structure

$$(52)\quad \boxed{L = \int_0^L dx\, \mathcal{L}\left(q, \frac{\partial q}{\partial t}, \frac{\partial q}{\partial x}\right)}$$

$\mathcal{L}$ – Lagrange density

What happens to the Euler-Lagrange equations in the continuum limit?

$$\frac{\partial L}{\partial q_j} = -k\left[-(q_{j+1} - q_j) + (q_j - q_{j-1})\right] - g\, q_j$$

$$= \underbrace{k\ell}_{=\varkappa}\left[\frac{q_{j+1} - q_j}{\ell} - \frac{q_j - q_{j-1}}{\ell}\right] - \frac{g\ell}{\ell} q_j$$

$$\xrightarrow[\ell \to 0]{} \varkappa\left(\left.\frac{\partial q}{\partial x}\right|_{x+\frac{\ell}{2}} - \left.\frac{\partial q}{\partial x}\right|_{x-\frac{\ell}{2}}\right) - \gamma\ell q$$

$$= -\ell \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial x}\right)} + \ell \frac{\partial \mathcal{L}}{\partial q}$$

and similarly

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = m\ddot{q}_j = \mu \ell \frac{\partial^2 \varphi}{\partial t^2} - \ell \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)}$$

This then gives

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \rightarrow \ell\left[-\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} + \frac{\partial \mathcal{L}}{\partial \varphi}\right]$$

Thus the Euler Lagrange equation become

(53) $$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\rightarrow \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x^\mu}\right)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

The principle of minimum action then reads

(54) $$\int_{t_1}^{t_2} dt \int dx\, \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}\right) = 0$$

(c) Euler-Lagrange equation for fields, Hamilton formalism

The result for the harmonic chain can be generalized to the Lagrange density for arbitrary fields $\phi^i(x^\mu)$

$$\mathcal{L} = \mathcal{L}\left[\phi, \frac{\partial}{\partial x^\mu}\phi\right]$$

$$0 = \delta \int_{t_1}^{t_2} dt\, L = \delta \int_{t_1}^{t_2} dt \int dV\, \mathcal{L}[\phi, \partial_\mu \phi]$$

$$= \int_{t_1}^{t_2} dt \int dV \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)\right]$$

partielle Integration

$$= \int_{t_1}^{t_2} dt \int dV \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \left(\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi\right]$$

$$+ \underbrace{\int_{t_1}^{t_2} dt \int dV \underbrace{\frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]}_{\text{4-divergence}}}_{\text{becomes a "surface" integral } = 0 \text{ since } \delta\phi(t_2) = \delta\phi(t_1) = 0}$$

Thus we find the Euler Lagrange equations for fields

(55) $$\boxed{\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x^\mu}\right)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$ for all x^μ

canonical momentum $\Rightarrow$ for the chain

(56) $$\dot{q}_j \rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j}$$ $$\boxed{\pi(x,t) = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q(x,t)}{\partial t}\right)}}$$

or in general $(c=1)$

(57) $$\boxed{\pi(x^\mu) = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x^0}\right)}}$$

problem!

Hamiltonian density

not Lorentz-invariant

(58) $$\boxed{\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi - \mathcal{L}}$$

$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$

[remark: there are generalizations of $\mathcal{H}$ which are Lorentz tensors] canonical energy-momentum tensor

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \partial_\beta \phi - g^{\alpha\beta}\mathcal{L}$$

here $T^{00} = \mathcal{H}$

Question: What independent variables does $\mathcal{H}$ depend upon?

$$d\mathcal{H} = \cancel{d\dot{\phi}\pi} + \dot{\phi}\, d\pi - \frac{\partial \mathcal{L}}{\partial \phi} d\phi - \cancel{\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{\pi} d\dot{\phi}}$$

$$= \dot{\phi}\, d\pi - \frac{\partial \mathcal{L}}{\partial \phi} d\phi$$

Thus

(59) $$\mathcal{H} = \mathcal{H}(\phi, \pi)$$

One also sees from $d\mathcal{H}$ that

(60) $$\boxed{\dot{\phi} = \frac{\partial}{\partial t}\phi = \frac{\partial \mathcal{H}}{\partial \pi}}$$ 1st canonical equation

equivalent to $\dot{q} = \frac{\partial H}{\partial p}$

Using the Euler-Lagrange equations (55) one finds

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x^j}\right)} = \frac{\partial}{\partial x^0} \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial}{\partial t} \pi$$

$j = 1,2,3$

So

$$\dot{\pi} = \underset{\displaystyle -\frac{\partial \mathcal{H}}{\partial \phi}}{\underset{\shortparallel}{\frac{\partial \mathcal{L}}{\partial \phi}}} - \frac{\partial}{\partial x^j} \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x^j}\right)}}_{-\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial \phi}{\partial x^j}\right)}}$$

Thus

$$\boxed{\dot{\pi} = -\left[\frac{\partial \mathcal{H}}{\partial \phi} - \frac{\partial}{\partial x^j} \frac{\partial \mathcal{H}}{\partial(\partial_j \phi)}\right]} \tag{61}$$

2nd canonical equation

equivalent to $\dot{p} = -\frac{\partial H}{\partial q}$. The second term somehow breaks the formal equivalence? ☹

$\Rightarrow$ Can we bring (60) and (61) into a symmetric form? And can we introduce something similar to Poisson brackets?

For this we introduce the notion of a

functional derivative

(here only $\phi(x)$ i.e. only one variable x)

functional $\quad F = F[\phi(x), \phi'(x)]$

(62)
$$\equiv \int dx \; f(\phi(x), \phi'(x))$$

Now we define a functional derivative

$$\frac{\delta F}{\delta \phi(y)} \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int dx \; f\left(\phi(x) + \varepsilon \delta(x-y), \frac{d}{dx}\left(\phi(x) + \varepsilon \delta(x-y)\right)\right)$$

$$= \lim_{\varepsilon \to 0} \int dx \left(\frac{\partial f}{\partial \phi} \delta(x-y) + \frac{\partial f}{\partial \phi'} \frac{d}{dx} \delta(x-y) \right)$$

$$\underset{\text{partial integration}}{=} \int dx \left(\frac{\partial f}{\partial \phi} - \frac{d}{dx} \frac{\partial f}{\partial \phi'} \right) \delta(x-y)$$

Thus

(63)
$$\boxed{\frac{\delta F}{\delta \phi(y)} = \frac{\partial f}{\partial \phi} - \frac{d}{dx} \frac{\partial f}{\partial \phi'} \Bigg|_{x=y}}$$

This allows us to write the canonical equation in a symmetric form: (H not $\mathcal{H}$)

$$(64) \quad \boxed{\frac{\partial \phi}{\partial t} = \frac{\delta H}{\delta \pi}} \qquad \boxed{\frac{\partial \pi}{\partial t} = -\frac{\delta H}{\delta \phi}}$$

Poisson brackets:

let

$$F = \int dV \, f(\phi, \pi, \partial_j \phi, x^\mu) \qquad j = 1,2,3$$

and

$$G = \int dV \, g(\phi, \pi, \partial_j \phi, x^\mu)$$

then define the Poisson bracket

$$(65) \quad \boxed{\{F, G\} \equiv \int dV \left(\frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi} \frac{\delta G}{\delta \phi} \right)}$$

With the help of a little trick we can write the field $\phi(x^\mu)$ and the conjugate field $\pi(x^\mu)$ as functional

$$\phi(\vec{y}, t) = \int dV \, \phi(\vec{x}, t) \, \delta(\vec{x} - \vec{y})$$

$$\pi(\vec{y}, t) = \int dV \, \pi(\vec{x}, t) \, \delta(\vec{x} - \vec{y})$$

Then we can write the canonical equations for fields in a very similar form as in canonical mechanics

$$\boxed{\begin{aligned} &(66) \quad \frac{\partial \phi(\vec{y},t)}{\partial t} = \{\phi(\vec{y},t), H\} \\ &(67) \quad \frac{\partial \pi(\vec{y},t)}{\partial t} = \{\pi(\vec{y},t), H\} \end{aligned}}$$

canonical equations of classical field theory

proof:
$$\frac{\delta \phi(\vec{y},t)}{\delta \phi(\vec{x},t)} = \frac{\partial \phi}{\partial \phi}\delta(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y})$$

so
$$\{\phi(\vec{y},t), H\} = \int d^3\vec{x} \left(\underbrace{\frac{\delta \phi(\vec{y},t)}{\delta \phi(\vec{x},t)}}_{=\delta(\vec{x}-\vec{y})} \frac{\delta H}{\delta \pi} - \underbrace{\frac{\delta \phi}{\delta \pi}}_{=0} \frac{\delta H}{\delta \phi} \right)$$

$$= \int d^3\vec{x}\, \delta(\vec{x}-\vec{y}) \frac{\partial \phi(\vec{x},t)}{\partial t} = \frac{\partial}{\partial t}\phi(\vec{y},t)$$

□

As in canonical mechanics one finds

the <u>fundamental Poisson brackets for fields</u>

$$\boxed{\begin{aligned}&\{\phi(\vec{x},t),\phi(\vec{y},t)\} = \{\pi(\vec{x},t),\pi(\vec{y},t)\} = 0\\ &\{\phi(\vec{x},t),\pi(\vec{y},t)\} = \delta(\vec{x}-\vec{y})\end{aligned}} \tag{68}$$

proof:

$$\{\phi(\vec{x},t),\pi(\vec{y},t)\} = \int d^3z \left(\frac{\delta\phi(\vec{x})}{\delta\phi(\vec{z})}\frac{\delta\pi(\vec{y})}{\delta\pi(\vec{z})} - \underbrace{\frac{\delta\phi(\vec{x})}{\delta\pi(\vec{z})}\frac{\delta\pi(\vec{y})}{\delta\phi(\vec{z})}}_{=0} \right)$$

$$= \int d^3z\, \delta(\vec{x}-\vec{z})\,\delta(\vec{y}-\vec{z})$$

$$= \delta(\vec{x}-\vec{y}) \qquad \square$$

I.4 Lagrange density of some important fields

What are the general properties a Lagrange density should have?