Quantum Field Theory
SS 2024

4 SWS + 2 SWS exercises

0. Introduction

- **Why do we need a quantum theory of fields?**

shortcomings in the theory so far:

(i) classical theory: $\vec{E} = -\frac{1}{q} m \ddot{\vec{r}}_q$

fields defined through force on charged point masses with "charge" q

quantum mechanics:

$\vec{r}_q \longrightarrow \hat{\vec{r}}_q$ how about $\vec{E}$?

(ii) relativistic quantum mechanics not consistently interpretable as single-particle theory

- ## What is a quantum field?

classical field $\mathbb{R}^3 \otimes \mathbb{R} \longrightarrow \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$

$\vec{r}\,, \; t \qquad\qquad \phi(\vec{r}, t)$

quantum field $\mathbb{R}^3 \otimes \mathbb{R} \longrightarrow$ linear operators

$\vec{r}, t$

$\Rightarrow$ position $\vec{r}$ which in quantum mechanics was described as an operator is now together with the time t a

continuous "index"

$$\hat{q}_i \longrightarrow \hat{O}(x)$$

discrete index in q mechanics

continuous "index" $x = (\vec{r}, t)$

rem: the transition to continuous "indices" is the most trouble creating feature of QFT

- ## How should a quantum theory of fields look like?

it is a quantum theory in first place!

So the postulates of quantum mechanics should hold!

- System will be described by a state vector $|\psi\rangle$ in Hilbert space (or its generalizations) $\mathcal{H}$

- observables are self-adjoint linear operators in $\mathcal{H}$

$$\hat{O} : \mathcal{H} \longrightarrow \mathcal{H} \in \mathcal{L}(\mathcal{H})$$

- measurement of an observable leads to a projection of the statevector to the eigenspace corresponding to the measurement result

- time evolution is described by a unitary

$$\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$$

where $\hat{H}$ is the Hamiltonian

- all observables can be expressed in terms of some fundamental set of operators obeying fundamental commutation relations

There are different ways to construct a quantum theory of fields consistently. We will use here the analogy between classical (non-relativistic) mechanics and (non-relativistic) quantum mechanics. An alternative are path integrals

$\boxed{\text{classical mechanics}}$	$\boxed{\text{quantum mechanics}}$
$L(q_j, \dot{q}_j, t)$	
$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$	$\hat{H}$
$q_j, \; p_j = \frac{\partial L}{\partial \dot{q}_j}$	$\hat{q}_j \; , \; \hat{p}_j$
$\{q_i, q_j\} = \{p_i, p_j\} = 0$	$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$
$\{q_i, p_j\} = \delta_{ij}$	$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$
Poisson algebra	Heisenberg algebra
$\dot{q}_j = \{q_j, H\}$	$\frac{d}{dt}\hat{q}_j = -\frac{i}{\hbar}[\hat{q}_j, \hat{H}]$
$\dot{p}_j = \{p_j, H\}$	$\frac{d}{dt}\hat{p}_j = -\frac{i}{\hbar}[\hat{p}_j, \hat{H}]$

or for $f = f(q_j, p_j, t)$

$$\frac{d}{dt} f = \{f, H\} + \frac{\partial f}{\partial t} \qquad \frac{d}{dt}\hat{f} = -\frac{i}{\hbar}[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t}$$

$\Rightarrow$ for this we need a formulation of classical field theories (such as Maxwell's electrodynamics) in the framework of a Lagrange - Hamilton formalism

- fundamental quantum fields are relativistic, i.e. Lorentz invariance will play a key role

  As we will see also for fields time will play a special role in the Hamilton formalism. The disadvantage is that the Hamilton formalism will not be explicitly covariant. Here an alternative is the path integral description which only uses Lagrange function. Here we will have to live with the drawback of the Hamilton formalism, however.

- mathematically consistent formulation

of QFT has problems. When theoretically calculating physical quantities we will often encounter divergencies. We will learn about a procedure to cure these (in almost all relevant cases): renormalization

- a special role will play field theories with a special symmetry: invariance under local gauge transformations

Gauge field theories

they are renormalizable

- What new physics to expect?

- particle number is no longer conserved
- pair creation / annihilation
- particles of large mass may decay in many particles of lower mass
- in the non-relativistic limit particle number of massive particles conserved
  $\Longrightarrow$ non-relativistic many-particle theory

- as in many-body quantum mechanics we will find

bosonic fields and fermionic fields

• Does QFT finally solve all problems?

No! - unification with gravity still open

- explanation of masses of fundamental particles?

- do we know all particles?
  $\Rightarrow$ no: dark matter?

• Structure of lecture

I) classical field theory
II) canonical quantization
III) some important results for free fields
IV) QED with interactions: perturbation theory
V) renormalization

$$\boxed{\hbar = c = 1}$$