VII.2 Spontaneous symmetry breaking and Higgs mechanism

A fundamental problem of the Yang-Mills theory is the absence of masses of the gauge bosons required by gauge invariance. We now will discuss how these masses can be generated by the so-called Higgs mechanism

(A) Spontaneous symmetry breaking

Before discussing the Higgs mechanism we will briefly examine a general phenomenon called spontaneous symmetry breaking which is relevant in the theory of phase transitions.

idea: A theory whose Hamiltonian is invariant under a certain symmetry group may not show this symmetry on the level of solutions

conditions:
- theory is invariant under symmetry group G, i.e. $[H, G] = 0$
- the ground state of H is degenerate and transforms non-trivially under the symmetry group

Classical theories

example: Scalar real (neutral) boson field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi^2)$$

invariant under parity

$$P: \quad \phi \rightarrow -\phi$$

- assume non-degenerate ground state $|0\rangle$

$$\Rightarrow \qquad P|0\rangle = e^{i\varphi}|0\rangle$$

$$\langle 0|\hat{\phi}|0\rangle = \langle 0|P^{-1} P \hat{\phi} P^{-1} P|0\rangle = \langle 0|P\hat{\phi}P^{-1}|0\rangle$$

$$= -\langle 0|\hat{\phi}|0\rangle \quad \Rightarrow \quad \underline{\langle 0|\hat{\phi}|0\rangle = 0}$$

- but if ground state is degenerate, e.g. for

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

potential $V(\phi)$ of (classical) Lagrangedensity has minimum at

$$\phi_0 = \pm\sqrt{\frac{\mu^2}{\lambda}} = \pm v$$

There are two minimum energy configurations (ground "states"): $\phi_0 = \pm v$; $|R\rangle, |L\rangle$

$$P|R\rangle = |L\rangle \neq |R\rangle$$

$$\Rightarrow \quad \langle R|\phi|R\rangle = \langle R|P^{-1}P\phi P^{-1}P|R\rangle = -\langle L|\phi|L\rangle$$

i.e. expectation value of the field in ground state is nonzero, despite the symmetry of the Lagrangian under $\phi \to -\phi$

This was an example of a spontaneously broken discrete symmetry. More interesting is a spontaneously broken continuous symmetry

$\Rightarrow$ two scalar (neutral) fields $\quad \vec{\phi} = (\phi_1, \phi_2)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} + \frac{1}{2}\mu^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2$$

with $\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2$. $\mathcal{L}$ has $O(2)$ symmetry i.e. rotations

$$R(\theta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}$$

leave $\mathcal{L}$ invariant

$\mu^2 < 0$: unique ground state $\quad \vec{\phi} = 0$

$\mu^2 > 0$: infinitely many degenerate ground states with

$$|\vec{\phi}|^2 = \phi_1^2 + \phi_2^2 = v^2$$

If we pick one reference state and called $|0\rangle$ among the ground states then

$$R(\theta)|0\rangle = |\theta\rangle \qquad \text{are all ground states}$$

quantum theory

In the quantum case there is strictly speaking no spontaneous symmetry breaking due to quantum tunneling (except in pathological cases of infinite potential barriers)

However the tunneling time scales as

$$T = \frac{2\pi}{\Delta E} \qquad \text{and} \quad \Delta E \sim \mathcal{O}(L^3)$$

i.e. in the thdynamic limit $L^3 \to \infty$: $T \to \infty$ and tunneling does not occur

(B) Goldstone theorem

In a covariant local field theory with a positive definite norm in Hilbert space a spontaneously broken continuous symmetry leads to a massless excitation (Goldstone boson).

classical scalar (neutral) field $\vec{\Phi}=(\phi_1, \phi_2)$

$$\mathcal{L}=\frac{1}{2}\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi}+\frac{\mu^2}{2}\vec{\Phi}\cdot\vec{\Phi}-\frac{\lambda}{4}(\vec{\Phi}\cdot\vec{\Phi})^2$$

has $O(N)$ symmetry. Minima (classical "ground state")

$$\frac{\partial V}{\partial\phi_e}=\mu^2\phi_e+\lambda\phi_e|\vec{\Phi}|^2=0$$

solutions — minimum for

a) $\phi_e\equiv 0$ — $\mu^2<0$

b) $|\vec{\Phi}|^2=\frac{\mu^2}{\lambda}=v^2$ — $\mu^2>0$

expand $V(\vec{\Phi})$ around minimum say $\vec{\Phi}_0=(0, v)$

$$V(\vec{\Phi})\cong V(\vec{\Phi}_0)+\frac{1}{2}\left.\frac{\partial^2 V}{\partial\phi_e\partial\phi_m}\right|_{\vec{\Phi}_0}(\phi_e-v\delta_{e2})(\phi_m-v\delta_{m2})$$

Mass matrix

$$M^2_{em}=\left.\frac{\partial^2 V}{\partial\phi_e\partial\phi_m}\right|_{\vec{\Phi}_0}=\begin{pmatrix}0 & 0\\ 0 & 2\mu^2\end{pmatrix}$$

$m\equiv 2\mu^2$ new field $(\phi_1, \underbrace{\phi_2-v}_{=x})$

$$V \cong \frac{m^4}{16\lambda} + \underbrace{\frac{1}{2} m^2 \chi^2}_{\text{mass term}} + \sqrt{\frac{m^2\lambda}{2}}\, \chi \left(\phi_1^2 + \chi^2 \right) + \frac{\lambda}{4} \left(\phi_1^2 + \chi^2 \right)^2$$

Since the minimum of χ is at $\chi = 0$ only the field χ is massive while ϕ_1 is massless.

Goldstone theorem holds if

(i) theory is explicitely covariant

and (ii) Hilbert space norm is positive definite

$\Rightarrow$ this is not the case e.g. for electromagnetism as one either has to choose a gauge where explicit covariance is no longer given (Coulomb gauge) or to work with a bigger Hilbert space with indefinite norm (Lorentz gauge) and where physical states are obtained with a supplementary condition.

(c) Higgs mechanism

To understand the Higgs mechanism we consider an $O(2)$ invariant two component real bose field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \frac{\mu^2}{2} \vec{\phi} \cdot \vec{\phi} - \frac{\lambda}{4} (\vec{\phi} \cdot \vec{\phi})^2$$

$\vec{\phi} = (\phi_1, \phi_2)$

coupled to a gauge field. Since $\mu^2 > 0$ the $O(2)$ symmetry is spontaneously broken and we choose

$$\vec{\phi}_0 = (v, 0) \qquad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Translation $\phi_1 = \chi + v$ gives the potential $m^2 = 2\mu^2$

$$V = \frac{m^4}{16\lambda} + \frac{1}{2} m^2 \chi^2 + \sqrt{\frac{m^2 \lambda}{2}}\, \chi \left(\phi_2^2 + \chi^2\right) + \frac{\lambda}{4}\left(\phi_2^2 + \chi^2\right)^2$$

The Goldstone boson is ϕ_2 which is massless. Lets consider its transformation under $O(2)$. For the original fields

$$\phi_1 \to \phi_1 + \delta\phi_1 \qquad \delta\phi_1 = -\alpha\,\phi_2$$
$$\phi_2 \to \phi_2 + \delta\phi_2 \qquad \delta\phi_2 = \alpha\,\phi_1$$

$(\sin\alpha \simeq \alpha \quad \cos\alpha \simeq 1) \Rightarrow$

$$\delta\chi = -\alpha\phi_2 \quad \text{but} \quad \delta\phi_2 = \alpha\chi + \underbrace{\alpha v}_{\text{translation!}}$$

The translation part shows why the Goldstone boson must be massless. Only in this case can the theory be $O(2)$ invariant.

introduce polar coordinates

$$\varrho = \sqrt{\phi_1^2 + \phi_2^2} \qquad \sin\theta = \frac{\phi_2}{\sqrt{\phi_1^2 + \phi_2^2}}$$

$O(2)$: $\quad \rho \to \rho \qquad \theta \to \theta + \alpha$

Since

$$\rho = \sqrt{\phi_2^2 + x^2 + 2xv + v^2} \approx v + x$$

$$\theta \approx \frac{\phi_2}{v+x} \approx \frac{\phi_2}{v}$$

One sees that the Goldstone mode corresponds to the angular degree of freedom θ.

Now lets consider the coupling to a gauge field. For this we introduce complex variables

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}} \rho e^{i\theta} \qquad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) = \frac{1}{\sqrt{2}} \rho e^{-i\theta}$$

$$\Rightarrow \quad \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

The $O(2)$ transformation is now a phase trafo $\phi \to e^{i\alpha}\phi$
Lets now introduce a minimum coupling to a gauge field A_μ

$$\partial_\mu \phi \to (\partial_\mu + igA_\mu)\phi$$

$$\mathcal{L} = (\partial_\mu - igA_\mu)\phi^* (\partial^\mu + igA^\mu)\phi + \mu^2 \phi^* \phi - \lambda(\phi^*\phi)^2$$
$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

special local gauge choice:

$$\phi \to \phi' = \phi e^{-i\theta(x)} = \phi'^*$$

unitary gauge

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \theta(x)$$

$\Rightarrow$ Lagrangian no longer contains Goldstone mode $\theta(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu - igA_\mu)\rho(\partial^\mu + igA^\mu)\rho + \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$
$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Now lets translate the fields which corresponds to a broken $O(2)$ symmetry

$$\rho = \chi + v \qquad \langle 0|\chi|0\rangle = 0$$

$$\mathcal{L} = \ldots + \boxed{\frac{1}{2}g^2v^2 A_\mu A^\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\boxed{m_A^2 = g^2v^2}$$

mass term of the gauge field!

Note: • It was important that we have fixed the gauge a priori such that there is no coupling of the gauge field to the Goldstone mode θ $A_\mu\partial^\mu\theta$

It is instructive to count the number of degrees of freedom

Symmetric case — 2 for massive boson ϕ
— 2 for massless gauge field A_μ

Symmetry broken — 1 for massive boson ξ
— 3 for gauge field A_μ
as it is now massive

The Higgs mechanism, i.e. the coupling of the gauge field A_μ to a massive Higgs boson ϕ (with negative mass) creates an effective mass for the gauge field after spontaneous symmetry breaking and thus solves a fundamental problem of Young Mills theories.

Nobel prize 2014

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