VII. Some remarks about the Standard model

VII.1. Gauge theories for fermion fields with more degrees of freedom: non-Abelian gauge theories

An important guide to construct QED was the requirement of local gauge invariance which has fixed the minimal coupling term between the Dirac field and a bosonic gauge field, the electromagnetic field

$$\psi_\alpha(x) \longrightarrow e^{-i\theta(x)}\psi_\alpha(x)$$

$e^{i\theta(x)}$ is element of the Abelian $U(1)$ Lie group
Invariance under this transformation required the existence of a gauge field coupled to ψ through the replacement

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu + ieA_\mu(x)$$

This resulted in

$$\mathcal{L}_D = \frac{1}{2}\bar{\psi}(i\gamma^\mu \overleftrightarrow{\partial_\mu} - m)\psi \longrightarrow \mathcal{L}' = \mathcal{L}_D - j^\mu A_\mu$$

The only bi-linear Lorentz invariant constructions for the Lagrangian of the gauge field are

$$\mathcal{L}_A = \lambda_1 F_{\mu\nu} F^{\mu\nu} + \lambda_2 G_{\mu\nu} G^{\mu\nu} + m_\gamma^2 A^\mu A_\mu$$

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \qquad G_{\mu\nu} = A_{\mu,\nu} + A_{\nu,\mu}$$

The only gauge invariant term is $\lambda_1 F_{\mu\nu} F^{\mu\nu}$ which fixes the minimum version of the theory

$$\mathcal{L} = \frac{1}{2} \bar{\Psi} (i\gamma^\mu \overleftrightarrow{\partial}_\mu - m) \Psi - j^\mu A_\mu + \lambda F_{\mu\nu} F^{\mu\nu}$$

It turns out that other fermionic elementary particles exist which are described by <u>additional degrees of freedom</u>:

<u>weak interaction</u>

Isospin $\longrightarrow$ doublett of Fermions

<u>strong interaction</u>

Color $\longrightarrow$ triplett of Fermions

In view of the success of local gauge invariance in QED we have want to request also local gauge invariance. Thus we have to ask what are the allowed gauge trafos for two- or three-component fermions?

weak interaction

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_1'(x) \\ \psi_2'(x) \end{pmatrix} = \underline{\underline{U}}(x) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

where $\psi_1(x)$ and $\psi_2(x)$ are the two Dirac Bispinors. $\underline{\underline{U}}(x)$ is now a unitary 2×2 matrix in the Isospin degrees

$$\underline{\underline{U}}(x) = e^{-i\frac{1}{2}\sigma_e \theta_e(x)}$$

$\sigma_e = \{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices and $\theta_1, \theta_2, \theta_3$ are three independent (space dependent) angles.

$$\underline{\underline{U}}(x) \in \underline{SU(2)}$$

which is a <u>non-Abelian Lie group</u> with the generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

strong interaction

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_1'(x) \\ \psi_2'(x) \\ \psi_3'(x) \end{pmatrix} = \underline{\underline{U}}(x) \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

$\underline{\underline{U}}(x)$ is 3×3 matrix in color degrees.

and

$$\underline{\underline{U}}(x) = e^{-i\frac{1}{2}\underline{\underline{\lambda}}_k \Theta_k(x)}$$

where $\underline{\underline{\lambda}}_1 \cdots \underline{\underline{\lambda}}_8$ are the 8 Gell-Mann matrices

$$\underline{\underline{\lambda}}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\lambda}}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \underline{\underline{\lambda}}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underline{\underline{\lambda}}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \underline{\underline{\lambda}}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The $\underline{\underline{U}}(x)$ are elements of the <u>non-Abelian Lie group</u>

$$\underline{\underline{U}}(x) \in \underline{SU(3)}$$

$\frac{1}{2}\underline{\underline{\lambda}}_K$ are the 8 generators of the $SU(n)$ which has n^2-1 generators

$$\boxed{\left[\underline{\underline{\lambda}}_a, \underline{\underline{\lambda}}_b\right] = 2i f_{abc} \underline{\underline{\lambda}}_c}$$

f_{abc} are the structure factors, which define a closed <u>Lie algebra</u>. Furthermore

$$Tr\{\underline{\underline{\lambda}}_a \underline{\underline{\lambda}}_b\} = 2\delta_{ab}$$

Non-Abelian gauge theory

In the case of QED requesting invariance under local $\underline{U(1)}$ phase transformations

$$(*) \qquad \psi(x) \longrightarrow e^{-ie\theta(x)}\psi(x)$$

required minimum coupling to a single gauge field A_μ corresponding to

$$\vec{\partial}_\mu \rightarrow \vec{D}_\mu = \vec{\partial}_\mu + ieA_\mu(x)$$

which has to undergo a gauge transformation simultaneously to (*)

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\theta(x)$$

In the case of a $\underline{SU(n)}$ transformation

$$\underline{\psi} \equiv \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} \rightarrow \underline{\underline{U}}(x)\begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} = e^{-i\frac{g}{2}\underline{\underline{\lambda}}_k\theta_k(x)}\begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix}$$

with (n^2-1) generators $\underline{\underline{\lambda}}_k$ the required gauge field is an $(n\times n)$ matrix which are non commuting objects

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \underline{\bar{\psi}}\, i\gamma^\mu \underline{\underline{U}}^+(\partial_\mu\underline{\underline{U}})\,\underline{\psi}$$

$$\boxed{\vec{\partial}_\mu \rightarrow \vec{\underline{\underline{D}}}_\mu = \vec{\partial}_\mu + i g \underline{\underline{A}}_\mu(x)} \qquad \underline{\underline{A}}_\mu \quad n \times n$$

decomposing $\underline{\underline{A}}_\mu$ in the complete basis set $\underline{\underline{\lambda}}_k$ gives (n^2-1) gauge fields $A^a_\mu(x)$ $\quad a = 1, \ldots, n^2-1$

$$\boxed{\underline{\underline{A}}_\mu(x) = \frac{1}{2} \sum_{a=1}^{n^2-1} \lambda_a A^a_\mu(x)}$$

the new interaction Lagrangian then reads

$$\boxed{\mathcal{L}_{WW} = -g \, \overline{\underline{\Psi}} \gamma^\mu \underline{\underline{A}}_\mu \underline{\Psi}}$$

The theory has now (n^2-1) Noether currents which are conserved due to invariance under global phase transformations

$$\boxed{j^\mu_e = g \, \overline{\underline{\Psi}} \gamma^\mu \underline{\underline{\lambda}}_e \underline{\Psi}} \qquad e = 1, \ldots, n^2-1$$

How about the free Lagrangian of the n^2-1 gauge fields? For this lets consider the gauge fields

$$\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu \qquad \text{not gauge invariant}$$

In contrast to the U(1) gauge theory where the definition of the field tensor (in Lorentz space) with the ordinary and the covariant derivative are the same, this is no longer the case here

$$\Rightarrow \quad \boxed{\underline{\underline{F}}_{\mu\nu} = \underline{\underline{D}}_\mu \underline{\underline{A}}_\nu - \underline{\underline{D}}_\nu \underline{\underline{A}}_\mu}$$

$$\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{\underline{A}}_\nu + iq \underline{\underline{A}}_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu - iq \underline{\underline{A}}_\nu \underline{\underline{A}}_\mu$$

$$\boxed{\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu + iq \underbrace{[\underline{\underline{A}}_\mu, \underline{\underline{A}}_\nu]}_{\neq 0 \text{ in general}}} \qquad (*)$$

<u>field tensor of SU(n)</u>
<u>gauge field</u>
(non linear in $\underline{\underline{A}}_\mu$!)

<u>gauge invariant Lagrange density of free gauge field:</u>

$$\boxed{\mathcal{L}_{gauge} = -\frac{1}{2} \, Tr \left\{ \underline{\underline{F}}_{\mu\nu} \underline{\underline{F}}^{\mu\nu} \right\}}$$

where the trace is over the n internal degrees of freedom (Isospin, Colour, ...)

Due to the non linear commutator term in (*) the free non-abelian theory is no longer quadratic in the field

$\Rightarrow$ intrinsic self-interaction of non-abelian gauge fields

The complete Lagrangian of a SU(N) local gauge invariant theory, called Yang-Mills theory reads:

$$\boxed{\mathcal{L} = \overline{\underline{\Psi}}(x)\left[\frac{i}{2}\gamma^\mu \overleftrightarrow{\underline{\underline{D}}}_\mu - m\right]\underline{\Psi}(x) - \frac{1}{2}\mathrm{Tr}\{\underline{\underline{F}}_{\mu\nu}\underline{\underline{F}}^{\mu\nu}\}}$$

There are different interaction terms:

ffg - coupling

SU(N) each fermion (f) has N degrees of freedom and couples to N^2-1 gauge bosons (g)

$$\boxed{\mathcal{L}_{ffg} = -g\,\overline{\underline{\Psi}}\gamma^\mu \frac{1}{2}\underline{\underline{\lambda}}_\ell \underline{\Psi} A^\ell_\mu} \qquad \ell = 1,\dots,N^2-1$$

vertex

ℓ,μ $\quad -ig\gamma^\mu \frac{1}{2}\underline{\underline{\lambda}}_\ell$ $\quad$ ℓth gauge boson polarisation μ

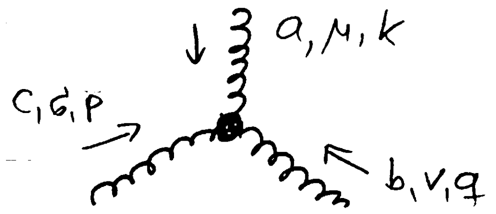
mixes internal degrees

The interactions contained in the free Lagrangian of the gauge field

$$\mathcal{L}_F = -\frac{1}{2}\partial^\mu A^{\ell\nu}(\partial_\mu A^\ell_\nu - \partial_\nu A^\ell_\mu) + \mathcal{L}_{3g} + \mathcal{L}_{4g} \quad \text{are:}$$

3g coupling

$$\boxed{\mathcal{L}_{3g} = \frac{1}{2} g f_{abc} A^{b\mu} A^{c\nu} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)}$$



a,b,c gauge field (Isospin, colour, ..)

μ, ν, σ Lorentz-index polarization

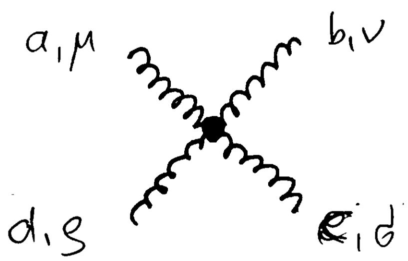
k, p, q 4-momenta

vertex

$$g f_{abc} \{ g_{\mu\nu} (q_\sigma - k_\sigma) + g_{\nu\sigma} (p_\mu - q_\mu) + g_{\sigma\mu} (k_\nu - p_\nu) \}$$

4g coupling

$$\boxed{\mathcal{L}_{4g} = -\frac{1}{4} g^2 f_{abe} f_{cde} A^a_\mu A^b_\nu A^{d\mu} A^{c\nu}}$$



vertex:

$$-ig^2 \{ f_{abe} f_{cde} (g_{\mu\sigma} g_{\nu\varsigma} - g_{\mu\varsigma} g_{\nu\sigma}) + f_{ace} f_{bde} (g_{\mu\nu} g_{\sigma\varsigma} - g_{\mu\varsigma} g_{\nu\sigma}) + f_{ade} f_{cbe} (g_{\mu\sigma} g_{\nu\varsigma} - g_{\mu\nu} g_{\varsigma\sigma}) \}$$