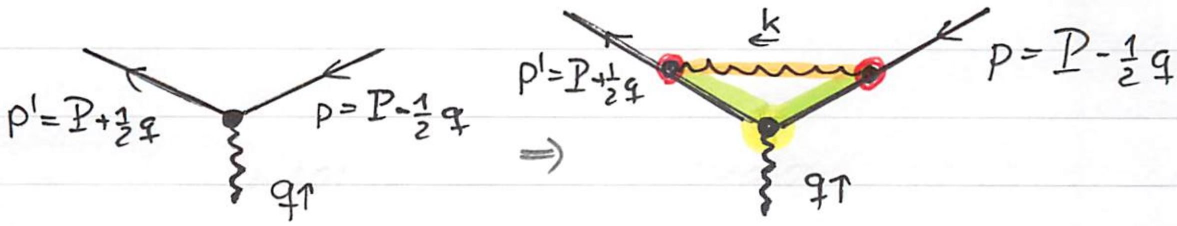
VI.4 Vertex correction and anomalous magnetic moment



$$-e\bar{u}(p')\gamma^\mu u(p) \Rightarrow -e\bar{u}(p')\Lambda^\mu(P,q)u(p)$$

$$P \equiv \frac{1}{2}(p+p') \qquad q \equiv p'-p$$

average 4-momentum   4-momentum transfer

$$\Lambda^\mu(P,q) = ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu(m-\not{p}'-\not{k})\gamma^\mu(m-\not{p}-\not{k})\gamma_\nu}{(m^2-(p'-k)^2-i\varepsilon)(m^2-(p-k)^2-i\varepsilon)(\lambda^2-k^2-i\varepsilon)}$$

λ^2 is a ficticious photon mass, which will be set to zero later

- transformation to normal form

$$A_1 = m^2-(p'-k)^2-i\varepsilon = 2p'k-k^2-i\varepsilon$$

We assume here that the outgoing fermion is a free fermion for which $m^2=p'^2$. Likewise the incoming fermion is assumed to be a free Dirac fermion i.e. $m^2=p^2$

$$\Rightarrow A_2 = m^2-(p-k)^2-i\varepsilon = 2pk-k^2-i\varepsilon$$
$$A_3 = \lambda^2-k^2-i\varepsilon$$

• denominator:

$$\frac{1}{A_1A_2A_3} = 2\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{1}{\underbrace{[A_1z_1+A_2z_2+A_3(1-z_1-z_2)]^3}_{=D(z_1,z_2)}}$$

$$D = 2(z_1p'+z_2p)\cdot k + \lambda^2(1-z_1-z_2) - k^2 - i\varepsilon$$

variable transformation $k = k' + z_1p' + z_2p$ removes linear terms in k'

$$D = (z_1+z_2)^2m^2 - z_1z_2q^2 + \lambda^2(1-z_1-z_2) - k'^2 - i\varepsilon$$

where we have used again that $p^2 = m^2 = p'^2$

• numerator after $k \to k'$

$$N^\mu = \gamma^\nu(m+\not{p}'(1-z_1)-z_2\not{p}-\not{k}')\gamma^\mu(m+\not{p}(1-z_2)-\not{p}'z_1-\not{k}')\gamma_\nu$$

$$= \gamma^\nu(m+\not{p}'(1-z_1)-z_2\not{p})\gamma^\mu(m+\not{p}(1-z_2)-\not{p}'z_1)\gamma_\nu$$

$$+ \gamma^\nu\not{k}'\gamma^\mu\not{k}'\gamma_\nu + \underbrace{\text{linear terms in } k'}_{\hookrightarrow 0 \text{ after integration } d^4k'}$$

further simplifications that use

- $\gamma^\mu\not{a}\gamma_\mu = -2\not{a}$ $\quad \gamma^\mu\not{a}\not{b}\gamma_\mu = 4a\cdot b$ $\quad \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$
- $(z_2-z_1)^1$ terms disappear in integrals $\int dz_1 \int dz_2$

- $\not{q}^1$ terms disappear after sandwiching $\Lambda^\mu(P, q)$ with $\bar{u}(p')$ and $u(p)$

- in integral $\int d^4k \quad k^\mu k^\nu \to g^{\mu\nu} k^2/4$

this then leads to

$$N^\mu = \gamma^\mu\left[-2m^2\left(1+(1-z_1-z_2)^2\right) - 2q^2(1-z_2)(1-z_1) + k'^2\right]$$

$$+ 4m(1-z_1-z_2)(p'+p)^\mu + 4m(1-z_1-z_2)\left(1-\tfrac{1}{2}(z_1+z_2)\right) i\sigma^{\mu\nu} q_\nu$$

where $\qquad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad$ (see Dirac theory)

$$\Rightarrow \quad \sigma^{\mu\nu} q_\nu = \frac{i}{2}[\gamma^\mu, \not{q}]$$

In the problem courses we will show that

$$\bar{u}(p') \frac{(p'+p)^\mu}{2m} u(p) = \bar{u}(p')\gamma^\mu u(p) - \bar{u}(p') \frac{i\sigma^{\mu\nu} q_\nu}{2m} u(p)$$

$\Rightarrow$

after integration d^4k', dz_1, dz_2 correction to "γ^μ"

$$N^\mu = \gamma^\mu \left\{ -2m^2\left[1 - 4(1-z_1-z_2) + (1-z_1-z_2)^2\right] - 2q^2(1-z_1)(1-z_2) + k'^2 \right\}$$

$$- i2m\,\sigma^{\mu\nu} q_\nu (1-z_1-z_2)(z_1+z_2)$$

gives rize to new term!

scalar functions of q^2

$$\Rightarrow \boxed{\Lambda^\mu(P,q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)}$$

vertex correction:

- modification of ordinary vertex factor $\sim \gamma^\mu$
  coupling of elm. field to <u>Dirac current</u>

- new coupling term $i\sigma^{\mu\nu}q_\nu$
  coupling of elm. field to <u>anomalous or Pauli current</u>

$$\boxed{\begin{aligned} F_1(q^2) &= 2ie^2\int\frac{d^4k'}{(2\pi)^4}\int_0^1 dz_1\int_0^{1-z_1}dz_2\,\frac{N_1}{(D[k'])^3} \\ F_2(q^2) &= -2ie^2\int\frac{d^4k'}{(2\pi)^4}\int_0^1 dz_1\int_0^{1-z_1}dz_2\,\frac{4m^2(z_1+z_2)(1-z_1-z_2)}{(D[k'])^3}\end{aligned}}$$

where

$$N_1 = k'^2 - 2q^2(1-z_1)(1-z_2) - 2m^2\left[1-4(1-z_1-z_2)+(1-z_1-z_2)^2\right]$$

one recognizes

$$F_2 \sim \int d^4k' \frac{1}{k'^6} < \infty \qquad \text{regular!}$$

but $F_1 \sim \int d^4k' \frac{k'^2}{k'^6} = \int d|k'| \frac{1}{|k'|} \to \infty$ singular!

- renormalization of singular term F_1: full charge renormalization

one recognizes that $F_1(q^2) - F_1(0)$ is regular

$\Rightarrow$

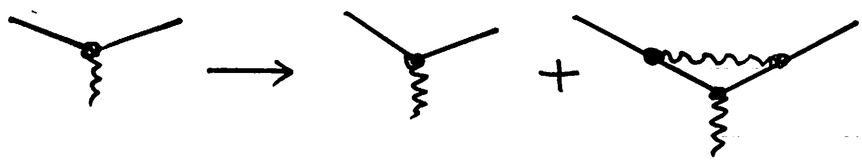
$$\Lambda^\mu(P,q) = \underbrace{\left[F_1(q^2) - F_1(0)\right]\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}}_{\text{regular part of dressed vertex}}$$

$$+ \underbrace{F_1(0)\gamma^\mu}_{\text{singular part}}$$

new renormalization constant Z_1

$$\boxed{F_1(0) = \frac{1}{Z_1} - 1}$$

vertex



$$\gamma^\mu = \underbrace{\gamma^\mu + \Lambda^\mu(P,q)}_{=\Gamma^\mu(P,q)}$$

$$\Gamma^\mu(P,q) = \underbrace{\left[F_1(q^2) - F_1(0)\right]\gamma^\mu + F_2(q)\frac{i\sigma^{\mu\nu}q_\nu}{2m}}_{\text{regular and } \to 0 \text{ for } q^2 \to 0} + \underbrace{\frac{1}{Z_1}\gamma^\mu}_{\text{divergent}}$$

$$\boxed{e_0 \longrightarrow e_0\sqrt{Z_3}\underbrace{Z_2}_{\text{wavefunction}} \longrightarrow e_0\sqrt{Z_3}\frac{Z_2}{Z_1}}$$

charge renormalization

- anomalous magnetic moment of the electron

Dirac equation in external elm field $\left(q^2 = q_0 q_0 - |\vec{q}|^2 \overset{!}{=} 0 \right)$

$$(i\gamma^\mu \partial_\mu - eA_\mu \gamma^\mu - m)\psi = 0$$

with regular part of vertex correction: $\left(q^2 \overset{!}{=} 0 \right)$

$$e\gamma^\mu A_\mu \rightarrow e\gamma^\mu A_\mu + eF_2(0)\, \frac{i\sigma^{\mu\nu} q_\nu A_\mu}{2m}$$

in Fourier space. If we transform to real space this is equivalent to $\quad i q_\nu A_\mu \rightarrow \frac{\partial}{\partial x_\nu} A_\mu$

$$e\gamma^\mu A_\mu \rightarrow e\gamma^\mu A_\mu + eF_2(0)\, \frac{\sigma^{\mu\nu} A_{\mu,\nu}}{2m} \qquad (\sigma^{\mu\nu} = -\sigma^{\nu\mu})$$

$$= e\gamma^\mu A_\mu + eF_2(0)\, \frac{\sigma^{\mu\nu} F_{\mu\nu}}{2m}$$

and thus the Dirac equation with the vertex correction reads:

$$\left(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - \underline{eF_2(0)\, \frac{\sigma^{\mu\nu} F_{\mu\nu}}{4m}} - m \right)\psi = 0$$

In the problem courses we will show that the correction term modifies the magnetic moment of the electron

gyromagnetic factor g $\qquad \boxed{\mu_{mag} = g \frac{e}{2m} s}$

Dirac	$g = 2$
QED	$g = 2 + \kappa$

$\boxed{\kappa = F_2(0) + \text{higher order}}$

$$F_2(0) = -2ie^2 \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \times$$

$$\times \int \frac{d^4k'}{(2\pi)^4} \frac{4m^2(z_1+z_2)(1-z_1-z_2)}{[\underbrace{m^2(z_1+z_2)^2 + \lambda^2(1-z_1-z_2)}_{\equiv B^2} - k'^2 - i\varepsilon]^3}$$

With the integrals given on page 183 we find (note $B^2 \neq 0$ due to λ^2)

$$\int \frac{d^4k'}{(2\pi)^4} \frac{1}{[B^2 - k'^2 - i\varepsilon]^3} = \frac{i}{32\pi^2 B^2}$$

Damit

$$F_2(0) = \frac{2e^2}{32\pi^2} 4m^2 \int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{(z_1+z_2)(1-z_1-z_2)}{m^2(z_1+z_2)^2 + \underbrace{\cancel{\lambda^2(1-z_1-z_2)}}_{\lambda \to 0} \to 0}$$

set $\quad \xi \equiv z_1 + z_2 \qquad \eta \equiv \frac{1}{2\xi}(z_1 - z_2)$

$$\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \longrightarrow \int_0^1 d\xi \int_{-1/2}^{1/2} d\eta \; \xi$$

$$\underline{\underline{F_2(0)}} = \frac{e^2}{4\pi^2} \int_0^1 d\xi \int_{-1/2}^{1/2} d\eta \, (1-\xi) = \frac{e^2}{\pi} \int_0^1 d\xi \, (1-\xi)$$

$$= \frac{e^2}{2\pi} = \underline{\underline{\frac{\alpha}{2\pi}}}$$ J. Schwinger 1948

$$\Rightarrow \quad \boxed{g-2 = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)}$$

<u>QED result for anomalous magnetic moment</u>

theory to α^4:

$$g-2 = \frac{\alpha}{2\pi} - 0.328478966\left(\frac{\alpha}{\pi}\right)^2 + 1.17611(42)\left(\frac{\alpha}{\pi}\right)^3$$
$$- 1.434(136)\left(\frac{\alpha}{\pi}\right)^4$$

$$\boxed{(g-2)_{Th} = 0.001159652140(28)}$$

$$\boxed{(g-2)_{Exp} = 0.0011596521884(43)}$$

$\frac{\Delta x}{x} \sim 10^{-12}$

phantastic precision!

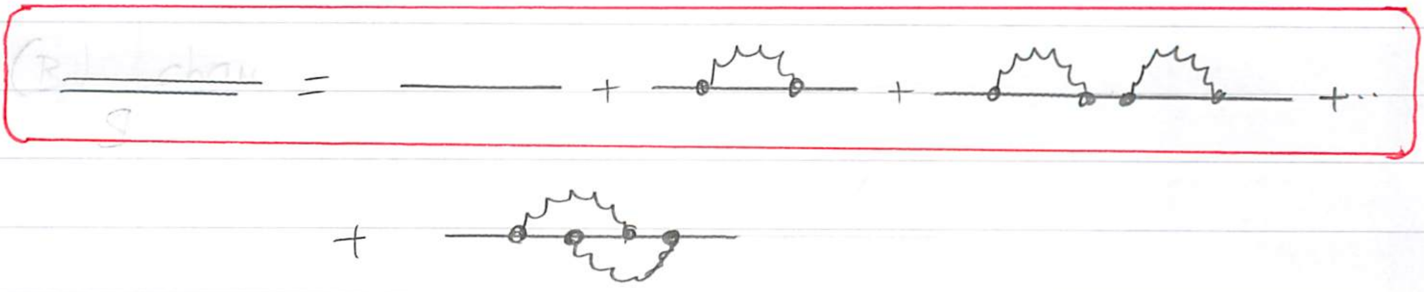
VI.5 Mass- and Charge Renormalization, Ward identities

(A) Mass renormalization

We have seen that the partial summation of all self energy diagrams of the electron has lead to a dressed electron propagator with pole at $p^2 = \bar{m}$, where

$$m + \Sigma(\bar{m}) = \bar{m}$$

Since it is physically impossible to observe the "naked" mass m, we will use $\bar{m}$ as the true mass of the electron. To simplify the notation we use "m" again instead of $\bar{m}$. At the same time we drop the divergent contribution to the self energy.



⇒

$= \Sigma$

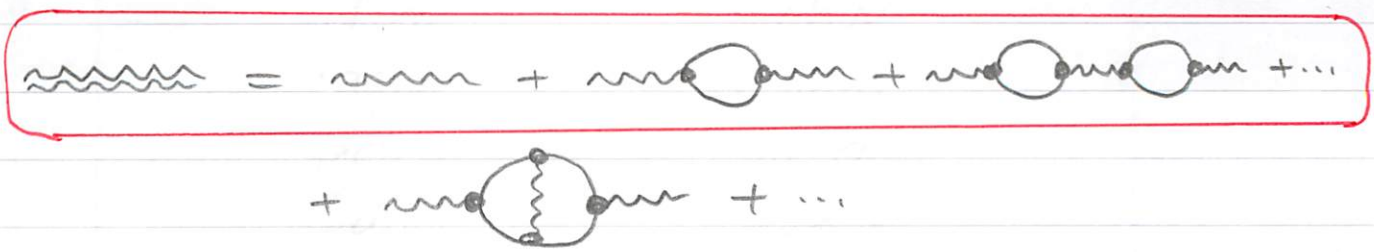
$$S(p) \to S'(p) = \frac{Z_2}{\not p - \underbrace{(m+\Sigma(\bar m))}_{=\bar m} - Z_2 \Sigma_R(\not p) + i\varepsilon}$$

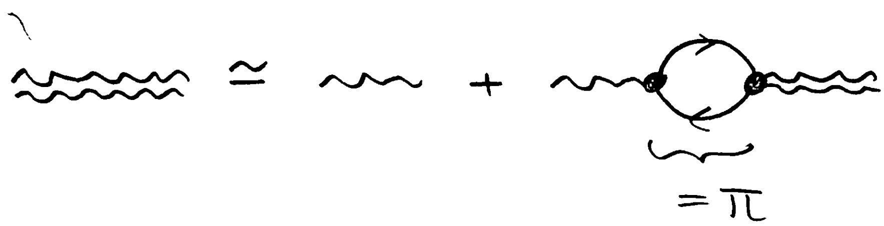
This also gave a contribution to the charge renormalization

$$e_0 \longrightarrow Z_2 e_0$$

(B) charge renormalization

We have seen that partial summation of all vacuum-polarization diagrams has lead to a dressed photon propagator which did not produce a mass but another contribution to the charge renormalization





$$D^{\mu\nu}(q) \to D'^{\mu\nu}(q) = \frac{i g^{\mu\nu}}{-q^2 - i\varepsilon} \; \frac{Z_3}{1 + Z_3 \bar{\pi}(q^2)}$$

where $\bar{\pi}(q^2) \xrightarrow[q^2 \to 0]{} 0$

This gave rize to

$$Z_2 e_0 \longrightarrow \sqrt{Z_3}\, Z_2 e_0$$

Finally we have seen that the vertex correction

$$\gamma^\mu \to \gamma^\mu + \Lambda^\mu(P,q) = \frac{1}{Z_1}\gamma^\mu + \text{regular terms vanishing for } q^2 \to 0$$

gave rize to

$$\sqrt{Z_3}\, Z_2 e_0 \longrightarrow \boxed{\sqrt{Z_3}\, \frac{Z_2}{Z_1} e_0 = e} \qquad (*)$$

One can show that the factors Z_1, Z_2, Z_3 can be absorbed into the charge e in all orders of perturbation theory in the same unique way given in $(*)$.

This leaves however the question if the charge e depends on the kind of charged fermion as Z_1, Z_2 depend on the mass of the fermion. So why is proton charge $|e|$?

(c) Ward identities

The last question can be answered using the Ward identities:

$$\boxed{\Lambda^\mu(p,0) = -\frac{\partial}{\partial p_\mu}\Sigma(p)}$$ Ward-Takahashi identity

note $q = p'-p = 0 \Rightarrow p'=p$ and $P = \frac{1}{2}(p+p') = p$

- $$\Lambda^\mu(P,q) = ie_0^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\nu \frac{1}{m-\not{p}'-\not{k}-i\varepsilon}\, \gamma^\mu \frac{1}{m-\not{p}+\not{k}-i\varepsilon} \cdot \gamma_\nu \frac{1}{\lambda^2-k^2-i\varepsilon}$$

and

$$\Sigma(p) = -ie_0^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\nu \frac{1}{m-\not{p}+\not{k}-i\varepsilon}\, \gamma_\nu \frac{1}{\lambda^2-k^2-i\varepsilon}$$

(later $\lambda^2 \to 0$)

- $$\frac{1}{m-\not{p}'+\not{k}} - \frac{1}{m-\not{p}+\not{k}} = \frac{1}{(m-\not{p}'+\not{k})}(\not{p}'-\not{p})\frac{1}{(m-\not{p}+\not{k})}$$

$$\Rightarrow \quad q_\mu \Lambda^\mu\left(\frac{p+p'}{2}, q\right) = \Sigma(p) - \Sigma(p')$$

in the limit $q \to 0$, i.e. $p' \to p$ one has

$$\Sigma(p') \simeq \Sigma(p) + \frac{\partial \Sigma(p)}{\partial p_\mu}\underbrace{(p'-p)_\mu}_{=q_\mu} + \cdots$$

$$\Rightarrow \quad \Lambda^\mu(p,0) = - \frac{\partial \Sigma(p)}{\partial p_\mu} \qquad \square$$

rem: one can show that this holds in all orders of perturbation theory

The Ward identity has an important consequence: In the low energy limit $p^2 \to m^2$ we find

$$\Lambda^\mu(p,0) = \gamma^\mu\left(\frac{1}{Z_1} - 1\right)$$

on the other hand

$$\Sigma(p) = \delta m + (m - \not{p})\left(\frac{1}{Z_2} - 1\right)$$

wind $\Lambda^\mu(p,0) = -\frac{\partial}{\partial p_\mu}\Sigma(p) \Rightarrow$

$$\boxed{Z_1 = Z_2}$$

thus the charge renormalization is just

$$\boxed{e = \sqrt{Z_3}\, e_0}$$

The renormalization of the charge is universal and does not depend on the mass of the fermion.

VI.6 Renormalizability of quantum field theories

Divergencies emerge in QFT's with Feynman rules due to the integration $\int d^d k$ in loop diagrams

<u>simplification:</u> theories without derivatives in interaction Lagrangian (like QED)

consider typical diagram

# of internal loops	ℓ
# of internal boson lines	n_B
# of internal fermion lines	n_F

<u>global degree of divergence</u> d - dimension of space

D = power of k in numerator − power of k in denominator

$$\boxed{D = \ell d - 2n_B - n_F}$$

if
- $D > 0$ diagram divergent
- $D = 0$ diagram log-divergent
- $D < 0$ diagram convergent

<u>more precisely:</u> a given diagram is convergent if <u>all</u> sub-diagrams are convergent

<u>Question:</u> Can we read off the degree of divergence of

all diagrams and diagram parts just from the interaction Lagrangian ?

consider $\mathcal{L}_I \sim (\bar{\Psi}\Theta\Psi)^{F/2}\phi^B$

F	# of fermion fields $(\Psi, \bar{\Psi})$ per vertex (even!)
B	# of boson fields (ϕ) per vertex
N_B	# of external boson lines in a diagram
N_F	# of external fermion lines in a diagram
n	order of perturbation

since each propagator couples to two vertices:

$$\Rightarrow \quad \begin{aligned} N_B + 2n_B &= nB \\ N_F + 2n_F &= nF \end{aligned}$$

number of 4-momenta not fixed by incoming/outgoing lines = # of loops ℓ

$$\ell = n_B + n_F - n + 1$$

each internal line has a free momentum, each vertex gives one constraint except that the total momentum is not fixed by the vertices.

depends on specific diagrams

$$\Rightarrow \boxed{D = nI + d - \frac{1}{2}(d-2)N_B - \frac{1}{2}(d-1)N_F}$$

order of perturbation

divergency index

$$\boxed{I = \frac{1}{2}(d-2)B + \frac{1}{2}(d-1)F - d}$$

divergency index

depends only on space dimension and interaction Lagrangian

d=4)

$$I = B + \frac{3}{2}F - 4$$

$$D = nI + 4 - N_B - \frac{3}{2}N_F$$

- $I > 0$ starting at a certain order of perturbation all diagrams are divergent
  theory is called non-renormalizable
- $I < 0$ only a finite number of sub-diagrams in a finite number of diagrams is divergent
  theory is called super-renormalizable
- $I = 0$ D independent on n. In $d=4$ $D < 0$ if $N_B + \frac{3}{2}N_F > 4$
  $\Rightarrow$ only small number of sub diagrams leads to divergent diagrams in all orders perturbation

theory is called superficially renormalizable

▶ example QED $\quad B = 1 \quad F = 2$

$$I = 1 + \frac{3}{2}2 - 4 = 0$$

$\Rightarrow$ superficially renormalizable

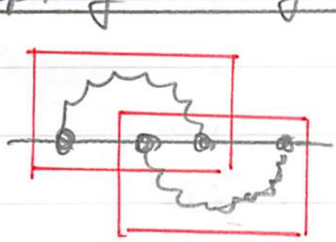
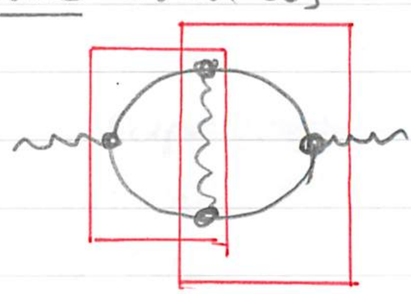
$\exists$ only finite number of diverging sub-diagrams for which $4 - N_B - \frac{3}{2}N_F \geq 0$

self energy  $D = 4 - 0 - \frac{3}{2}2 = 1$

vacuum pol.  $D = 4 - 2 = 2$

Vertex correction  $D = 4 - 1 - \frac{3}{2}2 = 0$

To prove renormalizability of a superficially renormalizable theory one has to take into account that there are overlapping divergent diagrams such as

 or 

In the case of QED all overlapping diagrams can be reduced to non-overlapping ones using Ward identities

$\Rightarrow$ proof of renormalizability of QED