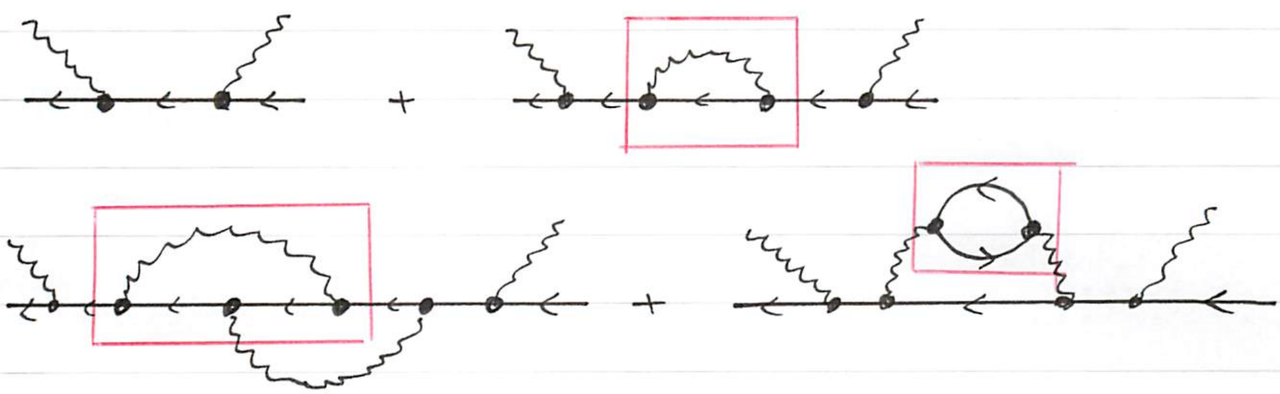
VI. Renormalization of QED

We will now consider the question of how to treat the loop diagrams in higher order perturbation theory. We will see that they formally diverge but nevertheless lead to reasonable and finite results as we can systematically remove these divergences. This will be done in two steps:

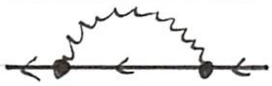
(i) Regularization = separation of divergencies

(ii) Renormalization = formal absorption of divergent expressions in the parameters of the theory such as m, e

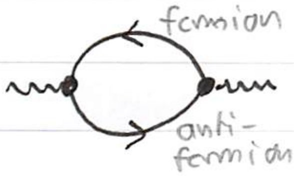
<u>example:</u> Compton-scattering, some typical digrams



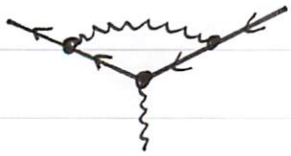
We here see 3 different types of loop diagrams:



self-energy of electron (fermion)

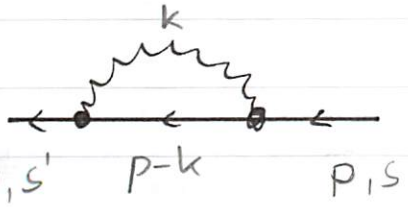


vacuum polarization



vertex correction

VI.1 Self energy of the electron



$$\mathcal{M}_\Sigma = \bar{u}(p,s')\,\Sigma(p)\,u(p,s)$$

according to Feynman rules

$$\boxed{\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (m + \not{p} - \not{k})\gamma_\mu}{(-k^2 - i\varepsilon)(m^2 - (p-k)^2 - i\varepsilon)}}$$

remember: integration is over d^4k, i.e. also for values $k^0 \neq |\vec{k}|$ i.e. off-energy shell!

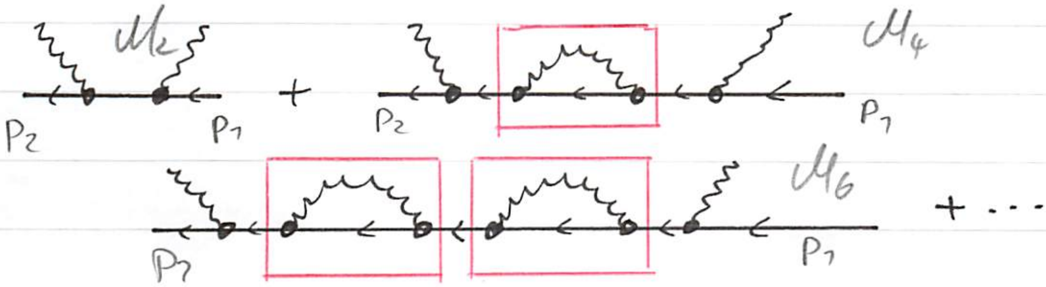
one recognizes that this integral diverges as for $|k| \to \infty$

$$\int d^4k \, \frac{k}{k^4} \sim \int^\infty d|k| \longrightarrow \infty$$

bevor discussing the renormalization of $\Sigma(p)$ we want to

discuss its physical meaning:
We realize that for a diagram of the type

there exist higher-order contributions of the same structure



Summing up all of these contributions (which is of course only a partial summation of all possible diagrams) yields

$$\mathcal{M}_2 = -ie^2\,\bar{u}(p_2)\,\not\epsilon_f\; iS(p)\;\not\epsilon_i\, u(p_1)$$

$$\mathcal{M}_4 = -ie^2\,\bar{u}(p_2)\,\not\epsilon_f\; iS(p)\left(-i\Sigma(p)\right) iS(p)\;\not\epsilon_i\, u(p_1)$$

$$\mathcal{M}_6 = -ie^2\,\bar{u}(p_2)\,\not\epsilon_f\; iS(p)\left(-i\Sigma(p)\right)iS(p)\left(-i\Sigma(p)\right) iS(p)\;\not\epsilon_i\, u(p_1)$$

$$\vdots$$

$$\mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}_{2n} = -ie^2\,\bar{u}(p_2)\,\not\epsilon_f\; iS'(p)\;\not\epsilon_i\, u(p_1)$$

where $S'(p)$ is the <u>dressed propagator</u>
of the electron field

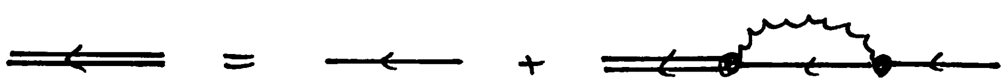
$$S'(p) = S(p)\Big[1 + \Sigma(p)S(p) + \Sigma(p)S(p)\Sigma(p)S(p) + \cdots\Big]$$

this leads to the <u>Dyson equation</u> of the e-propagator

$$\boxed{S'(p) = S(p) + S(p)\Sigma(p)S'(p)}$$

or alternatively $\qquad S(p) = \frac{1}{\not p - m + i\varepsilon} = \frac{\not p + m}{p^2 - m^2 + i\varepsilon}$

$$\boxed{S'(p) = S(p) + S'(p)\Sigma(p)S(p)}$$



remark: We will see later that there are also other contributions that lead to a dressing of the e-propagator

$$S'(p) = S(p)\frac{1}{1-\Sigma(p)S(p)} = \frac{1}{1-S(p)\Sigma(p)}S(p)$$

$\Sigma(p)$ is a 4×4 Dirac matrix and a Lorentz scalar. Thus it can be written in the form

$$\Sigma(p) = \underset{\uparrow\ \text{unity matrix}}{m}\, A(p^2) + \underset{\uparrow\ \text{all } \gamma\text{-matrices}}{\not p}\, B(p^2)$$

where $A(p^2)$ and $B(p^2)$ are scalars. This is the only possible decomposition as the only possibility to get a Lorentz scalar out of γ-matrices and the 4-momentum p which is the only Lorentz vector available is $\not p$. Further

$$\Rightarrow \qquad [\Sigma(p), S(p)] = 0 \qquad \text{(both contain } 1 \text{ or } \not p)$$

$$\Rightarrow S'(p) = \frac{1}{S^{-1}(p) - \Sigma(p)} = \frac{1}{\not{p} - m - \Sigma(p) + i\varepsilon}$$

We will now show that $\Sigma(p)$ has two effects:

(i) modification of the mass $m \rightarrow \bar{m}$
(ii) modification of the normalization of the propagator

Lets expand $\Sigma(p)$ in Taylor series in $(\not{p} - \bar{m})$
($\not{p}$ is Lorentz scalar but Dirac 4x4 Matrix) $\Sigma(p) = \Sigma(\not{p})$

$$\Sigma(\not{p}) = \Sigma(\bar{m}) + (\not{p} - \bar{m})\Sigma'(\bar{m}) + \frac{1}{2}(\not{p} - \bar{m})^2 \Sigma''(\bar{m}) + \cdots$$

$$\equiv \Sigma(\bar{m}) + (\not{p} - \bar{m})\Sigma'(\bar{m}) + \Sigma_R(\not{p})$$

Note that $\Sigma'(\bar{m}) = \left.\frac{d\Sigma(p)}{d\not{p}}\right|_{\not{p} = \bar{m}}$ is a Lorentz scalar and the above notation is a very compact one

$$\Sigma(\not{p}) = \begin{pmatrix} \Sigma_{11}(\not{p}) & \cdots & \Sigma_{1p}(\not{p}) \\ \vdots & & \\ \Sigma_{41}(\not{p}) & \cdots & \Sigma_{44}(\not{p}) \end{pmatrix} = \Sigma(\bar{m}) +$$

$$+ \begin{pmatrix} (\not{p} - \bar{m})_{11} \frac{\partial \Sigma_{11}}{\partial \not{p}_{11}} + (\not{p} - \bar{m})_{12} \frac{\partial \Sigma_{11}}{\partial \not{p}_{12}} + \cdots & \cdots \\ \vdots & \\ (\not{p} - \bar{m})_{11} \frac{\partial \Sigma_{41}}{\partial \not{p}_{11}} + (\not{p} - \bar{m})_{12} \frac{\partial \Sigma_{41}}{\partial \not{p}_{12}} + \cdots & \cdots \end{pmatrix}$$

$\Rightarrow$

$$S'(p) = \frac{1}{\underbrace{\not{p} - m - \Sigma(\bar{m})}_{\equiv -\bar{m}} - (\not{p} - \bar{m})\Sigma'(\bar{m}) - \Sigma_R(\not{p}) + i\varepsilon}$$

$$\boxed{\bar{m} \equiv m + \Sigma(\bar{m})}$$

$\Rightarrow$

$$S'(p) = \frac{1}{\underbrace{\not{p}(1 - \Sigma'(\bar{m}))}_{\equiv \frac{1}{Z_2}} - \bar{m}\left(1 - \Sigma'(\bar{m})\right) - \Sigma_R(\not{p}) + i\varepsilon}$$

$$\boxed{Z_2 \equiv \frac{1}{1 - \Sigma'(\bar{m})}}$$ divergent !

thus we find finally

$$\boxed{S'(p) = \frac{Z_2}{\not{p} - \bar{m} - Z_2\Sigma_R(\not{p}) + i\varepsilon}} \quad (*)$$

Looking at the free electron propagator $S(p) = \frac{1}{\not{p} - m + i\varepsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$ shows that the pole of $S(p)$ is just the mass m. We now show that the pole of $S'(p)$ in $(*)$ is $\bar{m}$ and that $\bar{m}$ has to be interpreted as <u>renormalized mass</u>

note $\quad \Sigma_R(\not{p}) = (\not{p} - \bar{m})^2 R \quad$ and $R \neq 0$ for $\not{p} = \bar{m}$

$$S'(p) = \frac{(\not p + \bar m) Z_2}{(p^2 - \bar m^2)\left[1 - (\not p - \bar m) Z_2 R\right]}$$

obviously $S'(p)$ has a pole at $p^2 = \bar m^2$.
Furthermore in the vicinity of the pole $\not p = \bar m + \mathcal{O}(p^2 - \bar m^2)$

$$\leadsto \quad S'(p) \xrightarrow[p^2 \approx \bar m^2]{} \frac{Z_2}{\not p - \bar m}$$

further procedure:

$\bar m \longrightarrow$ physical mass (later more)

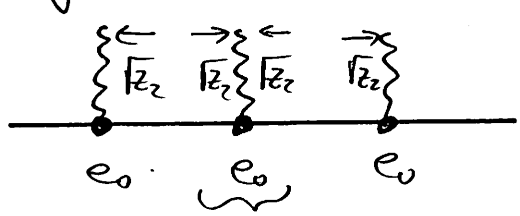
$Z_2 \longrightarrow$ eliminate from theory

elimination of Z_2; charge and wavefunction renormalization

- all <u>internal electron lines</u> connect to two vertices

$$Z_2 \to \sqrt{Z_2}\,\sqrt{Z_2}$$

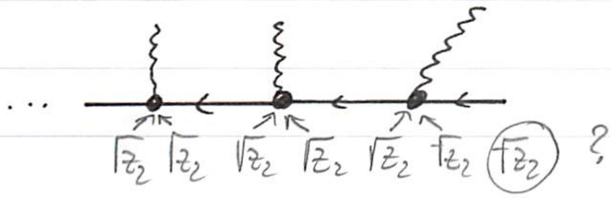
each vertex contains the electron charge. Thus we can absorb the Z_2's from internal electron lines in the charge e



remark: this is not yet the complete charge renormalization!

$$e_0 \to \sqrt{Z_2}\, e_0 \sqrt{Z_2} = Z_2 e_0 \to e$$

- in <u>external electron (positron) lines</u> a factor of $\sqrt{Z_2}$ remains however

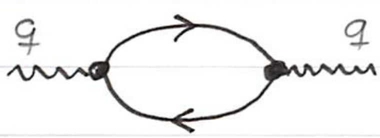


> rule **(10)** for Feynmandiagrams:
>
> multiply every <u>external fermion line</u> with <u>$\sqrt{Z_2}$</u>

<u>wavefunction renormalization</u>

physical background: incoming fermions can be considered as created in some QED process in the past

VI. 2. Vacuum polarization



$$\mathcal{M}_\Pi = (\varepsilon_f)^*_\mu \, \Pi^{\mu\nu}(q) \, (\varepsilon_i)_\nu$$

$$\boxed{\Pi^{\mu\nu}(q) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\{\gamma^\mu (m+\not{p}) \gamma^\nu (m+\not{p}-\not{q})\}}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)}}$$

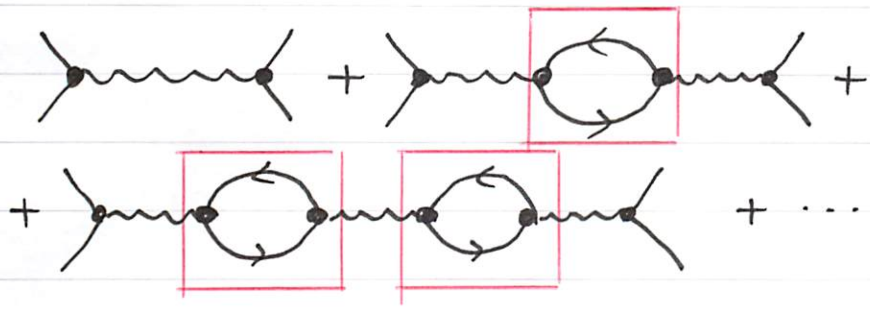
-1 since closed fermionloop

"Tr" $\hat{=}$ summation over all spins of internal fermion / anti-fermion pair

Also this integral is UV-divergent as

$$\int d^4p \, \frac{p^2}{p^4} \sim \int^{\infty} d|p| \, |p| \rightarrow \infty$$

As in the case of the self-energy we see that higher order contributions of the same structure exist



(*)

We will show later, that

$$\boxed{\Pi^{\mu\nu}(q) = \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2)}$$

where $\Pi(q^2)$ is a Lorentz scalar

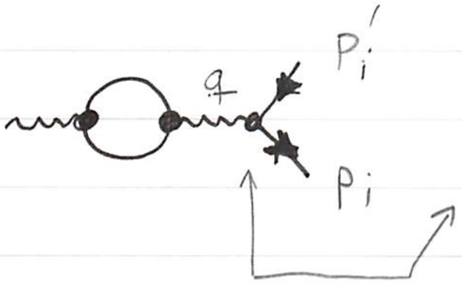
$$\Rightarrow \quad q_\mu \Pi^{\mu\nu} = \Pi^{\mu\nu} q_\nu = 0$$

Similarly to the selfenergy the higher contributions in (*) can be summed up (also here this summation is only a partial summation of higher-order terms!). For this we use

$$\frac{i g^{\mu\lambda}}{-q^2 - i\varepsilon} \left(g_{\lambda\lambda'} q^2 - q_\lambda q_{\lambda'}\right) \Pi(q^2) \frac{i g^{\lambda'\nu}}{-q^2 - i\varepsilon} =$$

$$= \frac{i}{-q^2 - i\varepsilon} \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi(q^2) \frac{i}{-q^2 - i\varepsilon} = \frac{-i g^{\mu\nu}}{-q^2 - i\varepsilon} \Pi(q^2)$$

Here we have used that the term containing $q^\mu q^\nu$ disappears when carrying out the contraction of the connecting fermion lines:



$$\bar{u}(p_i')\gamma^\mu v(p_i)\, q_\mu$$
$$= \bar{u}(p_i')\gamma^\mu v(p_i)\,(p_i - p_i')_\mu =$$

$$= \bar{u}(p_i')(\not{p}_i - \not{p}_i')v(p_i) = \bar{u}(p_i')(m-m)v(p_i) = 0$$

I.e.

$$D'^{\mu\nu}(q) = D^{\mu\nu}(q) + D^{\mu\nu}(q)\left(-\Pi(q^2)\right)$$
$$+ D^{\mu\nu}(q)\left(-\Pi(q^2)\right)\left(-\Pi(q^2)\right) + \cdots$$

$$\Rightarrow \quad \boxed{\begin{aligned} D'^{\mu\nu}(q) &= D^{\mu\nu}(q)\,\frac{1}{1+\Pi(q^2)} \\ &= \frac{g^{\mu\nu}}{q^2+i\varepsilon}\,\frac{1}{(1+\Pi(q^2))} \end{aligned}}$$

<u>dressed photon propagator</u>

One recognizes that the pole of the photon propagator is still at $q^2=0$. Gauge invariance does not allow the generation of an effective mass.

expanding $\Pi(q^2)$ around the pole at $q^2=0$

$$\Pi(q^2) = \Pi(0) + \bar{\Pi}(q^2) \quad \text{where } \bar{\Pi}(0) = 0$$

yields with $\boxed{Z_3 \equiv \frac{1}{1+\Pi(0)}}$ divergent

$$\boxed{D'^{\mu\nu}(q^2) = \frac{g^{\mu\nu}}{-q^2 - i\varepsilon} \, \frac{Z_3}{1 + \underbrace{Z_3 \bar{\Pi}(q^2)}_{\text{only relevant for large } q^2}}}$$

in vicinity of pole

$$D'^{\mu\nu}(q^2) \xrightarrow[q^2 \to 0]{} \frac{g^{\mu\nu} Z_3}{-q^2 - i\varepsilon}$$

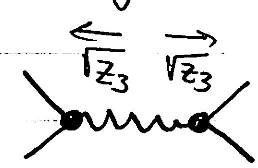
(i) no mass renormalization! (gauge invariance)
(ii) modification of normalization of the propagator

<u>elimination of Z_3 by charge and wavefunction renormalization</u>

- all internal photon lines connect to two vertices

$$Z_3 \to \sqrt{Z_3}\sqrt{Z_3}$$

$\sqrt{Z_3}$ for every charge at the end of the photon line



$$e_0 \to Z_2 \sqrt{Z_3}\, e_0$$

- in external photon lines a factor $\sqrt{Z_3}$ remains

> rule (11) for Feynman diagrams:
>
> multiply every external photon line with $\sqrt{Z_3}$

VI.3 dimensional regularization

With the electron self energy and the vacuum polarization we have seen two important examples of loop diagrams and have seen the principle ideas behind the elimination of divergencies. Now we want to discuss a general mathematical procedure for the systematic separation between finite and infinite terms.

general structure of loop diagram expressions

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{N}{A_1 A_2 \cdots A_n}$$

A_i - denominator of propagators $\quad k^2 + i\varepsilon$

$\quad p^2 - m^2 + i\varepsilon$

N - polynomial in k^μ

There are two steps in the regularization:

(i) all integrals of the type of I can be written in a normal form !

the main problem is the denominator

$$\frac{1}{A_1 A_2} = \int_0^1 dz \frac{1}{\left[A_1 z + A_2(1-z)\right]^2} = \int_0^1 dz \frac{1}{\left[D(z)\right]^2}$$

where $D(z) \equiv A_1 z + A_2(1-z)$

$$\frac{1}{A_1 A_2 A_3} = 2\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{1}{\left[A_1 z_1 + A_2 z_2 + A_3(1-z_1-z_2)\right]^3}$$

$$= 2\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \frac{1}{\left[D(z_1,z_2)\right]^3}$$

where $D(z_1,z_2) \equiv A_1 z_1 + A_2 z_2 + A_3(1-z_1-z_2)$

Since A_1, A_2, A_3 are denominators of Feynman propagators
D has the form $(A_i = a_i^2 + b_i k - k^2)$

$$D = B^2 + 2k\cdot Q - k^2$$

with variable trafo $k = k' - Q$

$$D = B^2 + Q^2 - k'^2 = C^2 - k'^2$$

$\Rightarrow$ standard form of loop integrals:
Apart from finite integrals over Feynman parameter $z_1, z_2, ..$
we have to evaluate

$$\boxed{\int \frac{d^4k}{(2\pi)^4} \frac{N}{(C^2 - k^2 - i\varepsilon)^n}} \quad (*)$$

(ii) dimensional regularization

If in (*) $n > 2$ the loop integral is convergent. For $n \leq 2$ it is divergent. In different dimensions this result is different

(*) convergent if $d < 2n$

$\Rightarrow$ consider general d and then make $d \to 4$ this will allow to separate divergent terms in a systematic and simple way

for $d < 2n$ holds

$$\boxed{\int \frac{d^d k}{(2\pi)^d} \frac{1}{(C^2 - k^2 - i\varepsilon)^n} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{C^2}\right)^{n - \frac{d}{2}}}$$

$$\boxed{\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(C^2 - k^2 - i\varepsilon)^n} = -\frac{i g^{\mu\nu}}{2(4\pi)^{d/2}} \frac{\Gamma(n - 1 - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{C^2}\right)^{n - 1 - \frac{d}{2}}}$$

proof: none

here
$$\boxed{\begin{array}{l} \Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1) \\ \Gamma(1) = 1 \end{array}}$$
Gamma Function

for $\alpha = n \in \mathbb{N}$ $\quad \Gamma(n) = (n-1)!$

$$|\Gamma(0)| = |\Gamma(-1)| = |\Gamma(-2)| = \cdots = \infty \quad !$$

however for non integer values of α $\Gamma(\alpha)$ is well defined

$$\boxed{\Gamma(\alpha) = \int_0^\infty dt\; t^{\alpha-1} e^{-t}}$$

$\Rightarrow$ consider the integrals as functions of the continuous parameter d and let $d \to 4$ at the end

Regularization of the vacuum polarization

$$\Pi^{\mu\nu}(q^2) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \frac{Tr\{\gamma^\mu(m+\not{p})\gamma^\nu(m+\not{p}-\not{q})\}}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)}$$

applying the rules for traces of γ matrices we find

$$\Pi^{\mu\nu}(q^2) = -4ie^2 \int \frac{d^4p}{(2\pi)^4} \cdot \frac{\overbrace{m^2 g^{\mu\nu} + 2p^\mu p^\nu - p^\mu q^\nu - q^\mu p^\nu - g^{\mu\nu} p\cdot(p-q)}^{= N^{\mu\nu}}}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)}$$

normal form

$$\frac{1}{(m^2-p^2-i\varepsilon)(m^2-(p-q)^2-i\varepsilon)} = \int_0^1 dz \frac{1}{\left[(m^2-p^2)(1-z)+(m^2-(p-q)^2)z\right]^2}$$

$$\Pi_d^{\mu\nu}(q) = -4ie^2 \int_0^1 dz \int \frac{d^dp}{(2\pi)^d} \frac{N^{\mu\nu}}{[m^2-p^2+2pqz-q^2z-i\varepsilon]^2}$$

with $p = k + qz$

$$\Pi_d^{\mu\nu}(q) = -4ie^2 \int_0^1 dz \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{N}^{\mu\nu}}{[m^2-k^2-q^2z(1-z)-i\varepsilon]^2}$$

where $\tilde{N}^{\mu\nu} = m^2 g^{\mu\nu} + 2k^\mu k^\nu + (q^\mu k^\nu + k^\mu q^\nu)(2z-1)$

$$-2q^\mu q^\nu z(1-z) - g^{\mu\nu}(k+qz)(k-q(1-z))$$

Note: • terms with odd exponents of k^μ vanish since the integration is over a symmetric area $(d\to 4)$

• for the same reason $\quad k^\mu k^\nu \longrightarrow g^{\mu\nu}\frac{k^2}{d}$

i.e. $\quad \tilde{N}^{\mu\nu} \longrightarrow g^{\mu\nu}\left(m^2 + \left(\frac{2}{d}-1\right)k^2 + q^2z(1-z)\right)$
$$- 2q^\mu q^\nu z(1-z)$$

Now we can carry out the integration d^dk. This yields

$$\Pi_d^{\mu\nu}(q) = \frac{4e^2}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \int_0^1 dz \frac{1}{(m^2-qz(1-z))^{2-d/2}}$$

$$\cdot \left[g^{\mu\nu}(m^2-q^2z(1-z)) \underbrace{\left(1-\frac{d}{2}\left(\frac{2}{d}-1\right)\frac{\Gamma(1-d/2)}{\Gamma(2-d/2)}\right)}_{=0} + (g^{\mu\nu}q^2-q^\mu q^\nu)2z(1-z) \right]$$

$$\Rightarrow \quad \underline{\underline{\Pi_d^{\mu\nu}(q^2) = \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \Pi_d(q^2)}}$$

This is the structure which we assumed in Sec. VI.2 with

$$\Pi_d(q^2) = \frac{\alpha}{\pi} \frac{\Gamma(3-\frac{d}{2})}{(2-\frac{d}{2})(4\pi)^{d/2-2}} \int_0^1 dz \frac{2z(1-z)}{(m^2 - q^2 z(1-z))^{2-d/2}}$$

This expression is well defined for all $d<4$.

Let's now consider the limit $d \to 4$

$$\Pi_d(q^2) = \underbrace{\Pi_d(0)}_{\text{divergent for } d\to 4} + \underbrace{\overline{\Pi}_d(q^2)}_{\text{finite for } d\to 4}$$

$$\boxed{\Pi_d(0) = \frac{\alpha}{\pi} \frac{2\Gamma(1+\varepsilon/2)}{\varepsilon(4\pi)^{-\varepsilon/2}} \int_0^1 dz \frac{2z(1-z)}{m^\varepsilon} \xrightarrow[\varepsilon\to 0]{} \infty}$$

$\varepsilon \equiv 4-d$

$$\overline{\Pi}_d(q^2) = \frac{\alpha}{\pi} \frac{4\Gamma(1+\frac{\varepsilon}{2})}{\varepsilon(4\pi)^{-\varepsilon/2}} \int_0^1 dz \, z(1-z) \cdot$$

$$\cdot \left\{ \frac{1}{[m^2 - q^2 z(1-z)]^{\varepsilon/2}} - \frac{1}{m^\varepsilon} \right\}$$

Using $\lim\limits_{\varepsilon \to 0} A^{\varepsilon} = 1 + \varepsilon \log A + \mathcal{O}(\varepsilon^2)$

$$\underset{(\varepsilon \to 0)}{\Rightarrow} \quad \boxed{\bar{\Pi}(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dz \; z(1-z) \log\left[1 - \frac{q^2}{m^2} z(1-z)\right]}$$

in the low energy regime $|\vec{q}|^2 \ll m^2$

$$\boxed{\bar{\Pi}(q^2) \longrightarrow \frac{\alpha}{15\pi} \frac{q^2}{m^2}}$$

The term $\bar{\Pi}(q^2)$ yields another contribution to the Lamb shift next to the Bethe term

$$D^{\mu\nu}(q^2) \longrightarrow D'^{\mu\nu}(q^2) = \frac{g^{\mu\nu}}{-q^2 - i\varepsilon} \; \frac{1}{1 + \bar{\Pi}(q^2)}$$

$$\Rightarrow \cdots \Rightarrow \qquad \underline{\underline{\Delta E_{1s,2s} \cong -27 \text{ MHz}}}$$

contribution of vacuum polarization to Lamb shift