(F) relation of scattering cross sections and decay rate to $|\mathcal{M}|^2$

Without goin into details we here state the rules connecting the reduced S-matrix element $\mathcal{M}$ to the

- differential n-body decay rate dW_n of a single particle
- differential cross section $d\sigma$ of scattering of two initial particle

$$\boxed{dW_n = \underbrace{(2\pi)^4 \delta^4\left(\sum_f p_f - p_i\right)}_{\text{4-momentum conservation}} \underbrace{\frac{1}{2E_i}}_{\text{energy of initial particle}} \left(\prod_f \underbrace{\frac{d^3k_f}{(2\pi)^3 2E_f}}_{\text{spatial momentum elements of final particles}}\right) |\mathcal{M}|^2}$$

$$\boxed{d\sigma = (2\pi)^4 \delta^4\left(\sum_f p_f - p_{i_1} - p_{i_2}\right) \frac{1}{4E_{i_1}E_{i_2}} \left(\prod_f \frac{d^3k_f}{(2\pi)^3 2E_f}\right) |\mathcal{M}|^2}$$

elastic scattering of 2 particles in center-of-mass frame

$$\Rightarrow \quad \left.\frac{d\sigma}{d\Omega}\right|_{\text{com}} = \frac{|\mathcal{M}|^2}{(2\pi)^2 16 \left(E_{i_1} + E_{i_2}\right)}$$

V. 4 Feynman rules of QED

We now want to establish the diagrammatic perturbation theory of QED. Starting point is the Hamiltonian in Coulomb gauge

$$\hat{H} = \hat{H}_{elm} + \hat{H}_D + \hat{H}_{int}$$

with

$$\hat{H}_D = \sum_{\mu=e,p} \int d^3r \, \hat{\psi}^+_\mu \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m_\mu\right) \hat{\psi}_\mu \qquad \mu \hat{=} e, p$$

electron, proton

$$\hat{H}_{elm} = \frac{1}{2}\int d^3r \left(\hat{\vec{E}}_\perp^2 + \hat{\vec{B}}_\perp^2\right)$$

$$\hat{H}_{int} = \hat{H}^1_{int} + \hat{H}^2_{int} \qquad \text{(Coulomb gauge!)}$$

where

$$\hat{H}^1_{int} = \frac{1}{4\pi}\int d^3r \int d^3r' \, \frac{\hat{j}^0_p(\vec{r},t)\, \hat{j}^0_e(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

instantaneous Coulomb interaction

$$\hat{H}^2_{int} = -\int d^3r \left(\hat{\vec{j}}_p(\vec{r},t) + \hat{\vec{j}}_e(\vec{r},t)\right) \hat{\vec{A}}_\perp(\vec{r},t)$$

radiative interaction

The Coulomb self interaction terms $\sim (j^0_p)^2$ and $\sim (j^0_e)^2$ is assumed to be absorbed in $\hat{H}_D$

(A) electron-proton scattering

2 Dirac fields   electron: $p_i \longrightarrow p_f$

proton: $q_i \longrightarrow q_f$

for simplicity   "p_i" $\hat{=}$ p_i, S_i i.e. spin degree absorbed

4 momentum

$\boxed{\hat{H}^1_{int}}$ yields contribution in 1st order

$$-i \langle p_f, q_f | \int_{-\infty}^{\infty} d\tau \, \hat{H}^1_{int}(\tau) \, | p_i, q_i \rangle \equiv I_1$$

with

$$\hat{j}^\mu_p = e \, \hat{\bar{\psi}}_p \gamma^\mu \hat{\psi}_p \qquad \text{proton } (e>0)$$

$$\hat{j}^\mu_e = -e \, \hat{\bar{\psi}}_e \gamma^\mu \hat{\psi}_e \qquad \text{electron}$$

$$I_1 = \frac{ie^2}{4\pi} \int d^3r \int d^3r' \, \frac{\langle q_f | \hat{\bar{\psi}}_p(\vec{r}) \gamma^0 \hat{\psi}_p(\vec{r}) | q_i \rangle \langle p_f | \hat{\bar{\psi}}_e(\vec{r}') \gamma^0 \hat{\psi}_e(\vec{r}') | p_i \rangle}{|\vec{r} - \vec{r}'|}$$

Since

$$\hat{\psi}(x) = \sum_{\vec{p}, s} \frac{1}{\sqrt{2E_p L^3}} \left(u(\vec{p}, s) e^{i\vec{p}\cdot\vec{r}} \hat{b}_{\vec{p}s} + v(\vec{p}, s) e^{-i\vec{p}\cdot\vec{r}} \hat{d}^+_{\vec{p}, s} \right)$$

$$\hat{\psi}_e(x') |p_i\rangle = \frac{1}{\sqrt{2L^3}} \sum_{\vec{p}, s} \frac{u(\vec{p}, s)}{\sqrt{E_p}} e^{i\vec{p}\cdot\vec{r}'} \underbrace{\hat{b}_{\vec{p}s} |p_i\rangle}_{e^{-iE_pt} \delta_{\vec{p}\vec{p}_i}}$$

in continuous limit $(p \hat{=} p, s)$

$$\vec{\psi}_e(x')|\vec{p}_i\rangle = \frac{1}{\sqrt{2(2\pi)^3}} \frac{u(p_i)}{\sqrt{E_{p_i}}} e^{i\vec{p}_i\cdot\vec{r}'} e^{-iE_{p_i}t}$$

$$\Rightarrow \quad I_1 = \frac{ie^2}{4\pi} \frac{1}{\sqrt{16 E_{\vec{p}_f} E_{\vec{p}_i} E_{\vec{q}_f} E_{\vec{q}_i}}} \frac{1}{(2\pi)^6} \int_{-\infty}^{\infty} d\tau\, e^{i(E_{p_f}+E_{q_f}-E_{p_i}-E_{q_i})\tau}$$

$$\cdot \int d^3r \int d^3r' \frac{e^{-i(\vec{p}_f-\vec{p}_i)\cdot\vec{r}'} e^{-i(\vec{q}_f-\vec{q}_i)\cdot\vec{r}}}{|\vec{r}-\vec{r}'|}$$

$$\cdot \left[\bar{u}(q_f)\gamma^0 u(q_i)\right] \cdot \left[\bar{u}(p_f)\gamma^0 u(p_i)\right]$$

introduce center-of-mass and relative coordinates

$$\vec{s} = \vec{r} - \vec{r}' \qquad \vec{R} = \frac{1}{2}(\vec{r} + \vec{r}')$$

spatial integral: $\bullet \quad = (2\pi)^3 \delta^{(3)}(\vec{p}_f + \vec{q}_f - \vec{p}_i - \vec{q}_i) \dfrac{4\pi}{|\vec{q}|^2}$

$$\vec{q} = \vec{q}_f - \vec{q}_i = \vec{p}_i - \vec{p}_f$$

time integral: $\bullet \quad = (2\pi)\, \delta^{(1)}(E_{p_f} + E_{q_f} - E_{p_i} - E_{q_i})$

$\Rightarrow$

$$I_1 = \frac{ie^2}{4\pi^2} \frac{\delta^{(4)}(q_f + p_f - q_i - p_i)}{\sqrt{16 E_f^p E_f^e E_i^p E_i^e}} \frac{1}{|\vec{q}|^2} \cdot$$

$$\cdot \left[\bar{u}(q_f)\gamma^0 u(q_i)\right] \cdot \left[\bar{u}(p_f)\gamma^0 u(p_i)\right]$$

H^2_{int} | contribution in 2nd order $\qquad (qx \equiv q_\mu x^\mu)$

$$I_2 = -\frac{1}{2} \langle p_f, q_f| \langle 0| \int d^4x_1 \int d^4x_2 \; (-e^2)$$
$$T\Big[\hat{A}^k_\perp(x_1)\, \hat{A}^\ell_\perp(x_2) \Big(:\bar{\hat{\psi}}_p(x_1)\gamma^k \hat{\psi}_p(x_1): :\bar{\hat{\psi}}_e(x_2)\gamma^\ell \hat{\psi}_e(x_2): + \text{"}\begin{matrix} x_1 \leftrightarrow x_2 \\ k \leftrightarrow \ell \end{matrix}\text{"} \Big)\Big] |0\rangle |p_i\, q_i\rangle$$

$$= e^2 \frac{1}{\sqrt{16 E^p_f E^p_i E^q_f E^q_i}} \frac{1}{(2\pi)^6} \int d^4x_1 \int d^4x_2 \langle 0| T \hat{A}^k_\perp(x_1)\, \hat{A}^\ell_\perp(x_2) |0\rangle$$
$$\cdot e^{i(q_f - q_i)x_1}\, e^{i(p_f - p_i)x_2}$$
$$\cdot \left[\bar{u}(q_f)\gamma^k u(q_i)\right]\left[\bar{u}(p_f)\gamma^\ell u(p_i)\right]$$

again center-of-mass and relative coordinates:

$$x = x_1 - x_2 \qquad X = \frac{1}{2}(x_1 + x_2) \qquad x = (t, \vec{x})$$

$\Rightarrow$

$$I_2 = \frac{e^2}{\sqrt{16 E^p_f E^p_i E^q_f E^q_i}} \frac{1}{(2\pi)^6} \left[\bar{u}(q_f)\gamma^k u(q_i)\right]\left[\bar{u}(p_f)\gamma^\ell u(p_i)\right]$$
$$\cdot \int d^4X\, e^{i(q_f - q_i + p_f - p_i)X} \quad \Big\} \to (2\pi)^4 \delta^4(q_f + p_f - p_i - q_i)$$
$$\cdot \int d^4x\, \langle 0| T \hat{A}^k(x)\hat{A}^\ell(0)|0\rangle\, e^{iqx} \quad \Big\} \text{ FTrafo of propagator}$$

$\Rightarrow$

$$\boxed{\boxed{I_2 = \frac{ie^2}{4\pi^2} \frac{\delta^{(4)}(q_f+p_f-p_i-q_i)}{\sqrt{16 E_f^p E_i^p E_f^q E_i^q}} \frac{\delta_{kl} - \frac{q_k q_l}{|\vec{q}|^2}}{q^2+i\varepsilon} \cdot \left[\bar{u}(q_f)\gamma^k u(q_i)\right] \cdot \left[\bar{u}(p_f)\gamma^l u(p_i)\right]}}$$

I_1 and I_2 are both of the same order in e^2 (fine-structure constant as $\hbar = c = 1$)

We now show that

$$(*) \quad \boxed{S = I_1 + I_2 = \frac{\delta^{(4)}(q_f+p_f-q_i-p_i)}{\sqrt{16 E_f^e E_f^p E_i^e E_i^p}} \frac{1}{(2\pi)^2} \mathcal{M}}$$

with

$$\boxed{\mathcal{M} = ie^2 \left[\bar{u}(q_f)\gamma_\mu u(q_i)\right]\left[\bar{u}(p_f)\gamma_\nu u(p_i)\right] \cdot \frac{(-i g^{\mu\nu})}{q^2+i\varepsilon}}$$

this is the full 4-dimensional propagator of the elm field! (= Lorentz gauge)

That $\mathcal{M}$ contains the full propagator must be the case as physical observables cannot depend on the gauge choice and we could have used Lorentz gauge from the beginning!

<u>proof of (*)</u>

$k=1,2,3 \to \mu = 0,1,2,3$
$l = 1,2,3 \to \nu = 0,1,2,3$

(i) $$\delta_{kl} - \frac{q_k q_l}{|\vec{q}|^2} = -g_{\mu\nu} - \frac{q_\mu q_\nu}{|\vec{q}|^2} + \frac{\delta_{\mu 0} q_0 q_\nu}{|\vec{q}|^2} + \frac{\delta_{\nu 0} q_0 q_\mu}{|\vec{q}|^2} - \delta_{\mu 0}\delta_{\nu 0}\frac{q^2}{|\vec{q}|^2}$$

$$\left(q^2 = q_0^2 - |\vec{q}|^2\right)$$

(ii) $$q_\mu \bar{u}(p_f)\gamma^\mu u(p_i) = (p_f - p_i)_\mu \bar{u}(p_f)\gamma^\mu u(p_i) = \bar{u}(p_f)(\not{p}_f - \not{p}_i)u(p_i)$$

now since $\bar{u}(p_f)$ and $u(p_i)$ are solutions of the free Dirac equation

$$(\not{p}_i - m_e)u(p_i) = 0$$
$$\bar{u}(p_f)(\not{p}_f - m_e) = 0$$

$$\Rightarrow \bar{u}(p_f)(\not{p}_f - \not{p}_i)u(p_i) = (m_e - m_e)\bar{u}(p_f)u(p_i) = 0$$

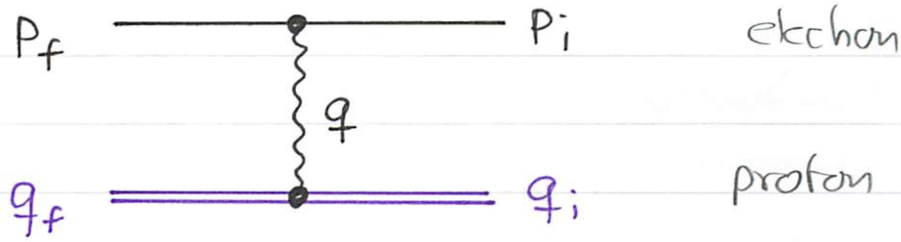
$$\Rightarrow I_2 = \frac{ie^2}{4\pi^2}\frac{\delta^{(4)}(q_f + p_f - q_i - p_i)}{\sqrt{16 E_f^p E_f^q E_i^p E_i^q}}\frac{1}{q^2 + i\varepsilon}\left(-g_{\mu\nu} - \delta_{\mu 0}\delta_{\nu 0}\frac{q^2}{|\vec{q}|^2}\right)\bar{u}(q_f)\gamma^\mu u(q_i)\,\bar{u}(p_f)\gamma^\nu u(p_i)$$

(iii) $$I_1 = \frac{ie^2}{4\pi^2} \frac{\delta^{(4)}(q_f+p_f-q_i-p_i)}{\sqrt{16 E_f^p E_f^q E_i^p E_i^q}} \frac{1}{q^2+i\varepsilon}$$

$$\cdot\, \delta_{\mu 0}\, \delta_{\nu 0} \frac{q^2}{|\vec{q}|^2}\, \bar{u}(q_f)\gamma^\mu u(q_i)\, \bar{u}(p_f)\gamma^\nu u(p_i)$$

this proves eq. (*) $\square$

diagrammatic representation of $\mathcal{M}$



(B) Feynman rules of QED

Generalizing the findings in (A) and in Sec. V1–V3

$$\boxed{S = -i(2\pi)^4 \delta^{(4)}\left(\sum_f p_f - \sum_i p_i\right) \prod_{j=1}^{n} f_j\, \mathcal{M}}$$

$$\boxed{f_j = \frac{1}{\sqrt{(2\pi)^3\, 2E_j}}}$$ incoming and outgoing particles

rules for calculating $\mathcal{M}$:

(i) Draw all topologically different, connected diagrams consisting of interaction vertices, Dirac lines and photon lines. The number of external lines corresponds to the number of incoming plus outgoing lines. The number of vertices corresponds to the order of perturbation theory

(ii) Each diagram corresponds to a mathematical expression according to the following rules. Add the results for all diagrams in a given order of perturbation theory.

(1) Energy and momentum of external lines are given by those of the incoming / outgoing particles. At each vertex there is energy-momentum conservation (4-momentum conservation). In tree-like diagrams this fixes all 4-momenta. In loop diagrams an integration $\int \frac{d^4k}{(2\pi)^4}$ has to be carried out for each loop.

(2) Every vertex at which a photon of polarisation μ is emitted or absorbed by a fermion of positive charge e gets a matrix factor

$$\boxed{-ie\gamma_\mu}$$

(3) Every internal photon line with 4-momentum q is described by the photon propagator

$$\boxed{\frac{-ig^{\mu\nu}}{q^2+i\varepsilon}} \qquad q^2 = q_\mu q^\mu$$

(4) Every internal Fermion line with 4-momentum p and Dirac indices α and β is described by the electron/positron propagator

$$\boxed{\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}} \qquad p^2 = p_\mu p^\mu$$

(5) For incoming and outgoing Fermion lines

- multiply from left with $\bar{u}(p,s)$ for outgoing fermion with 4-momentum p and spin s
- multiply from right with $u(p,s)$ for incoming fermion with 4-momentum p and spin s
- multiply from right with $v(p,s)$ for outgoing anti-fermion with 4-momentum p and spin s
- multiply from left with $\bar{v}(p,s)$ for incoming anti-fermion with 4-momentum p and spin s

(6) For incoming and outgoing photon lines

- multiply with $\vec{e}_m^{\,*}$ for outgoing photon of polarization m (formally e_μ^*)

- multiply with $\vec{e}_m$ for incoming photon of polarization m (formally e_μ)

(7) Symmetrize between all identical Bosons in the incoming or outgoing states

(8) Provide a factor (-1) for each closed Fermion loop

(9) Multiply loop diagrams with n identical Boson lines with $1/n!$

(C) Spin averages

Often one wants to calculate differential cross sections for unpolarized electrons / protons. For this one needs to average over spin degrees of freedom

e.g. $\quad \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2$

For the electron - proton scattering:

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^2}{q^4} \sum_{s_f s_i} \bar{u}(q_f, s_f)\gamma^\mu u(q_i, s_i)\, \bar{u}(q_i, s_i)\gamma^\nu u(q_f, s_f)$$

$$\cdot \sum_{\sigma_f \sigma_i} \bar{u}(p_f, \sigma_f)\gamma_\mu u(p_i, \sigma_i)\, \bar{u}(p_i, \sigma_i)\gamma_\nu u(p_f, \sigma_f)$$

Here we have used:

$$\left[\bar{u}(q_f, s_f)\gamma^\mu u(q_i, s_i)\right]^* = \left[u^+(q_f, s_f)\gamma^0\gamma^\mu u(q_i, s_i)\right]^*$$

$$= u^+(q_i s_i)(\gamma^\mu)^+\gamma^0 u(q_f s_f)$$

$$= u^+(q_i s_i)\gamma^0\gamma^\mu u(q_f s_f) = \bar{u}(q_i s_i)\gamma^\mu u(q_f s_f)$$

One can show that

$$\boxed{\begin{aligned} \frac{1}{2m}\sum_s u(p,s)\bar{u}(p,s) &= \Lambda_+(p) \\ -\frac{1}{2m}\sum_s v(p,s)\bar{v}(p,s) &= \Lambda_-(p) \end{aligned}}$$

$\Lambda_\pm(p) = \dfrac{m \pm \not{p}}{2m}$ are projectors to positive (+) and negative (-) energy states

proof: excercises

Thus

$$\sum_{s_i s_f} \bar{u}(q_f s_f)\gamma^\mu \underline{u(q_i s_i)\bar{u}(q_i s_i)}\gamma^\nu u(q_f, s_f)$$

$$= \sum_{s_f} \bar{u}(q_f s_f)\gamma^\mu \underline{(m_p + \not{q}_i)}\gamma^\nu u(q_f s_f)$$

$$= \sum_{s_f} \bar{u}_\alpha(q_f, s_f)\left[\gamma^\mu(m_p + \not{q}_i)\gamma^\nu\right]_{\alpha\beta} u_\beta(q_f s_f)$$

$$= \underline{\sum_{s_f} u_\beta(q_f, s_f)\bar{u}_\alpha(q_f, s_f)}\left[\gamma^\mu(m_p + \not{q}_i)\gamma^\nu\right]_{\alpha\beta}$$

$$= \underline{(m_p + \not{q}_f)_{\beta\alpha}}\left[\gamma^\mu(m_p + \not{q}_i)\gamma^\nu\right]_{\alpha\beta}$$

$$= \mathrm{Tr}\left\{(m_p + \not{q}_f)\gamma^\mu(m_p + \not{q}_i)\gamma^\nu\right\}$$

$\Rightarrow$ $\boxed{\text{spin sums} \;\hat{=}\; \text{trace of product of } \gamma\text{-matrices}}$

$\boxed{\text{theorem 1: the trace of a product of an odd number of } \gamma\text{-matrices vanishes}}$

proof: $\mathrm{Tr}\{\not{a}_1 \not{a}_2 \cdots \not{a}_n\} = \mathrm{Tr}\{\overbrace{\gamma^5\gamma^5}^{=1} \not{a}_1\not{a}_2 \cdots \not{a}_n\}$

$\underset{\text{n fold permutation}}{=} (-1)^n \mathrm{Tr}\{\gamma^5 \not{a}_1\not{a}_2 \cdots \not{a}_n \gamma^5\} \underset{\text{cyclic property of trace}}{=} (-1)^n \mathrm{Tr}\{\not{a}_1 \cdots \not{a}_n\}$ □

note: $\gamma^\mu = \not{e}_\mu \qquad (e_\mu)_\nu = \delta_{\mu\nu}$

theorem 2:		
	0:	$Tr\{1\} = 4$
	2:	$Tr\{\not{a}\not{b}\} = 4\, a_\mu b^\mu$
	4:	$Tr\{\not{a}\not{b}\not{c}\not{d}\} = 4\,(a_\mu b^\mu c_\nu d^\nu - a_\mu c^\mu b_\nu d^\nu + a_\mu d^\mu b_\nu c^\nu)$

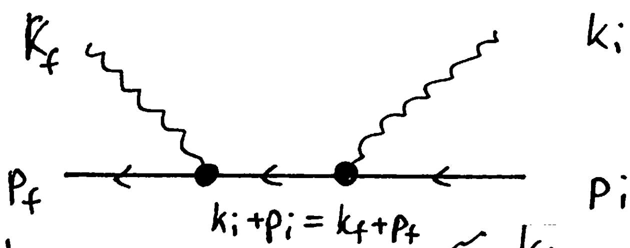
(D) Compton scattering

An important example is the scattering of a photon with an electron

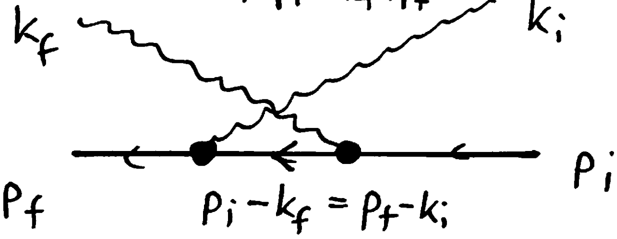
$$\gamma + e \longrightarrow \gamma + e$$

the reduced S-matrix element has two contributions in 2nd order

(a)



(b)



4-momentum conservation $\qquad p_i + k_i = p_f + k_f$

$$\mathcal{M} = -e^2 \bar{u}(p_f) \Big\{ \gamma^\mu e_\mu^{f*} \frac{-i(m+\not p_i + \not k_i)}{m^2-(p_i+k_i)^2-i\varepsilon} \gamma^\nu e_\nu^i$$

$$+ \gamma^\mu e_\mu^{i*} \frac{-i(m+\not p_i - \not k_f)}{m^2-(p_i-k_f)^2-i\varepsilon} \gamma^\nu e_\nu^f \Big\} u(p_i)$$

<u>remark:</u> each of the two terms taken alone is not gauge invariant; the sum is however

for the calculation of the scattering cross section one can use that in the Lab frame (electron at rest)

$$p_i = (m, \vec{0}) \qquad \{e^\mu\} = (0, \vec{e})$$

$$e_i k_i = e_f k_f = e_i p_i = e_f p_f = 0$$

$$k_i^2 = k_i^{0\,2} - \vec{k}_i^{\,2} = k_f^2 = k_f^{0\,2} - \vec{k}_f^{\,2} = 0$$

i.e. $\Rightarrow$ $(p_i+k_i)^2 = m^2 + 2m\omega_i \qquad (p_i-k_f)^2 = m^2 - 2m\omega_f$

$$\mathcal{M} = -e^2 \bar{u}(p_f) \Big\{ \not e^f \frac{m + \not p_i + \not k_i}{-2m\omega_i} \not e^i$$

$$+ \not e^i \frac{m + \not p_i - \not k_f}{2m\omega_f} \not e^f \Big\} u(p_i)$$

spin averaging yields in the Lab-frame the famous <u>Klein-Nishina formula</u> $\quad k = |\vec{k}_i| \quad k' = |\vec{k}_f|$

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \frac{\alpha^2}{4m^2} \left(\frac{k'}{k}\right)^2 \left[4 \, (\vec{e}_i \cdot \vec{e}_f)^2 + \frac{k}{k'} + \frac{k'}{k} - 2 \right]}$$

In the limit of low-energy scattering

$$k' \ll m \quad k \simeq k'$$

and averaging over polarization of photons yields

$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol.} = \frac{\alpha^2}{m^2}\frac{1}{2}\left(1+\cos^2\theta\right)$$

$$\sigma_{unpol} = \frac{8\pi}{3}\frac{\alpha^2}{m^2}$$ Thomson scattering cross section