For more complex processes in higher order perturbation theory, the calculation of the reduced matrix element \( M \) is rather complicated. We are in need of a simple set of rules for the systematic calculation of \( M \)

\[ \Downarrow \]

Feynman rules

We will also profit a graphical representation of these rules

Feynman diagrams

Relevant example of interaction:

\[ L_{int} = -\frac{g}{2} \bar{\psi} \psi \phi \]

\( \psi \) - Dirac field \( \phi \) - Scalar field

Each operator \( \bar{\psi}, \psi, \phi \) either creates or annihilates a particle/anti-particle. \( L_{int} \) is of \( \phi^3 \) structure. Application of \( L_{int} \) with \( n \) interacting fields (here \( n = 3 \)):

- \( n \)-Particle input
- \( e \)-Particle output (Osseen)
for $\psi, \bar{\psi}, \phi$

\[ l = 0 \]
3 particle annihilation

\[ l = 1 \]
2 particle annihilation

\[ l = 2 \]
particle emission, 2 particle decay

\[ l = 3 \]
8 particle creation

$\psi, \bar{\psi}$ Dirac field
$\phi$ scalar field
$p, \bar{p}$ particle with momentum $p$
outgoing particle (p) and anti-particle ($\bar{p}$)
incoming particle of $p$ with 4-momentum $k$
\[ M = i \left( -i \lambda_1 \right) = \lambda_1 \]

(ii) Relativistic scattering in 1+1 st order \( \lambda_1, \lambda_2 \)

Propagator of field \( \varphi \) with 4-momentum
\[ \Delta q = q' - q = p - p' \]

- \( -i \lambda_1 \)
- \( -i \lambda_2 \)

Vertices from interaction operator
\[ -i H_{1\pi} \]

\[ M = i \left( -i \lambda_1 \right) \left( -i \lambda_2 \right) \frac{i}{\Delta q^2 - m_0^2 + i \varepsilon} \]

---

(I) Draw all topologically distinct diagrams that correspond to a given scattering process in the considered order of perturbation theory. The order of perturbation corresponds to the number of vertices.

(II) Find the mathematical expression for every diagram and odd expressions for all diagrams.

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Feynman rules: translation between graphs and mathematical expressions for 

\begin{align*}
\text{rule 0:} & \quad - \text{a factor of } i \\
\text{rule 1:} & \quad - \text{a factor of } -i2k \text{ for every vertex} \\
& \quad \text{at every vertex 4-momentum is conserved} \\
\text{rule 2:} & \quad - \text{for every internal line a factor} \\
& \quad \text{corresponding to the propagator of} \\
& \quad \text{the field with 4-momentum } \Delta Q. \\
& \quad \Delta Q \text{ is determined from 4-momentum} \\
& \quad \text{conservation at vertex}
\end{align*}

Remark: These rules are for \( \phi^3 \) theory. Most of them are generic. Some need modifications. And there will be more

(D) relativistic scattering in 2nd order of \( \lambda \)

In previous examples incoming and outgoing particles distinguishable. Now: identical particles in incoming and outgoing channels

\( |\varphi_{\text{in}}\rangle = \varphi_{1p} \varphi_{1q} |10\rangle \)

\( |\varphi_{\text{out}}\rangle = \varphi_{1p'} \varphi_{1q'} |10\rangle \)
\[ S = \frac{(-i)^2}{2} \lambda_n^2 \int d^4x_1 \int d^4x_2 \cdot \langle 0 | \hat{a}_{\mathbf{q}_1} \hat{a}_{\mathbf{q}_1}^\dagger | \hat{\phi}_n^+(x_1) \hat{\phi}_n^+(x_2) \phi(x_1) \phi(x_2) | 0 \rangle \]

\[ \langle 0 | \hat{a}_{\mathbf{p}_1} \hat{a}_{\mathbf{p}_1}^\dagger | \hat{\phi}_n^+(x_1) \hat{\phi}_n^+(x_2) \phi(x_1) \phi(x_2) | 0 \rangle \]

\[ \mathcal{M} = \frac{\lambda_i^2}{(p'-p)^2 - m_0^2 + i\varepsilon} + \frac{\lambda_i^2}{(q'-p)^2 - m_0^2 + i\varepsilon} \]

\text{from } \text{(A)} \quad \text{from } \text{(B)}

\[ \begin{array}{ccccc}
1 & p' & p & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
p' & 1 & p &q & 1 \\
\end{array} \]

\text{rule 4: symmetrize between identical bosons in the outgoing channels}
alternatively rule 4 could be formulated for

\[ \begin{array}{c}
\text{remarle}:
\end{array} \]

\[ \begin{array}{c}
\text{in giving identical bosons but not for both}
\end{array} \]

\[ \begin{array}{c}
\text{\textbullet~ how about terms from } C \text{?}
\end{array} \]

\[ \begin{array}{c}
correspond to unconnected diagrams:
\end{array} \]

\[ \begin{array}{c}
\text{connected-graph theorem}
\end{array} \]

\[ \begin{array}{c}
\text{In a consistent perturbation theory of the } S \text{-matrix,}
\end{array} \]

\[ \begin{array}{c}
\text{unconnected diagrams do not contribute. Unconnected}
\end{array} \]

\[ \begin{array}{c}
\text{diagrams are those which do not connect to external}
\end{array} \]

\[ \begin{array}{c}
\text{lines}
\end{array} \]

\[ \begin{array}{c}
\text{idea of a proof: make use of normalization}
\end{array} \]

\[ \begin{array}{c}
\mathcal{S} = \langle \beta | \hat{U}_{\text{I}}(\infty, -\infty) | \alpha \rangle \rightarrow \mathcal{S} = \frac{\langle \beta | \hat{U}_{\text{I}}(\infty, -\infty) | \alpha \rangle}{\langle \beta | \hat{U}_{\text{I}}(\infty, -\infty) | \alpha \rangle}
\end{array} \]
\[ \langle p'q' | U | p \rangle = \] 
\[ \begin{align*} 
& \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \ldots 
\end{align*} \]

\[ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} \]

\[ \langle 01 | \tilde{U} | 10 \rangle = 1 + \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[<->, dashed] (0,0) arc (180:0:1); 
\end{tikzpicture}} + \ldots \]

\[ \langle p'q' | U | pq \rangle = \left\{ \begin{align*} 
& \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \ldots \right\} \]

\[ \frac{\langle p'q' | U | pq \rangle}{\langle 01 | \tilde{U} | 10 \rangle} = \left\{ \begin{align*} 
& \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[->] (0,0) -- (0.8,0) node[midway, above] {p'}; 
\draw[->] (0,0) -- (0,0.8) node[midway, right] {p}; 
\draw[->] (0,0) -- (-0.8,0) node[midway, below] {q}; 
\draw[->] (0,0) -- (0,-0.8) node[midway, left] {q'}; 
\end{tikzpicture}} 
+ \ldots \right\} \left\{ 1 + \text{\begin{tikzpicture}[baseline=(current bounding box.center)] 
\draw[<->, dashed] (0,0) arc (180:0:1); 
\end{tikzpicture}} + \ldots \right\} \]
(E) loop diagrams in 2nd order of $\lambda$

Consider incoming neutral particle ($\hat{\phi}$) and outgoing neutral particle ($\hat{\bar{\phi}}$) in second order of $\lambda$

$$S = \frac{(-i)^2 \lambda^2}{2 (3!)^2} \int d^4x_1 d^4x_2 \cdot$$

- $\langle 0 | \hat{A}_k T \left[ : \hat{\bar{\phi}}(x_1) \hat{\phi}(x_1) \hat{\bar{\phi}}(x_2) \hat{\phi}(x_2) \hat{\bar{\phi}}(x_3) \hat{\phi}(x_3) \hat{\bar{\phi}}(x_4) \hat{\phi}(x_4) \right] \hat{A}^*_k | 10 \rangle$

- 3 possibilities

- 3 possibilities

$$S = \frac{(-i)^2 \lambda^2}{2} \int d^4x_1 d^4x_2 \langle 0 | T \hat{A}_k (\infty) \hat{\bar{\phi}}(x_1) | 10 \rangle$$

$$\langle 0 | T \hat{\bar{\phi}}(x_1) \hat{\phi}(x_2) | 10 \rangle^2 \langle 0 | T \hat{\bar{\phi}}(x_2) \hat{A}^*_k | -\infty \rangle | 10 \rangle$$

+ " $x_1 \leftrightarrow x_2$ "

$$= \frac{(-i)^2 \lambda^2}{4} \frac{1}{\sqrt{2^2 E_k E_{k'} (2\pi)^6}}$$

$$\int d^4x_1 d^4x_2 e^{-i k \cdot x_2} e^{i k \cdot x_1} \langle 0 | T \hat{\bar{\phi}}(x_1) \hat{\phi}(x_2) | 10 \rangle^2$$

+ " $x_1 \leftrightarrow x_2$ "
center of mass and relative coordinates

\[ R = \frac{1}{2} (x_1 + x_2) \quad r = x_1 - x_2 \]

\[ S = \left( \frac{\lambda}{4} \right)^2 \lambda^2 \frac{1}{12^2 (2\pi)^6 \omega \omega'} \]

\[ \int d^4r \int d^4R \ e^{ik(R+\frac{1}{2}r)} e^{-i\omega'(R-\frac{1}{2}r)} \langle 0| T \tilde{\phi}(r) \phi(0) |0 \rangle^2 \]

\[ + \ " r \leftrightarrow -r " \]

Integration \[ \int d^4R \] yields

\[ S = -\frac{\lambda^2}{4} \int f_k f_{k'} (2\pi)^4 \delta^{(4)}(k-k') \]

\[ \cdot \int d^4r \ e^{ikr} \langle 0| T \tilde{\phi}(r) \phi(0) |0 \rangle^2 \]

\[ + \ " r \leftrightarrow -r " \]

Making use of the properties of the Fourier halo (convolution)

\[ \int d^4r \ e^{ikr} \langle h(r) \rangle^2 = \int \frac{d^4k''}{(2\pi)^4} \tilde{h}(k'') \tilde{h}(k-k'') \]

\[ \Rightarrow \]

\[ S = -\frac{\lambda^2}{2} \int f_k^2 \delta^{(4)}(k-k') \cdot (2\pi)^4 \int \frac{d^4k''}{(2\pi)^4} \frac{1}{k''^2 - m^2 + i\epsilon} \frac{1}{(k-k'')^2 - m^2 + i\epsilon} \]
\[ M = i \left( -i \pi \right)^2 \frac{1}{2!} \int \frac{d^4 k'}{(2\pi)^4} \frac{i}{k'^2 - m^2 + i\varepsilon} \frac{i}{(k' - k'' - k''')^2 - m^2 + i\varepsilon} \]

**Rule 5:** Integrate over all 4-momenta which are not fixed by energy-momentum conservation and external lines with weight

\[ \int \frac{d^4 k'}{(2\pi)^4} \]

**Rule 7:** Multiply loop diagrams with n internal lines of identical bosons with a factor \( \frac{1}{n!} \)

**Remark:** Loop diagrams yield in general divergent contributions and are the origin of the mathematical problems of the theory. This will be cured using renormalization. (See later.)