For more complex processes in higher order perturbation theory the calculation of the reduced Matrix element $\mathcal{M}$ is rather complicated! We are in need of

a simple set of rules for the systematic calculation of $\mathcal{M}$

$\Downarrow$

$\boxed{\text{Feynman rules}}$

We will also find a graphical representation of these rules

$\boxed{\text{Feynman diagrams}}$

Feynmandiagrams

relevant example of interaction $\mathcal{L}_{int} = -\mathcal{H}_{int} = -g : \bar{\psi}\psi\phi :$

ψ - Dirac field $\quad \phi$ - scalar field

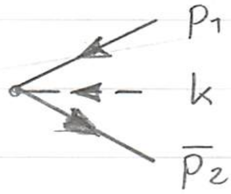
Each operator $\bar{\psi}, \psi, \phi$ either creates or annihilates a particle / anti-particle. $\mathcal{H}_{int}$ is of ϕ^3 structure. Application of $\mathcal{H}_{int}$ with n interacting fields (here $n=3$):

$n-\ell$ particle input $\qquad$ ℓ particle output $(0 \leq \ell \leq n)$

for $\bar{\psi}\psi\phi$

time ←

$\boxed{\ell=0}$



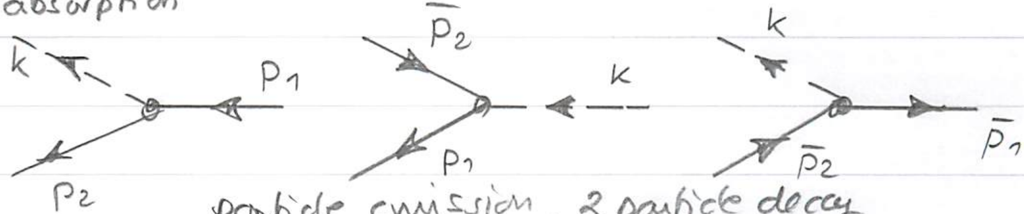
3 particle annihilation

$\boxed{\ell=1}$

particle absorption

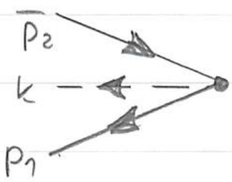
2 particle annihilation

$\boxed{\ell=2}$

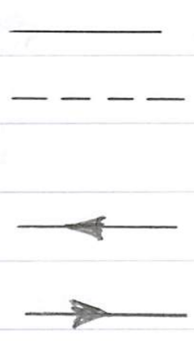


particle emission, 2 particle decay

$\boxed{\ell=3}$



3 particle creation



$\psi, \bar{\psi}$ Dirac field

ϕ scalar field

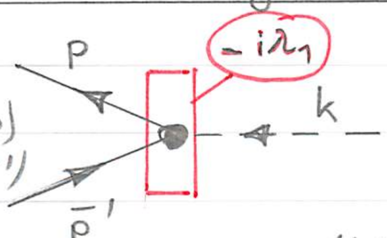
p particle with momentum p

$\bar{p}$ antiparticle with momentum p

example: ϕ^3 theory

(i) <u>relativistic decay in 1st order of λ_1</u>

outgoing particle (p) and anti-particle ($\bar{p}'$)

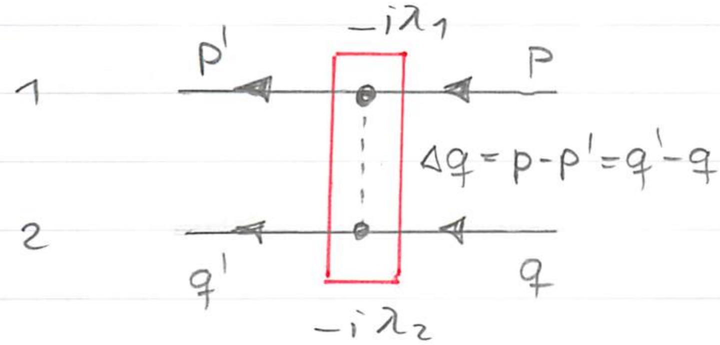


incoming particle of φ with 4-momentum k

$-i\lambda_1$ Vertex   from interaction operator $-i\mathcal{H}_{1I}$

$$\boxed{\mathcal{M} = i(-i\lambda_1) = \lambda_1}$$

(ii) relativistic scattering in 1+1 'st order λ_1, λ_2



 Propagator of field φ with 4-momentum $\Delta q = q' - q = p - p'$

$-i\lambda_1$, $-i\lambda_2$ vertices from interaction operator $-i\mathcal{H}_{1I}$ and $-i\mathcal{H}_{2I}$

$$\boxed{\mathcal{M} = \underbrace{i}_{\text{rule } 0}\underbrace{(-i\lambda_1)(-i\lambda_2)}_{\text{rule } 1}\underbrace{\frac{i}{\Delta q^2 - m_0^2 + i\varepsilon}}_{\text{rule } 2}}$$

(I) Draw all topologically distinct diagrams that correspond to a given scattering process in the considered order of perturbation theory. The order of perturbation corresponds to the number of vertices.

(II) Find the mathematical expression for every diagram and add expressions for all diagrams

Feynman rules: translation between graphs and mathematical expressions for $\mathcal{M}$

<u>rule 0:</u> - a factor of i

<u>rule 1:</u> - a factor of $-i\lambda_k$ for every vertex
at every vertex 4-momentum is conserved

<u>rule 2:</u> - for every <u>internal</u> line a factor corresponding to the propagator of the field with 4-momentum Δq.
Δq is determined from 4-momentum conservation at vertex

Remark: These rules are for ϕ^3 theory. Most of them are generic. Some need modifications, And there will be more

(D) relativistic scattering in 2nd order of λ_1

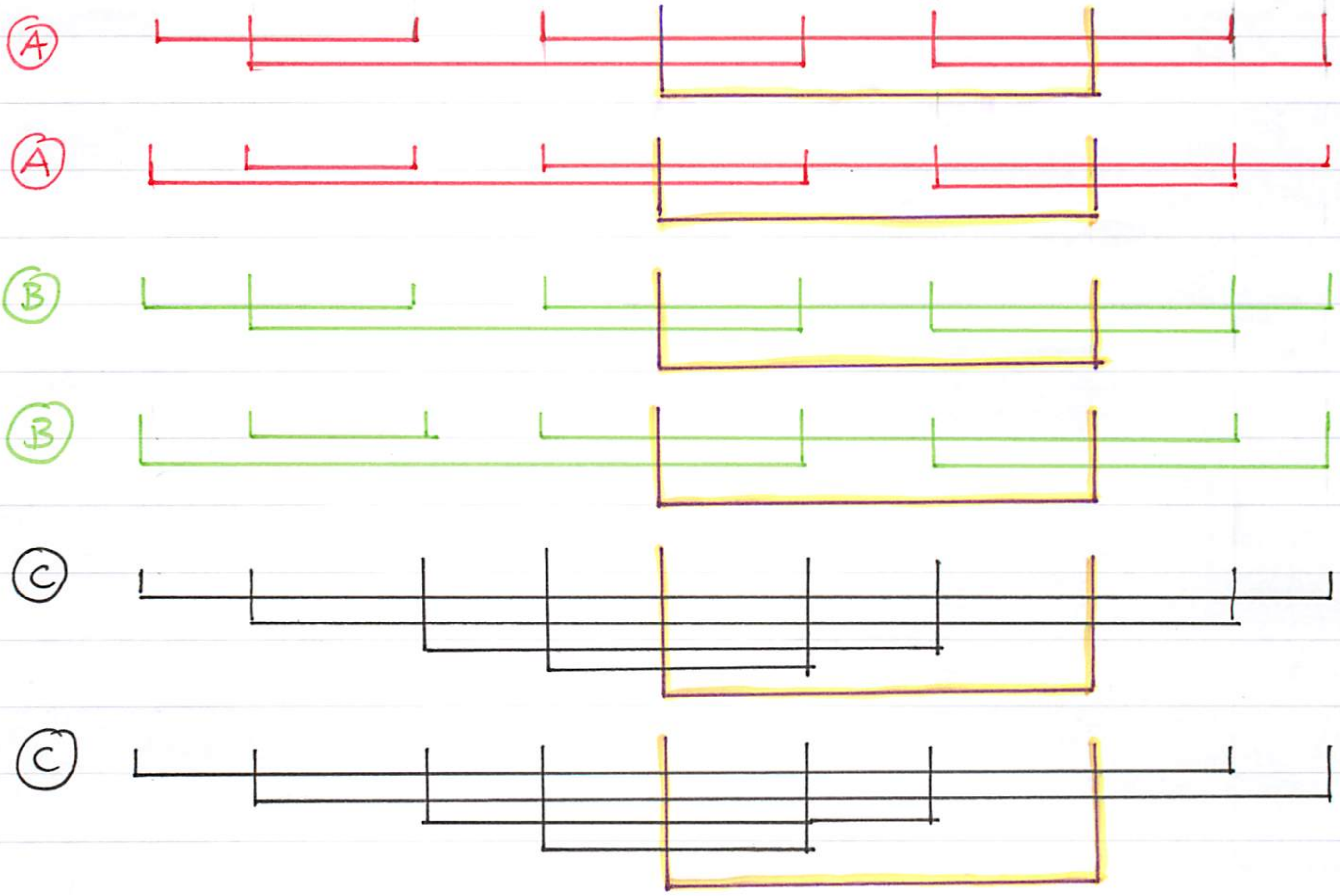
In previous examples incoming and outgoing particles distinguisable. Now: identical particles in incoming and outcoming channels

$$|\Psi_{in}\rangle = \hat{a}^+_{1p}\,\hat{a}^+_{1q}\,|0\rangle$$

$$|\Psi_{out}\rangle = \hat{a}^+_{1p'}\,\hat{a}^+_{1q'}\,|0\rangle$$

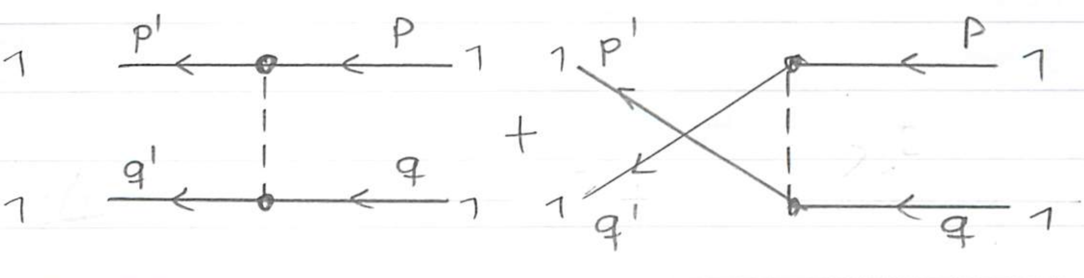
$$S = \frac{(-i)^2}{2} \lambda_1^2 \int d^4x_1 \int d^4x_2 \cdot$$

$$\cdot \langle 0 | \hat{a}_{1p'} \hat{a}_{1q'} \, T\left[: \hat{\Phi}_1^{+}(x_1) \hat{\Phi}_1(x_1) \hat{\varphi}(x_1) :: \hat{\Phi}_1^{+}(x_2) \hat{\Phi}_1(x_2) \hat{\varphi}(x_2) : \right] \hat{a}_{1p}^{+} \hat{a}_{1q}^{+} | 0 \rangle$$



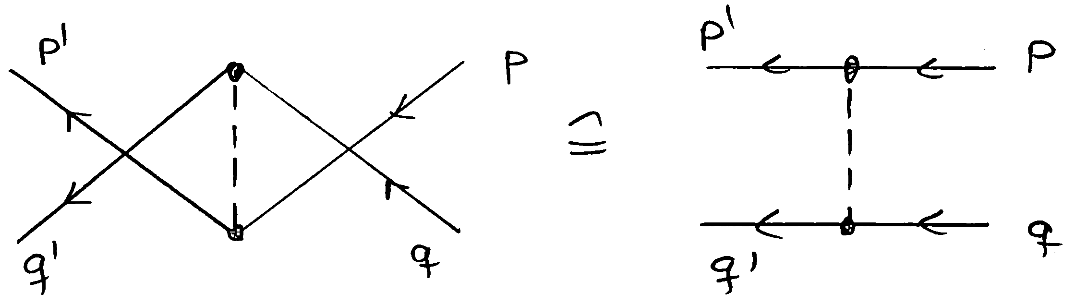
$$\mathcal{M} = \frac{\lambda_1^2}{(p'-p)^2 - m_0^2 + i\varepsilon} + \frac{\lambda_1^2}{(q'-p)^2 - m_0^2 + i\varepsilon}$$

from A from B



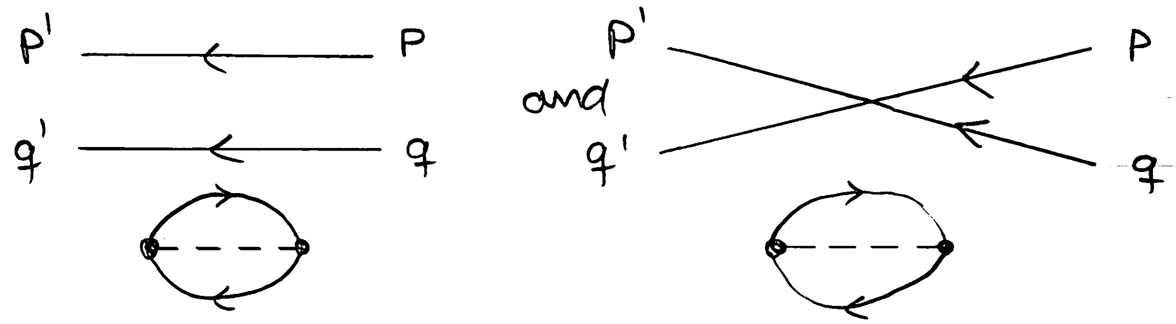
rule 4: symmetrize between identical bosons in the outgoing channels

remarks: alternatively rule 4 could be formulated for incoming identical bosons but not for both



- how about terms from (C) ?

correspond to unconnected diagrams:

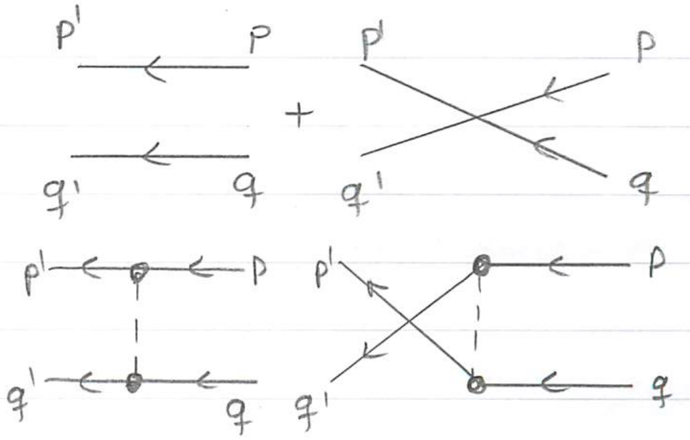


> **connected-graph theorem**
>
> In a consistent perturbation theory of the S-matrix unconnected diagrams do not contribute. Unconnected diagrams are those which do not connect to external lines

idea of a proof: make use of normalization

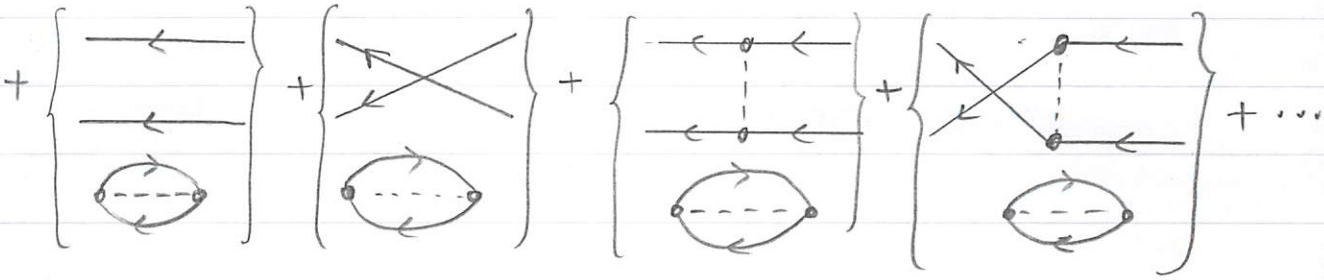
$$S_{\beta\alpha} = \langle\beta|\hat{U}_I(\infty,-\infty)|\alpha\rangle \longrightarrow S_{\beta\alpha} = \frac{\langle\beta|\hat{U}_I(\infty,-\infty)|\alpha\rangle}{\langle 0|\hat{U}_I(\infty,-\infty)|0\rangle}$$

$\langle p' q' | \hat{U}_I | p, q \rangle =$



0'th

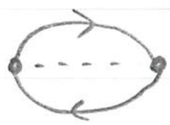
2nd



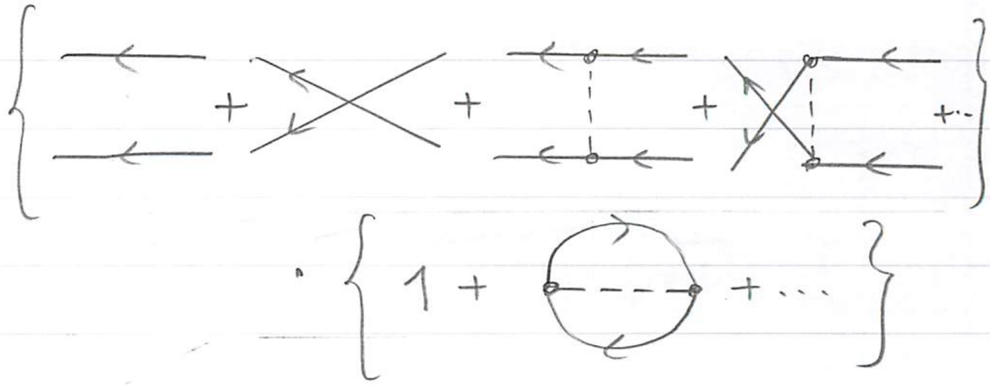
2nd

4th

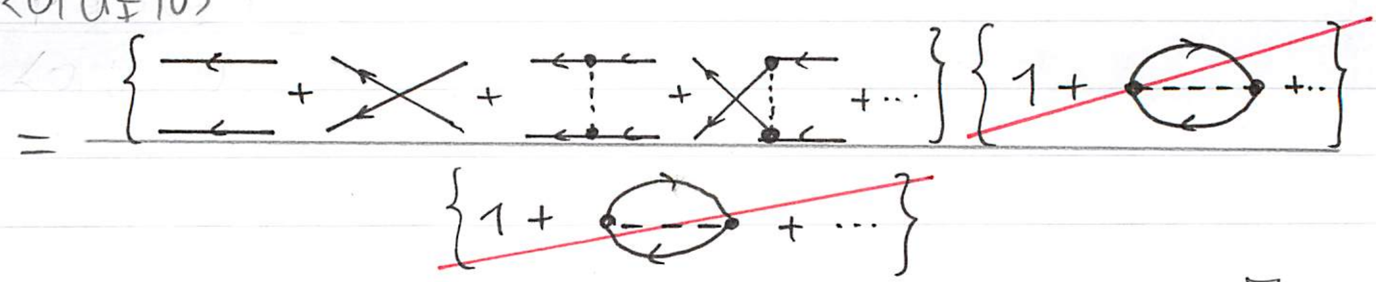
on the other hand

$\langle 0 | \hat{U}_I | 0 \rangle = 1 +$  $+ \cdots$

$\langle p' q' | \hat{U}_I | p q \rangle =$



$\dfrac{\langle p' q' | \hat{U}_I | p q \rangle}{\langle 0 | \hat{U}_I | 0 \rangle} =$



$\square$

(E) loop diagrams in 2nd order of λ

consider incoming neutral particle ($\hat{\varphi}$) and outgoing neutral particle ($\hat{\varphi}$) in second order of λ

$$S = \frac{(-i)^2}{2} \frac{\lambda^2}{(3!)^2} \int d^4x_1 \int d^4x_2 \,\cdot$$

2 possibilities

$$\cdot \langle 0 | \hat{a}_k T[:\hat{\varphi}(x_1)\hat{\varphi}(x_1)\hat{\varphi}(x_1)::\hat{\varphi}(x_2)\hat{\varphi}(x_2)\hat{\varphi}(x_2):] \hat{a}^{+}_{k'} |0\rangle$$

3 possibilities

3 possibilities

3 possibilities

3 possibilities

$$S = \frac{(-i)^2}{2} \frac{\lambda^2}{2} \int d^4x_1 \int d^4x_2 \; \langle 0 | T \, \hat{a}_k(\infty) \hat{\varphi}(x_1) |0\rangle$$

$$\langle 0 | T \hat{\varphi}(x_1) \hat{\varphi}(x_2) |0\rangle^2 \langle 0 | T \hat{\varphi}(x_2) \hat{a}^{+}_{k'}(-\infty) |0\rangle$$

$$+ \;\; \text{"} x_1 \leftrightarrow x_2 \text{"}$$

$\Rightarrow$

$$S = \frac{(-i)^2 \lambda^2}{4} \frac{1}{\sqrt{2^2 E_k E_{k'} (2\pi)^6}} \,\cdot$$

$$\cdot \int d^4x_1 \int d^4x_2 \; e^{-ik'x_2} e^{ikx_1} \langle 0 | T \hat{\varphi}(x_1) \hat{\varphi}(x_2) |0\rangle^2$$

$$+ \;\; \text{"} x_1 \leftrightarrow x_2 \text{"}$$

center of mass and relative coordinates

$$R = \frac{1}{2}(x_1 + x_2) \qquad r = x_1 - x_2$$

$$S = \frac{(-i)^2 \lambda^2}{4} \frac{1}{\sqrt{2^2 (2\pi)^6 E_k E_{k'}}} \cdot$$

$$\cdot \int d^4r \int d^4R \, e^{ik(R+\frac{1}{2}r)} e^{-ik'(R-\frac{1}{2}r)} \left(\langle 0 | T \hat{\varphi}(r) \hat{\varphi}(0) | 0 \rangle \right)^2$$

$$+ \text{"} r \leftrightarrow -r \text{"}$$

integration $\int d^4R$ yields

$$S = -\frac{\lambda^2}{4} f_k f_{k'} (2\pi)^4 \delta^{(4)}(k-k') \cdot$$

$$\cdot \int d^4r \, e^{ikr} \left(\langle 0 | T \hat{\varphi}(r) \hat{\varphi}(0) | 0 \rangle \right)^2$$

$$+ \text{"} r \leftrightarrow -r \text{"}$$

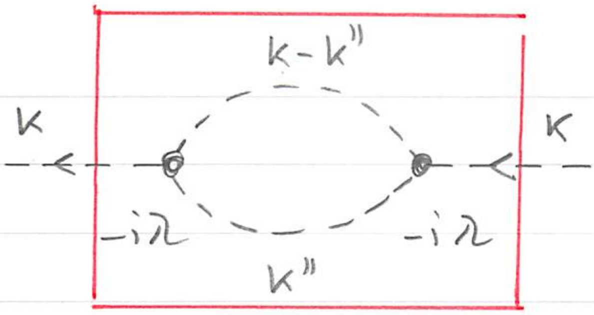
making use of the properties of the Fourier trafo (convolution)

$$\int d^4r \, e^{ikr} \left(h(r) \right)^2 = \int \frac{d^4k''}{(2\pi)^4} \tilde{h}(k'') \tilde{h}(k-k'')$$

$\Rightarrow$

$$S = -\frac{\lambda^2}{2} f_k^2 \, \delta^{(4)}(k-k') \cdot$$

$$\cdot (2\pi)^4 \int \frac{d^4k''}{(2\pi)^4} \frac{1}{k''^2 - m_0^2 + i\varepsilon} \frac{1}{(k-k'')^2 - m_0^2 + i\varepsilon}$$



loop diagram

$$\mathcal{M} = i(-i\lambda)^2 \frac{1}{2!} \int \frac{d^4k'}{(2\pi)^4} \frac{i}{k'^2 - m_0^2 + i\varepsilon} \cdot \frac{i}{(k-k')^2 - m_0^2 + i\varepsilon}$$

rule 7 rule 5

rule 5: integrate over all 4-momenta which are not fixed by energy-momentum conservation and external lines with weight

$$\int \frac{d^4k'}{(2\pi)^4}$$

rule 7: multiply loop diagrams with n internal lines of identical bosons with a factor

$$1/n!$$

remark: loop diagrams yield in general divergent contributions and are the origin of the mathematical problems of the theory. This will be cured using renormalization. (see later)