V.2 Contractions and propagators of field eqs.

We will now see that the contractions resulting from the application of Wick's theorem have a direct physical interpretation

(A) elm. field in Coulomb gauge

fundamental fields (classical)

$$\vec{A}_\perp(\vec{r},t) \qquad \vec{E}_\perp(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}_\perp(\vec{r},t)$$

Euler Lagrange equations = field equations in the presence of an external (prescribed) current

$$\Box \vec{A}_\perp(\vec{r},t) = \vec{j}_\perp(\vec{r},t) = \int d^3r' \, \vec{j}(\vec{r}',t)\, \overleftrightarrow{\delta}_\perp^{(3)}(\vec{r}-\vec{r}')$$

The Greens function or Propagator of this equation can be defined as

$$\boxed{\Box D_c^{kl}(\vec{r},t;\vec{r}',t') = -\delta_\perp^{kl}(\vec{r}-\vec{r}')\,\delta(t-t')}$$

We will now proof that

$$\boxed{D_c^{kl}(\vec{r},t;\vec{r}',t') = -i\langle 0|T\hat{A}_\perp^k(\vec{r},t)\hat{A}_\perp^l(\vec{r}',t')|0\rangle}$$

where $\hat{\vec{A}}_\perp$ is the free transverse vector potential

$$\partial_t \langle 0| T \hat{A}_\perp^k(\vec r,t)\, \hat{A}_\perp^\ell(\vec r\,',t')|0\rangle =$$

notation "$A_\perp \to A$"

$$= \partial_t \Big\{ \theta(t-t')\langle 0|\hat{A}^k(\vec r,t)\hat{A}^\ell(\vec r\,',t')|0\rangle + \theta(t'-t)\langle 0|\hat{A}^\ell(\vec r\,',t')\hat{A}^k(\vec r,t)|0\rangle \Big\}$$

$$= \underbrace{\delta(t-t')\langle 0|[\hat{A}^k(\vec r,t),\hat{A}^\ell(\vec r\,',t')]|0\rangle}_{=0}$$

$$+\theta(t-t')\langle 0|\partial_t \hat{A}^k(\vec r,t)\hat{A}^\ell(\vec r\,',t')|0\rangle + \theta(t'-t)\langle 0|\hat{A}^\ell(\vec r\,',t')\partial_t\hat{A}^k(\vec r,t)|0\rangle$$

$\Rightarrow$

$$\partial_t^2 \langle 0|T A^k A^\ell|0\rangle = \delta(t-t')\langle 0|[\underbrace{\partial_t \hat{A}^k(\vec r,t)}_{-\hat{E}^k},\hat{A}^\ell(\vec r\,',t')]|0\rangle$$

$$+\theta(t-t')\langle 0|\partial_t^2 \hat{A}^k\cdot\hat{A}^\ell|0\rangle + \theta(t'-t)\langle 0|\hat{A}^\ell\cdot\partial_t^2\hat{A}^k|0\rangle$$

$$= -i\,\delta(t-t')\,\delta_\perp^{\ell k}(\vec r-\vec r\,') +$$

$$+\theta(t-t')\langle 0|\Delta\hat{A}^k\hat{A}^\ell|0\rangle + \theta(t'-t)\langle 0|A^\ell\Delta A^k|0\rangle$$

$\Rightarrow$

$$\boxed{\begin{aligned}&\Box_x \langle 0|T\hat{A}^k(x)\hat{A}^\ell(x')|0\rangle = \\ &= (\partial_t^2-\Delta_{\vec r})\langle 0|T\hat{A}^k(x)\hat{A}^\ell(x')|0\rangle = -i\,\delta_\perp^{k\ell}(\vec r-\vec r\,')\,\delta(t-t')\end{aligned}}$$

$\Box$

In the ∞ extended space, i.e. without boundary conditions one recognises that

$$\boxed{D_c^{ke}(\vec{r},t;\vec{r}',t') = D_c^{ke}(\vec{r}-\vec{r}',t-t')}$$

In the future we will often use the Fourier representation instead of one in realspace and time

Def: $X(k) = X(k_0,\vec{k}) = X(\omega,\vec{k}) =$

$$\equiv \int_{V_\infty} d^3r \int_{-\infty}^{\infty} dt \; e^{ik_\mu x^\mu} X(\vec{r},t)$$

where $k_\mu x^\mu = \omega t - \vec{k}\cdot\vec{r}$

In F-space we find

$$-(\omega^2 - \vec{k}^2)\, D_c^{ke}(\omega,\vec{k}) = -\int d^3r\, e^{-i\vec{k}\vec{r}}\, \delta_\perp^{ke}(\vec{r})$$

We had

$$\delta_\perp^{ke}(\vec{r}) = \delta_{ke}\,\delta(\vec{r}) - \delta_\parallel^{ke}(\vec{r})$$

$$= \delta_{ke}\,\delta(\vec{r}) - \frac{1}{4\pi}\left(\nabla\circ\nabla\frac{1}{|\vec{r}|}\right)^{ke}$$

F-Trafo

$$\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \delta_\perp^{ke}(\vec{r}) = \delta_{ke} - \frac{k_k k_e}{|\vec{k}|^2}$$

$$\boxed{D_C^{ij}(k) = D_{C,ij}(k) = \frac{\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}}{\omega^2 - \vec{k}^2 + i\varepsilon}}$$

Here "$+i\varepsilon$" was introduced to move the pole at $\omega = |\vec{k}|$ into the complex plane. The positive sign corresponds to the choice of the retarded GF.

(B) elm. field in Lorentz gauge

If one wants to keep explicit Lorentz invariance in the formulation of QED one needs to add a gauge fixing term to the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} \partial_\mu A^\mu \partial_\nu A^\nu \qquad (*)$$

then

$$\pi^0 = -\frac{1}{\alpha} \partial_\mu A^\mu$$

The Lorentz gauge is then realized by restricting the allowed quantum states such that

$$\partial_\mu \hat{A}^\mu |\psi\rangle = 0 \qquad \text{Gupta-Bleuler}$$

The Euler-Lagrange equations following from (*) are

$$\Box A^\nu - \left(1 - \frac{1}{\alpha}\right) \partial^\nu \partial_\mu A^\mu = 0$$

For the choice $\alpha = 1$ (Fermi- or Feynmann gauge) the equations of motion in an external (prescribed) field are

$$\Box A^\nu = j^\nu$$

fundamental commutation relations of quantized fields:

$$\boxed{\left[\hat{A}^\mu(\vec{r},t), \hat{\pi}^\nu(\vec{r}',t)\right] = i g^{\mu\nu} \delta(\vec{r}-\vec{r}')}$$

(i.e. for $\mu=0, \nu=0$ the sign is reversed)

Following the same procedure as in (A) one finds that

$$\boxed{D_{\mu\nu}(x-x') = -i\langle 0| T \hat{A}_\mu(x) \hat{A}_\nu(x') |0\rangle}$$

is Greensfunction (or propagator) of the wave equation

$$\boxed{\Box_x D_{\mu\nu}(x-x') = g_{\mu\nu} \delta^{(4)}(x-x')}$$

Fourier trafo with respect to $x = (t, \vec{r}) \to k = (\omega, \vec{k})$ yields

$$\boxed{D^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{\omega^2 - \vec{k}^2 + i\varepsilon} = \frac{g^{\mu\nu}}{-k_\mu k^\mu - i\varepsilon}}$$

(c) Dirac field

Now let's consider the Dirac equation in an external elm. field

$$(i\gamma^\mu \partial_\mu - m)\psi = -e\gamma^\mu A_\mu \psi$$

We define the Greensfunction (propagator) as the 4×4 matrix $S(x,x')$

$$\boxed{(i\gamma^\mu \partial_\mu - m)_x \, S(x,x') = \mathbb{1}_{4\times 4}\, \delta(x-x')}$$

Fourier-trafo yields (noting that $S(x,x') = S(x-x')$)

$$(\gamma^\mu k_\mu - m)\, S(k) = \mathbb{1}_{4\times 4}$$

$$\Rightarrow \quad \boxed{S(k) = \frac{1}{\gamma^\mu k_\mu - m + i\varepsilon} = \frac{1}{\not{k} - m + i\varepsilon}}$$

where a term "$+i\varepsilon$" was added to move the pole into the complex plane

Notation $\quad \not{a} \equiv \gamma^\mu a_\mu \quad$ "Feynman dagger"

In the problem courses we will prove that

$$(*) \quad \boxed{S(x,x') = -i\langle 0|\, T\, \hat{\psi}(x)\, \hat{\bar{\psi}}(x')\, |0\rangle}$$

Note that S is a 4×4 matrix and (*) has to we read as

$$S_{\mu\nu} = -i\langle 0|T\,\hat{\psi}_\mu(x)\,\hat{\bar{\psi}}_\nu(x')|0\rangle$$

In order to see the matrix structure more clearly it is convenient to use the following equivalence

$$\boxed{S(k) = \frac{1}{\not{k}-m+i\varepsilon} = \frac{\not{k}+m}{k_\mu k^\mu - m^2 + i\varepsilon}}$$

(keeping only leading terms in "$i\varepsilon$")

note that $\quad (\not{k}+m)_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\,k_\mu + m\,\delta_{\alpha\beta}$

V.3 Diagrammatic perturbation theory: an introduction for ϕ^3-theory

We will now introduce a systematic approach to calculate S-matrix elements in perturbation theory. For this we consider the so-called ϕ^3 theory of a neutral and charged Klein-Gordon field:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{int}$$

$$\mathcal{L}_0 = :\frac{1}{2}\frac{\partial\hat{\varphi}}{\partial x^\mu}\frac{\partial\hat{\varphi}}{\partial x^\mu} - m_0^2\hat{\varphi}^2: \qquad \text{neutral KG field } \hat{\varphi} = \hat{\varphi}^+$$

$$\mathcal{L}_{1,2} = :\frac{\partial \hat{\Phi}_i^+}{\partial x^\mu}\frac{\partial \hat{\Phi}_i}{\partial x^\mu} - m_i^2 \hat{\Phi}_i^+ \hat{\Phi}_i: \quad \text{charged KG field}$$

interaction

$$\mathcal{L}_{int} = -\lambda_1 :\hat{\Phi}_1^+(x)\hat{\Phi}_1(x)\hat{\varphi}(x): - \lambda_2 :\hat{\Phi}_2^+(x)\hat{\Phi}_2(x)\hat{\varphi}(x):$$

$$- \frac{\lambda}{3!} :\hat{\varphi}^3(x): \quad = - \mathcal{H}_{int}$$

We here have used normal order to have $\langle 0|\mathcal{H}_{int}|0\rangle = 0$

Mode decompositions of fields:

$$\hat{\Phi}_i(x) = \sum_p \left(\hat{d}_{ip}\,\Phi_{ip}^{(+)}(x) + \hat{c}_{ip}^+\,\Phi_{i-p}^{(-)}(x)\right)$$

$$\hat{\varphi}(x) = \sum_k \left(\hat{a}_k\,\varphi_k^{(+)}(x) + \hat{a}_k^+\,\varphi_{-k}^{(-)}(x)\right)$$

K.-G. wave functions:

$$\varphi_k^{(\pm)}(x) = \frac{1}{\sqrt{2E_k L^3}}\, e^{i(\vec{k}_n \vec{r} \mp E_n t)}$$

similarly $\Phi_{ip}^{(\pm)}(x)$

time evolution operator in interaction picture:

$$\hat{U}_I = 1 - i\int d^4x\, \hat{\mathcal{H}}_I(x) + \frac{(-i)^2}{2!}\int d^4x_1 \int d^4x_2\, T\left[\hat{\mathcal{H}}_I(x)\,\hat{\mathcal{H}}_I(x')\right] + \cdots$$

(A) relativistic decay

consider the following problem:

neutral particle with 4-momentum k $\qquad |k\rangle = \hat{a}_k^+ |0\rangle$

$$\downarrow$$

particle-antiparticle pair of type 1 with 4-momenta $p, \bar{p}'$ $\qquad |p, \bar{p}'\rangle = \hat{d}_{1p}^+ \hat{c}_{1p'}^+ |0\rangle$

overbar to indicate antiparticle

$$S = \langle p \bar{p}' | \hat{U} | k \rangle = ?$$

There is a contribution in 1st order perturbation in λ_1

$$S = -i \langle 0 | \hat{d}_{1p} \hat{c}_{1p'} \int d^4x \, \lambda_1 : \hat{\Phi}_1^+(x) \hat{\Phi}_1(x) \hat{\varphi}(x) : \hat{a}_k^+ |0\rangle$$

this can be written as a time-ordered product with

$$\hat{d}_{1p} \rightarrow \hat{d}_{1p}(+\infty) \qquad \hat{a}_k^+ \rightarrow \hat{a}_k^+(-\infty)$$

$$S = -i \int d^4x \, \lambda_1 \, \langle 0 | T \left[\hat{d}_{1p}(\infty) \hat{c}_{1p'}(\infty) : \hat{\Phi}_1^+(x) \hat{\Phi}_1(x) \hat{\varphi}(x) : \hat{a}_k^+(-\infty) \right] |0\rangle$$

non-vanishing contractions!

the fields are bose fields $\Rightarrow$

$$S = -i\lambda_1 \int d^4x \, \langle 0| T\left[\hat{d}_{1p}(\infty)\hat{\Phi}_1^+(x)\right]|0\rangle$$

$$\cdot \langle 0| T\left[\hat{c}_{1p'}(\infty)\hat{\Phi}_1(x)\right]|0\rangle$$

$$\cdot \langle 0| T\left[\hat{\varphi}(x)\hat{a}_k^+(-\infty)\right]|0\rangle$$

(all contractions contain operators at $t = \pm\infty$)

now

$$\langle 0| T\left[\hat{d}_{1p}(\infty)\hat{\Phi}_1^+(x)\right]|0\rangle = \langle 0|\hat{d}_{1p}(\infty)\hat{\Phi}_1^+(x)|0\rangle$$

$$= \Phi_{1p}^{(+)*}(x) = \frac{1}{\sqrt{2E_pL^3}}\, e^{ipx} \qquad (px = E_pt - \vec{p}\cdot\vec{r})$$

$$\langle 0| T\left[\hat{c}_{1p'}(\infty)\hat{\Phi}_1(x)\right]|0\rangle = \langle 0|\hat{c}_{1p'}(\infty)\hat{\Phi}_1(x)|0\rangle$$

$$= \Phi_{1-p'}^{(-)}(x) = \frac{1}{\sqrt{2E_{p'}L^3}}\, e^{ip'x}$$

$$\langle 0| T\left[\hat{\varphi}(x)\hat{a}_k^+(-\infty)\right]|0\rangle = \langle 0|\hat{\varphi}(x)\hat{a}_k^+(-\infty)|0\rangle$$

$$= \varphi_k^{(+)}(x) = \frac{1}{\sqrt{2E_kL^3}}\, e^{-ikx}$$

$$\Rightarrow \quad S = -i\lambda_1 \int d^4x \; \Phi_{1p}^{(+)*}(x)\,\Phi_{1-p'}^{(-)}(x)\,\varphi_k^{(+)}(x)$$

$$= -i\lambda_1 \int d^4x \; \frac{e^{i(p+p'-k)x}}{\sqrt{2^3E_pE_{p'}E_kL^9}}$$

in the continuum limit

$$\boxed{S = -i\,\underline{(2\pi)^4\,\delta^{(4)}(p+p'-k)}\,\prod_j \underline{f_j}\;\underline{\mathcal{M}}} \qquad (*)$$

- energy-momentum conservation between incoming and outgoing plane wave states

- $$\boxed{f_j = \frac{1}{\sqrt{(2\pi)^3\,2E_j}}}$$ amplitude factor for incoming and outgoing waves

- $\boxed{\mathcal{M}}$ reduced scattering matrix element in the given order of perturbation theory

  here $\mathcal{M} = \lambda_1$

(*) is the general form of the S-matrix for decay processes !

(B) relativistic scattering in 1+1 order perturbation

input: charged particle of type 1 p

+ — " — of type 2 q

$|pq\rangle = d^{\dagger}_{1p}\,d^{\dagger}_{2q}\,|0\rangle$

$\downarrow$

charged particle of type 1 p'

+ — " — of type 2 q'

$|p'q'\rangle = d^{\dagger}_{1p'}\,d^{\dagger}_{2q'}\,|0\rangle$

need 2nd order of $\mathcal{H}_I$

there are 2 terms $(x_1 \leftrightarrow x_2)$

$$S = \frac{(-i)^2}{2} \, 2 \lambda_1 \lambda_2 \int d^4x_1 \int d^4x_2$$

$$\langle 0 | \hat{a}_{1p'} \hat{a}_{2q'} T\left[:\hat{\Phi}_1^+(x_1) \hat{\Phi}_1(x_1) \hat{\varphi}(x_1): :\hat{\Phi}_2^+(x_2) \hat{\Phi}_2(x_2) \hat{\varphi}(x_2): \right] \hat{a}_{1p}^+ \hat{a}_{2q}^+ | 0 \rangle$$

non-vanishing contractions

$$\Rightarrow \quad S = -\lambda_1 \lambda_2 \int d^4x_1 \int d^4x_2 \; \Phi_{1p'}^{(+)*}(x_1) \, \Phi_{1p}^{(+)}(x_1) \; \Phi_{2q'}^{(+)*}(x_2) \, \Phi_{2q}^{(+)}(x_2)$$

$$\cdot \langle 0 | T[\hat{\varphi}(x_1) \hat{\varphi}(x_2)] | 0 \rangle$$

relative and center-of-mass coordinates:

$$R \equiv \frac{1}{2}(x_1 + x_2) \qquad r = (x_1 - x_2) \qquad x = (t, \vec{r})$$

$$\int d^4x_1 \int d^4x_2 \longrightarrow \int d^4R \int d^4r$$

due to the translational invariance of the free KG-equation

$$\langle 0 | T[\hat{\varphi}(x_1) \hat{\varphi}(x_2)] | 0 \rangle = \langle 0 | T[\hat{\varphi}(r) \hat{\varphi}(0)] | 0 \rangle$$

with this we find

$$S = -\lambda_1 \lambda_2 \int d^4R \int d^4r \; \frac{e^{i\{(p'-p)(R+\frac{1}{2}r) + (q'-q)(R-\frac{1}{2}r)\}}}{\sqrt{2^4 (2\pi)^{3\cdot 4} \, E_p E_{p'} E_q E_{q'}}}$$

$$\cdot \langle 0 | T[\hat{\varphi}(r) \hat{\varphi}(0)] | 0 \rangle$$

R integration can be carried out:

$$S = -\lambda_1\lambda_2 \frac{(2\pi)^4 \delta^{(4)}(p'+q'-p-q)}{\sqrt{(2(2\pi)^3)^4 E_p E_q E_{p'} E_{q'}}}$$

$$\cdot \int d^3r\, e^{i(q'-q)r} \langle 0| T[\tilde{\varphi}(r)\tilde{\varphi}(0)] |0\rangle$$

this has a similar form as for the decay

$$\boxed{S = -i\,(2\pi)^4 \delta^{(4)}(p'+q'-p-q) \prod_j f_j\, \mathcal{M}}$$

now with $\Delta q r = \Delta q_\mu r^\mu$

$$\boxed{\mathcal{M} = -i\lambda_1\lambda_2 \int d^4r\, e^{i\Delta q r} \langle 0| T[\tilde{\varphi}(r)\tilde{\varphi}(0)] |0\rangle}$$

$$\Delta q = q' - q \underset{\uparrow}{=} p - p'$$ 4-momentum transfer

4-momentum conservation

- energy momentum conservation between incoming and outgoing particles
- $f_j = \frac{1}{\sqrt{2(2\pi)^3 E_j}}$ amplitude factors for all incoming and outgoing waves
- reduced scattering matrix element

In the problem courses it will be shown that

$$-i \int d^4x \, e^{i\Delta q x} \langle 0| T[\hat{\varphi}(x) \hat{\varphi}(0)] |0\rangle =$$

$$= \frac{1}{\Delta q^2 - m_0^2 + i\varepsilon}$$

$\Rightarrow$

$$\mathcal{M} = \frac{-\lambda_1 \lambda_2}{m_0^2 - (q'-q)^2 - i\varepsilon}$$

(C) Feynman diagrams: introduction

We have seen in (A) and (B) important simple examples for the calculation of S-matrix elements where incoming and outgoing states are plane-waves i.e. eigenstates of the 4-momentum.

S-Matrix elements for relativistic decay / scattering with plane-wave, free particle asymptotic states:

$$S = -i\,(2\pi)^4 \delta^{(4)}\Big(\sum_{in} p - \sum_{out} p'\Big) \prod_j f_j \,\mathcal{M}$$

$(2\pi)^4 \delta^{(4)}\Big(\sum_{in} p - \sum_{out} p'\Big)$ energy-momentum conservation between incoming and outgoing states

$f_j = \frac{1}{\sqrt{(2\pi)^3 2E_j}}$ amplitude factors of all incoming and outgoing waves

$\mathcal{M}$ reduced S-matrix element