V. QED with interactions: perturbation theory

In the following we will consider Quantumelectrodynamics as the most important (for the purpose of this lecture) example of an interacting quantum field theory. In general wee need at least two sorts of fermions electrons and protons. In fully covariant notation the Lagrangian reads

$$\mathcal{L}_{QED} = \mathcal{L}_e + \mathcal{L}_p - \frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} - \hat{j}^{\mu}\hat{A}_{\mu}$$

where

$$\mathcal{L}_e = \frac{1}{2}\,\hat{\bar{\psi}}_e\left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m_e\right)\hat{\psi}_e$$

$$\mathcal{L}_p = \frac{1}{2}\,\hat{\bar{\psi}}_p\left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m_p\right)\hat{\psi}_p$$

$$\hat{j}^\mu = e\left(\hat{\bar{\psi}}_p\gamma^\mu\hat{\psi}_p - \hat{\bar{\psi}}_e\gamma^\mu\hat{\psi}_e\right)$$

In Coulomb-gauge we eliminate A^0 and find

$$\mathcal{L}_{QED} = \mathcal{L}_e + \mathcal{L}_p + \frac{1}{2}\left(\hat{\vec{E}}_\perp^2 - \hat{\vec{B}}^2\right) + \vec{j}\cdot\hat{\vec{A}}_\perp$$
$$- \frac{1}{2}\int\frac{d^3r'}{4\pi}\;\frac{\rho(\vec{r},t)\cdot\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}$$

This then gives the Hamiltonian where the Coulomb self-interaction terms were dropped

$$\boxed{\hat{H}_{QED} = \hat{H}^0_e + \hat{H}^0_p + \hat{H}^0_{elm} + \frac{1}{4\pi}\int d^3r \int d^3r' \frac{\hat{J}^0_p(\vec{r},t)\hat{J}^0_e(\vec{r}',t)}{|\vec{r}-\vec{r}'|} - \int d^3r \left(\hat{\vec{J}}_p(\vec{r},t) + \hat{\vec{J}}_e(\vec{r},t)\right)\cdot\vec{A}_\perp(\vec{r},t)}$$

$J^0_p = \rho_p \qquad J^0_e = \rho_e$

$$\boxed{\hat{H}^0_e = \int d^3r \, \hat{\psi}^+_e(\vec{r},t)\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m_e\right]\hat{\psi}_e(\vec{r},t)}$$

and similarly for H^0_p

$$\boxed{\hat{H}^0_{elm} = \frac{1}{2}\int d^3r \left[\hat{\vec{E}}^2_\perp(\vec{r},t) + \vec{B}^2(\vec{r},t)\right]}$$

While $\hat{H}^0_e$, $\hat{H}^0_p$ and $\hat{H}^0_{elm}$ are all bi-linear in field operators the interaction terms

$$\boxed{H_{Coulomb} = \frac{1}{4\pi}\int d^3r \int d^3r' \frac{\hat{J}^0_p(\vec{r},t)\hat{J}^0_e(\vec{r}',t)}{|\vec{r}-\vec{r}'|}}$$

$$\boxed{H_{rad} = -\int d^3r \left(\vec{J}_p(\vec{r},t) + \vec{J}_e(\vec{r},t)\right)\vec{A}(\vec{r},t)}$$

no longer are and thus do not allow a general analytic treatment. Here we need a systematic approximation method to calculate S-matrix elements!

IV.1 Wick theorem

In the following we want to calculate transition probability amplitudes perturbatively, i.e.

$$\langle\beta|\hat{U}_I(\infty,-\infty)|\alpha\rangle = \langle\beta|T\exp\left\{-i\int_{-\infty}^{\infty}d\tau\,\hat{H}_I(\tau)\right\}|\alpha\rangle$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}d\tau_1\cdots\int_{-\infty}^{\infty}d\tau_n\,(-i)^n\langle\beta|T\hat{H}_I(\tau_1)\hat{H}_I(\tau_2)\cdots\hat{H}_I(\tau_n)|\alpha\rangle$$

Thus we need to calculate terms like

$$\langle\beta|T\hat{H}_I(\tau_1)\cdots\hat{H}_I(\tau_n)|\alpha\rangle$$

We will mostly be concerned about scattering processes where $|\alpha\rangle$ and $|\beta\rangle$ are incoming or outgoing plane waves, i.e.

$$|\alpha\rangle = \hat{a}^+_{\vec{q}}|0\rangle \qquad \text{or generalizations}$$

This means it is sufficient to calculate vacuum expectation values of the type

$$\langle 0|T\hat{a}_{\vec{q}'}(\infty)\hat{H}_I(\tau_1)\cdots\hat{H}_I(\tau_n)\hat{a}^+_{\vec{q}}(-\infty)|0\rangle$$

As typically $\hat{H}_I \sim \hat{\psi}^+\hat{A}\hat{\psi}$ we need to find an efficient way to calculate expressions like

$$\langle 0|T\hat{a}(\tau_1)\hat{a}(\tau_2)\hat{a}^+(\tau_2)\cdots|0\rangle = ?$$

For this we can use the Wick theorem

<u>Def:</u> time ordered product of two field operators ($x_0 = t_0$)

$$T\left[\hat{\phi}(x)\,\hat{\phi}^{(+)}(y)\right] \equiv \begin{cases} \hat{\phi}(x)\,\hat{\phi}^{(+)}(y) & x_0 > y_0 \\ \eta\,\hat{\phi}^{(+)}(y)\,\hat{\phi}(x) & x_0 < y_0 \end{cases}$$

$\eta = +1$ Bose $\qquad \eta = -1$ Fermi

<u>Def:</u> time-ordered product of many field operators

$$T\left[\hat{A}(x)\hat{B}(y)\hat{C}(z)\right] = \begin{cases} \hat{A}(x)\,\hat{B}(y)\,\hat{C}(z) & x_0 > y_0 > z_0 \\ \eta\,\hat{A}(x)\,\hat{C}(z)\,\hat{B}(y) & x_0 > z_0 > y_0 \\ \eta'\,\hat{B}(y)\,\hat{A}(x)\,\hat{C}(z) & y_0 > x_0 > z_0 \\ \text{etc.} & \end{cases}$$

$\eta, \eta' = (-1)^P$ $\quad$ P : number of permutations between two fermionic operators required to reach time-ordered configuration

<u>Def:</u> normal ordered product of two field operators

$$:\hat{\phi}(x)\,\hat{\phi}^{(+)}(y): \;\equiv\; \hat{\phi}(x)\hat{\phi}^{(+)}(y) - \langle 0|\,\hat{\phi}(x)\hat{\phi}^{(+)}(y)\,|0\rangle$$

Bose fields:

$$:\hat{a}\hat{a}^{+}: = \hat{a}\hat{a}^{+} - \langle 0|\hat{a}\hat{a}^{+}|0\rangle = \hat{a}\hat{a}^{+} - 1 = \hat{a}^{+}\hat{a}$$

Fermi fields

$$:\hat{c}\hat{c}^{+}: = \hat{c}\hat{c}^{+} - \langle 0|\hat{c}\hat{c}^{+}|0\rangle = -\hat{c}^{+}\hat{c} + 1 - 1 = -\hat{c}^{+}\hat{c}$$

Normal ordering means commuting creation operators to the left and annihilation operators to the right

<u>Def:</u> normal ordering of products of multiple creation and annihilation operators:

permute all creation operators to the left and all annihilation operators to the right

prefactor $(-1)^p$ $\quad p =$ number of times fermionic operators need to be exchanged to reach normally ordered expression

<u>example:</u>
$$:\hat a \hat a^+ \hat a^+ \hat a \hat a: \; = \hat a^{+2} \hat a^3$$

$$:\hat c_1 \hat c_1^+ \hat c_2^+: \; = \hat c_1^+ \hat c_2^+ \hat c_1$$
$$= -\hat c_2^+ \hat c_1^+ \hat c_1$$

<u>Def:</u> contraction of two operators which are linear in creation and annihilation operators

$$\overbracket{\hat A(x) \hat B(y)} \equiv \langle 0 | T \hat A(x) \hat B(y) | 0 \rangle$$

let
$$\hat\phi(x) = \hat\phi^{(-)}(x) + \hat\phi^{(+)}(x)$$

contains only creation operators

contains only annihilation operators

$\Rightarrow$ obviously

$$\overbracket{\hat{\phi}^{(+)}\hat{\phi}^{(+)}} = \overbracket{\hat{\phi}^{(-)}\hat{\phi}^{(-)}} = 0$$

$$\overbracket{\hat{\phi}^{(\pm)}\hat{\psi}^{(\pm)}} = \overbracket{\hat{\phi}^{(\mp)}\hat{\psi}^{(\pm)}} = 0$$

Wick theorem

The time-ordered product (T) of operators linear in creation and annihilation operators is the same than the sum of all normal ordered products with all possible pairwise contractions including the term without contractions. In addition each term gets a prefactor ± 1 depending on the number of commutations of fermionic operators needed in normal ordering.

remark: in many-body theory there are other, more general forms of a Wick theorem

i.e. $$T(\hat{\phi}_1\hat{\phi}_2 \cdots \hat{\phi}_n) = :\hat{\phi}_1 \cdots \hat{\phi}_n:$$

$$+ :\overbracket{\hat{\phi}_1\hat{\phi}_2}\hat{\phi}_3 \cdots \hat{\phi}_n: \pm :\overbracket{\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3}\hat{\phi}_4 \cdots \hat{\phi}_n: \pm \cdots$$

$$\pm :\overbracket{\hat{\phi}_1\hat{\phi}_2}\,\overbracket{\hat{\phi}_3\hat{\phi}_4}\hat{\phi}_5 \cdots \hat{\phi}_n: \pm \cdots$$

idea of a proof:

(i) fix times and put operators in time order

(ii) replace successively neighboring operators according to
$$\hat{\phi}_1\hat{\phi}_2 = :\hat{\phi}_1\hat{\phi}_2: - \langle 0|\hat{\phi}_1\hat{\phi}_2|0\rangle$$

Since the operators were initially put into time order
$$\langle 0|\hat{\phi}_1\hat{\phi}_2|0\rangle = \langle 0|T\hat{\phi}_1\hat{\phi}_2|0\rangle = \overbrace{\hat{\phi}_1\hat{\phi}_2}$$

an important consequence:

$$\boxed{\begin{aligned} &\langle 0|T(\hat{\phi}_1\cdots\hat{\phi}_{2n+1})|0\rangle = 0 && \text{odd number} \\ &\langle 0|T(\hat{\phi}_1\cdots\hat{\phi}_{2n})|0\rangle = \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}\cdots\overbrace{\phi_{2n-1}\phi_{2n}} \\ &\quad + \eta\,\overbrace{\phi_1\phi_3}\,\overbrace{\phi_2\phi_4}\cdots\overbrace{\phi_{2n-1}\phi_{2n}} + \cdots && \text{even number} \end{aligned}}$$

I.e. the vacuum expectation value of a T-product of $2n$ operators can be written as a sum of products of all possible n pairwise contractions

number of terms: $\dfrac{(2n)!}{2^n\, n!} = (2n-1)(2n-3)$

$\Rightarrow$ all can be reduced to $\overbrace{\phi(x)\phi(y)}$!