$$\boxed{\hat{P}\,\hat{a}_{\vec{k},\pm}\,\hat{P}^{+} = -\,\hat{a}_{-\vec{k},\mp}}$$

rem.: If parity is a symmetry $\Rightarrow$ $\hat{P}\mathcal{L}(\vec{r})\hat{P}^{+} \overset{*}{=} \mathcal{L}(-\vec{r})$
- $\mathcal{L}$ must be scalar which excludes certain types of interaction Lagrangians

* physics is identical in spatially inverted system where $\vec{r} \to -\vec{r}$

(B) Charge conjugation

Transfers negative energy solutions to positive energy solutions under change of sign of charge. It can however also be applied to neutral systems and is thus better called "field conjugation"

We define charge conjugation by a unitary $\hat{C}^{-1} = \hat{C}^{+}$

Scalar field

$$\boxed{\hat{C}\hat{\phi}(x)\hat{C}^{+} = \eta^{c}_{\phi}\,\hat{\phi}^{+}(x)}$$

$$\hat{C}^{2} = 1 \quad \rightsquigarrow \quad \eta^{c}_{\phi} = \pm 1$$

Vector field

$$\boxed{\hat{C}\,\hat{\vec{A}}(x)\,\hat{C}^{+} = -\vec{A}(x)}$$

$$\vec{p} - e\vec{A} \xrightarrow{\hat{C}} \vec{p} - e\vec{A} \quad \text{as} \quad \hat{C} e \hat{C}^{+} \to -e$$

Dirac field (bi-spinor) $\qquad \hat{C}\hat{\psi}\hat{C}^{+} = ?$

consider (classical) Dirac equation in external field

$$(*) \qquad i\partial_t \psi = \left[\alpha_j \cdot (-i\partial_j - eA^j) + eA^0 + \beta m\right]\psi$$

this equation has a symmetry

$$\psi \to \psi^c = \underline{\underline{C}}\beta\psi^*$$

where $\boxed{\underline{\underline{C}} = \begin{pmatrix} 0 & -i\sigma_y \\ -i\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \\ \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \end{pmatrix}}$

$$\boxed{\underline{\underline{C}} = -\underline{\underline{C}}^T = -\underline{\underline{C}}^{-1} \qquad \underline{\underline{C}}^2 = -1 \qquad \underline{\underline{C}}\, d\, \underline{\underline{C}}^{-1} = -d^*}$$

$$d \in \{\alpha_i, \beta\}$$

multiplying complex conjugate of (*) with $\underline{\underline{C}}\beta \Rightarrow$

$$-i\partial_t \psi^c = C\beta\left[\alpha_j^* (i\partial_j - eA^j) + eA^0 + \beta m\right]\psi^*$$

$$= \left[\alpha_j (i\partial_j - eA^j) + eA^0 - \beta m\right]\psi^c$$

$$\Rightarrow \quad i\partial_t \psi^c = \left[\alpha_j(-i\partial_j + eA^j) - eA^0 + \beta m\right]\psi^c$$

identical to (*) with $e \to -e$ "charge conjugation"
exchange of positive and negative energy solutions

$$\boxed{\hat{C}\hat{\psi}\hat{C}^{+} = \eta_{\psi}^{c}\,\underline{\underline{C}}\,\beta\,\hat{\psi}^{*}(x) = \eta_{\psi}^{c}\,\underline{\underline{C}}\,\overline{\psi}^{T}}$$

note: $\hat{\psi}^{*} = (\hat{\psi}^{+})^{T}$ and $\beta\hat{\psi}^{+} = \overline{\psi}$

The free Dirac Lagrangian is invariant under the charge conjugation

$$\hat{C}\mathcal{L}(x)\hat{C}^{+} = \mathcal{L}(x)$$

What is the action of $\hat{C}$ on creation and annihilation operators?

$\boxed{\hat{\phi}}$ <u>Klein-Gordon field</u>

$$\hat{C}\hat{\phi}(x)\hat{C}^{+} = \eta_{\phi}^{c}\,\hat{\phi}^{+}(x) \qquad kx = \vec{k}\vec{r} - \omega t$$

$$= \eta_{\phi}^{c}\sum_{k}\frac{1}{\sqrt{2E_k L^3}}\left(\hat{a}^{+}_{\vec{k}}e^{ikx} + \hat{c}_{\vec{k}}e^{-ikx}\right)$$

$$= \hat{C}\sum_{k}\frac{1}{\sqrt{2E_k L^3}}\left(\hat{a}_{\vec{k}}e^{-ikx} + \hat{c}^{+}_{\vec{k}}e^{ikx}\right)\hat{C}^{+}$$

$$\boxed{\begin{aligned}\hat{C}\hat{a}_{\vec{k}}\hat{C}^{+} &= \eta_{\phi}^{c}\,\hat{c}_{\vec{k}}\\ \hat{C}\hat{c}_{\vec{k}}\hat{C}^{+} &= \eta_{\phi}^{c}\,\hat{a}_{\vec{k}}\end{aligned}}$$

$\hat{C}$ interchanges particles and anti-particles without changing the momentum.

$\boxed{\hat{\psi}}$ <u>Dirac field</u>

positive energy solution $\quad u(\vec{k},s) = \sqrt{E_k+m}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{k}}{E_k+m}\end{pmatrix}\chi^{(s)}$

negative energy $\quad v(\vec{k},s) = \sqrt{E_k+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E_k+m} \\ 1 \end{pmatrix}(-i\sigma_y\chi^{(s)})$

$(-i\sigma_y)^* = -i\sigma_y$

$$\underline{\underline{C}}\beta\, v^* = \sqrt{E_k+m}\begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0\end{pmatrix}\begin{pmatrix} \frac{\vec{\sigma}^*\cdot\vec{k}}{E_k+m} \\ -1 \end{pmatrix}(-i\sigma_y\chi^{(s)})$$

$$= \sqrt{E_k+m}\begin{pmatrix} 1 \\ -\sigma_2\frac{\vec{\sigma}^*\cdot\vec{k}}{E_k+m}\sigma_2 \end{pmatrix}\chi^{(s)} \underset{\uparrow}{=} u$$

$\sigma_2\vec{\sigma}^*\sigma_2 = -\vec{\sigma}$

i.e.

$$\boxed{\begin{aligned} \underline{\underline{C}}\beta\, v^*(\vec{k},s) &= u(\vec{k},s) \\ \underline{\underline{C}}\beta\, u^*(\vec{k},s) &= v(\vec{k},s)\end{aligned}}$$

exchange of positive and negative energy solutions

$$\hat{C}\hat{\psi}\hat{C}^+ = \hat{C}\sum_{\vec{k}s}\frac{1}{\sqrt{2E_kL^3}}\left(u(\vec{k},s)e^{-ikx}\hat{b}_{k,s} + v(\vec{k},s)e^{ikx}\hat{d}^+_{k,s}\right)\hat{C}^+$$

$$= \eta^C_\psi\,\underline{\underline{C}}\beta\sum_{ks}\frac{1}{\sqrt{2E_kL^3}}\left(u^* e^{ikx}\hat{b}^+_{ks} + v^* e^{-ikx}\hat{d}_{ks}\right)$$

$$= \eta^C_\psi \sum_{k,s} \frac{1}{\sqrt{2E_k L^3}} \left(v(\vec{k},s) e^{ikx} \hat{b}^\dagger_{\vec{k},s} + u(\vec{k},s) e^{-ikx} \hat{d}_{\vec{k},s} \right)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \hat{C}\, \hat{b}_{\vec{k},s}\, \hat{C}^\dagger &= \eta^C_\psi\, \hat{d}_{\vec{k},s} \\ \hat{C}\, \hat{d}_{\vec{k},s}\, \hat{C}^\dagger &= \eta^C_\psi\, \hat{b}_{\vec{k},s} \end{aligned}}$$

$\boxed{\hat{\vec{A}}}$ Maxwell field

$$\hat{C}\, \hat{\vec{A}}(x)\, \hat{C}^\dagger = -\hat{\vec{A}}(x)$$

$$\hat{C} \sum_{k,\alpha} \frac{1}{\sqrt{2\omega L^3}} \left(\hat{a}_{\vec{k},\alpha}\, \vec{e}^{\,\alpha}_{\vec{k}}\, e^{-ikx} + \hat{a}^\dagger_{\vec{k},\alpha}\, \vec{e}^{\,\alpha *}_{\vec{k}}\, e^{ikx} \right) \hat{C}^\dagger$$
$$= -\hat{\vec{A}}(x)$$

$$\boxed{\hat{C}\, \hat{a}_{\vec{k},\alpha}\, \hat{C}^\dagger = -\hat{a}_{\vec{k}\alpha}}$$

application example: positronium decay

$$e^+ + e^- \longrightarrow 2, 3, 4, 5 \cdots \gamma \text{ quanta}$$

QED is invariant under charge transformation

$$e^+ + e^- \quad \begin{array}{l} \nearrow \; {}^3S_1 \qquad \overset{(+)}{f}(s,s') = \overset{(+)}{f}(s',s) \quad \text{symmetric} \\ \searrow \; {}^1S_0 \qquad \overset{(-)}{f}(s,s') = -\overset{(-)}{f}(s',s) \quad \text{antisymmetric} \end{array}$$

$${}^3S_1, \; {}^1S_0 \qquad |\psi\rangle = \sum_{s,s'} \int d^3p \, \overset{(\pm)}{f}(p,s,s') \, \hat{b}^+_{p,s} \, \hat{d}^+_{-p,s} |0\rangle$$

$$\hat{C}|\psi\rangle = \pm |\psi\rangle$$

i.e. triplet positronium ${}^3S_1 \rightarrow 3\ (5,7,9..)$ γ quanta

singlet positronium ${}^1S_0 \rightarrow 2\ (4,6,8,...)$ γ quanta

(C) Time reversal

time reversal

$$\hat{T}\hat{U}\hat{T}^{-1} \overset{!}{=} \hat{U}^{-1}$$

for this $\hat{T}$ must be anti-unitary

unitary $\quad |\psi\rangle \rightarrow \hat{U}|\psi\rangle \qquad \langle \hat{U}\psi | \hat{U}\phi \rangle = \langle \psi | \phi \rangle$

anti-unitary $\quad |\psi\rangle \rightarrow \hat{U}_A|\psi\rangle \qquad \langle \hat{U}_A\psi | \hat{U}_A\phi \rangle = (\langle \psi | \phi \rangle)^* = \langle \phi | \psi \rangle$

like unitary with $i \to -i$

we should have

$$-i\frac{\partial \phi}{\partial t} = [H, \phi] \xrightarrow{\hat{T}} -i\frac{\partial \phi'}{\partial t} = [H', \phi']$$

if $\hat{T}$ were unitary

$$\hat{T}\left(-i\frac{\partial \phi}{\partial t}\right)\hat{T}^{-1} = -i\frac{\partial \phi'}{\partial(-t)} = i\frac{\partial \phi'}{\partial t}$$

$$\hat{T}[H, \hat{\phi}]\hat{T}^{-1} = [H', \hat{\phi}'] \quad \lightning$$

$\Rightarrow$ $\hat{T}$ must be anti-unitary (i.e. $\hat{T}^{-1} \neq \hat{T}^{+}$)

$$\boxed{\begin{aligned} \hat{T}\hat{\phi}(\bar{r},t)\hat{T}^{-1} &= \eta_\phi^T\, \phi(\bar{r},-t) \\ \hat{T}\hat{\psi}(\bar{r},t)\hat{T}^{-1} &= \eta_\phi^T\, \underline{I}\, \hat{\psi}(\bar{r},-t) \\ \hat{T}\hat{A}^\mu(\bar{r},t)\hat{T}^{-1} &= -\Lambda(T)^\mu{}_\nu\, \hat{A}^\nu(\bar{r},-t) \end{aligned}}$$

<u>remarks:</u>

- $\Lambda(T)^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

- minus sign for $\hat{A}^\mu$ $\to$ time reversal reverses current $j^\mu A_\mu \to j^\mu A_\mu$

- $$\boxed{\underline{\underline{T}} = \underline{\underline{C}}\gamma^5}$$

With this we find

$$\hat{T}\mathcal{L}_{Dirac}(x)\hat{T}^{-1} = \mathcal{L}_{Dirac}(\vec{r}, -t)$$

action of $\hat{T}$ on annihilation and creation operators:

$\boxed{\hat{\phi}}$ Klein-Gordon field

$$\hat{T}\hat{\phi}(x)\hat{T}^{-1} = \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}L^3}} \left(e^{ikx}\hat{T}\hat{a}_{\vec{k}}\hat{T}^{-1} + e^{-ikx}\hat{T}\hat{c}^{\dagger}_{\vec{k}}\hat{T}^{-1}\right)$$

$\vec{k} \to -\vec{k}$ compensates $t \to -t$ in expansion

$$\stackrel{!}{=} \eta^T_\phi \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}L^3}} \left(e^{ikx}\hat{a}_{-\vec{k}} + e^{-ikx}\hat{c}^{\dagger}_{-\vec{k}}\right)$$

$$\boxed{\begin{aligned} \hat{T}\hat{a}_{\vec{k}}\hat{T}^{-1} &= \eta^T_\phi \hat{a}_{-\vec{k}} \\ \hat{T}\hat{c}_{\vec{k}}\hat{T}^{-1} &= \eta^T_\phi \hat{c}_{-\vec{k}} \end{aligned}}$$

like parity

$\boxed{\hat{\psi}}$ Dirac field

one finds for the bi-spinors after some algebra

$$\boxed{\begin{aligned} \underline{\underline{T}}\, u(\vec{k}, s) &= (-1)^{\frac{1}{2}-s} u^*(-\vec{k}, -s) \\ \underline{\underline{T}}\, v(\vec{k}, s) &= (-1)^{\frac{1}{2}-s} v^*(-\vec{k}, -s) \end{aligned}}$$

$$\Rightarrow \quad \hat{T}\,\hat{\psi}(x)\hat{T}^{-1} = \hat{T}\sum_{\vec{k},s}\frac{1}{\sqrt{2E_{\vec{k}}L^3}}\left(u(\vec{k},s)e^{-ikx}\hat{b}_{ks} + v(\vec{k},s)e^{ikx}\hat{d}^{\dagger}_{ks}\right)\hat{T}^{-1}$$

$$= \sum_{\vec{k},s}\frac{1}{\sqrt{2E_{\vec{k}}L^3}}\Big(u^*(\vec{k},s)e^{ikx}\hat{T}\hat{b}_{ks}\hat{T}^{-1}$$

$$+ v^*(\vec{k},s)e^{-ikx}\hat{T}\hat{d}^{\dagger}_{k,s}\hat{T}^{-1}\Big)$$

$$\overset{!}{=} \eta_\psi^T \sum_{\vec{k},s}\frac{1}{\sqrt{2E_{\vec{k}}L^3}}\left(u(-\vec{k},-s)e^{ikx}\hat{b}_{-\vec{k},s} + v(-\vec{k},-s)e^{-ikx}\hat{d}^{\dagger}_{-\vec{k},s}\right)$$

$\vec{k}\to-\vec{k}$ to compensate $t\to -t$ in kx

$$\Rightarrow \quad \boxed{\begin{aligned}\hat{T}\,\hat{b}_{\vec{k},s}\hat{T}^{-1} &= (-1)^{\frac{1}{2}+s}\,\hat{b}_{-\vec{k},-s}\\ \hat{T}\,\hat{d}^{\dagger}_{\vec{k},s}\hat{T}^{-1} &= (-1)^{\frac{1}{2}+s}\,\hat{d}^{\dagger}_{-\vec{k},-s}\end{aligned}}$$

$\boxed{\hat{\vec{A}}(x)}$ <u>Maxwell field</u>

$\vec{k}\to-\vec{k}$ to compensate $t\to -t$

$$\hat{T}\hat{\vec{A}}(x)\hat{T}^{-1} = -\sum_{\vec{k}\alpha}\frac{1}{\sqrt{2\omega L^3}}\left(\vec{e}^{\,\alpha}_{-\vec{k}}\,\hat{a}_{-\vec{k},\alpha}e^{ikx} + \vec{e}^{\,\alpha *}_{-\vec{k}}\,\hat{a}^{\dagger}_{-\vec{k}\alpha}e^{-ikx}\right)$$

$$= \sum_{\vec{k}\alpha}\frac{1}{\sqrt{2\omega L^3}}\left(\vec{e}^{\,\alpha *}_{\vec{k}}\,\hat{T}\hat{a}_{\vec{k}\alpha}\hat{T}^{-1}e^{ikx} + \vec{e}^{\,\alpha}_{\vec{k}}\,\hat{T}\hat{a}^{\dagger}_{\vec{k}\alpha}\hat{T}^{-1}e^{-ikx}\right)$$

using circular polarization basis

$$\vec{e}^{\,\pm *}_{-\vec{k}} = - \vec{e}^{\,\pm}_{\vec{k}}$$

$\Rightarrow$ in circular basis $\alpha = \pm$

$$\boxed{\hat{T}\, \hat{a}_{\vec{k},\alpha}\, \hat{T}^{-1} = \hat{a}_{-\vec{k},\alpha}}$$

(D) PCT theorem

If the Lagrangian density is hermitian, normal ordered and Lorentz invariant and constructed from fields quantized according the spin-statistics theorem then

$$\hat{\Theta} = \hat{P}\hat{C}\hat{T}$$

is a symmetry of the theory

$$\hat{\Theta}\, \hat{\phi}(x)\, \hat{\Theta}^{-1} = \eta_\phi\, \hat{\phi}^{+}(-x)$$
$$\hat{\Theta}\, \hat{\psi}(x)\, \hat{\Theta}^{-1} = \eta_\psi\, \gamma^5 \gamma^0\, \overline{\psi}^{\,T}(-x)$$
$$\hat{\Theta}\, A^\mu(x)\, \hat{\Theta}^{-1} = -A^\mu(-x)$$

consequences:

$$\boxed{\text{masses of stable particles and anti-particles are the same; the same holds for unstable particles that cannot decay into other } \underline{\text{single}} \text{ particle states}}$$!

proof for stable particles:

$$H|k\rangle = E_k |k\rangle \qquad E_k = \sqrt{m^2+k^2}$$
$$H|\bar{k}\rangle = \bar{E}_k |\bar{k}\rangle \qquad \bar{E}_k = \sqrt{\bar{m}^2+k^2}$$

PCT invariance $\quad [\hat{O}, H] = 0$

$$\hat{O}\hat{H}|k\rangle = E_k \hat{O}|k\rangle = E_k|\bar{k}\rangle$$
$$= \hat{H}\hat{O}|k\rangle = H|\bar{k}\rangle = \bar{E}_k|\bar{k}\rangle$$

$$\Rightarrow \quad E_k = \sqrt{m^2+k^2} \overset{!}{=} \bar{E}_k = \sqrt{\bar{m}^2+k^2} \qquad \underline{m = \bar{m}} \quad \square$$