IV. Symmetries of quantized fields I

IV.1 discrete and continuous symmetries

Symmetries in quantum mechanics are groups of transformations of states or operators that leave matrix elements invariant. $\Rightarrow$ unitary transformations

$$|\psi'\rangle = \hat{U}(a)\,|\psi\rangle \qquad a\text{ - some parameter}$$

$$\hat{U}^{-1}(a) = \hat{U}^{+}(a) \qquad \text{unitary}$$

if a is continuous $\quad \hat{U}(a\to 0) \to \mathbb{1}$

$$\Rightarrow \hat{U}(a) = e^{ia\hat{X}} \qquad \hat{X} = \hat{X}^{+}$$

it holds
$$\boxed{\begin{aligned}\hat{U}(a)\,\hat{U}(b) &= e^{ia\hat{X}} e^{ib\hat{X}} = e^{i(a+b)\hat{X}} \\ &= \hat{U}(a+b) \qquad \text{group!}\end{aligned}}$$

group isomorphism $\quad \{a\} \longleftrightarrow \{\hat{U}(a)\}$

equation of motion of transformed state

$$i\frac{\partial}{\partial t}|\psi'\rangle = \frac{\partial}{\partial t}\hat{U}|\psi\rangle = \hat{U}\hat{H}|\psi\rangle = \hat{U}\hat{H}\hat{U}^{-1}|\psi'\rangle$$

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^{-1}$$

iff $\boxed{[\hat{H}, \hat{U}(a)] = 0} \iff \hat{H}' = \hat{H}$

invariance of equation of motion!

<u>Def:</u> A group of unitary transformations (unitary representation of the transformation group $\{a\}$) is called a <u>symmetry group</u> if $\hat{U}(a)$ commutes with $\hat{H}$ for all values of a

$\Rightarrow$ if $|\phi_E\rangle$ is eigenstate of $\hat{H}$ $\Rightarrow$ $U|\phi_E\rangle = |\phi_E'\rangle$ is also eigenstate if $\hat{U}$ is symmetry group

if there are two unitary operators $\hat{U}_1, \hat{U}_2$ with

$$[\hat{U}_1, \hat{U}_2] \neq 0 \quad \text{but} \quad [\hat{U}_1, \hat{H}] = [\hat{U}_2, \hat{H}] = 0$$

$\Rightarrow$ $\exists$ linear independent eigenvectors of $\hat{H}$ for the same eigenvalue $\Rightarrow$ <u>degeneracy</u>

Symmetric groups

$\swarrow \quad \downarrow \quad \searrow$

discrete — continuous (Lie groups) — mixed

a-discrete — a-continuous

IV.2 continuous symmetries and Lie groups

Def: a continuous group whose elements are unitary operators $\hat{U}(a_1, \dots, a_n)$ which are analytic functions of n continuous parameter $a_1, a_2, \dots, a_n$ which uniquely determine the elements of the group is called a Lie group

we set $$\hat{U}(\vec{a}=0) = \mathbb{1}$$

Def: The n quantities

$$\hat{L}_\mu = i \left.\frac{\partial \hat{U}(\vec{a})}{\partial a_\mu}\right|_{\vec{a}=0}$$

are called generators of the Lie group

expansion in the vicinity of $\vec{a}=0$ yields

$$\hat{U}(\delta\alpha_\mu) = \hat{U}(0) + \left.\frac{\partial \hat{U}}{\partial \alpha_\mu}\right|_{\vec{\alpha}=0} \delta\alpha_\mu = \mathbb{1} - i\hat{L}_\mu \delta\alpha_\mu$$

finite transformation $\Delta\alpha_\mu = N\delta\alpha_\mu$

$$\hat{U}(\Delta\alpha_\mu) = \left[\hat{U}(\delta\alpha_\mu)\right]^N = \left[1 - i\frac{\hat{L}_\mu \Delta\alpha_\mu}{N}\right]^N$$

$N \to \infty$

$$\boxed{\hat{U}(\alpha_\mu) = e^{-i\hat{L}_\mu \alpha_\mu}}$$

If the α_μ are chosen real $\quad \hat{L}_\mu^+ = \hat{L}_\mu$

consider $\hat{U}(\{\delta\alpha_\mu\})$ in 2nd order in $\delta\alpha_\mu$

$$\hat{U}(\{\delta\alpha_\mu\}) = \mathbb{1} - i\sum_\mu \hat{L}_\mu \delta\alpha_\mu - \frac{1}{2}\sum_{\mu\nu} \hat{L}_\mu \hat{L}_\nu \delta\alpha_\mu \delta\alpha_\nu$$

and now the sequence of transformations

$$\hat{U}^{-1}(\delta\beta_\mu)\hat{U}^{-1}(\delta\alpha_\mu)\hat{U}(\delta\beta_\mu)\hat{U}(\delta\alpha_\mu) = \qquad (\neq 1 !)$$

$$= \left(1 + i\delta\beta_\mu \hat{L}_\mu - \frac{1}{2}\delta\beta_\nu\delta\beta_\lambda \hat{L}_\nu\hat{L}_\lambda\right)\left(1 + i\delta\alpha_\sigma \hat{L}_\sigma - \frac{1}{2}\delta\alpha_\gamma\delta\alpha_\sigma \hat{L}_\gamma \hat{L}_\sigma\right)$$

$$\cdot \left(1 - i\delta\beta_\varepsilon \hat{L}_\varepsilon - \frac{1}{2}\delta\beta_\omega\delta\beta_\varsigma \hat{L}_\omega\hat{L}_\varsigma\right)\left(1 - i\delta\alpha_\eta \hat{L}_\eta - \frac{1}{2}\delta\alpha_\tau\delta\alpha_\varsigma \hat{L}_\tau\hat{L}_\varsigma\right)$$

$$= \cdots = 1 + \delta\alpha_\mu\delta\beta_\nu \underbrace{(\hat{L}_\mu\hat{L}_\nu - \hat{L}_\nu\hat{L}_\mu)}$$

must be a generator due to group property

$$\delta\alpha_\mu\delta\beta_\nu [\hat{L}_\mu, \hat{L}_\nu] \stackrel{!}{=} -i\delta\gamma_\lambda \hat{L}_\lambda$$

set $\qquad -i\delta\gamma_\lambda = c_{\mu\nu\lambda}\delta\alpha_\mu\delta\beta_\nu$

$$\boxed{[\hat{L}_\mu, \hat{L}_\nu] = c_{\mu\nu\lambda}\hat{L}_\lambda} \qquad \text{Lie algebra}$$

This defines a <u>Lie algebra</u>. The coefficients $c_{\mu\nu\lambda}$ are called <u>structure factors</u>

an important relation follows from Jacobi's identity

$$[[\hat{L}_i,\hat{L}_j],\hat{L}_k]+[[\hat{L}_j,\hat{L}_k],\hat{L}_i]+[[\hat{L}_k,\hat{L}_i],\hat{L}_j]=0$$

$$\boxed{C_{ijm}C_{mkn}+C_{jkm}C_{min}+C_{kim}C_{mjn}=0}$$

Def: The number of commuting generators of a Lie group is called rank of the group

example: (i) translational group

$$\hat{U}=e^{-i\vec{a}\cdot\vec{p}} \qquad \vec{p}=-i\vec{\nabla}$$

generators: $\hat{p}_x, \hat{p}_y, \hat{p}_z$ rank: 3

structure factors: $C_{abc}\equiv 0$

(ii) rotational group

$$\hat{u}=e^{-i\vec{\Theta}\cdot\vec{L}}$$

generators: $\hat{L}_x, \hat{L}_y, \hat{L}_z$ rank: 1

structure factor: $C_{xyz}=i\varepsilon_{xyz}$

theorem: (Cartan / Racah)

Let $C_{\alpha\beta\gamma}$ be the structure coefficients of a Lie algebra and

$$g_{\alpha\beta}=g_{\beta\alpha}=C_{\alpha\mu\nu}C_{\beta\nu\mu}$$

If $$\mathrm{Det}(g_{\alpha\beta}) \neq 0$$

the Lie group is semi-simple. If the group has rank ℓ there are ℓ invariant operators called <u>Casimir operators</u> $\hat{C}_\lambda$; $\lambda = 1, 2, \dots, \ell$. They are functions of the generators and commute with all generators of the group.

<u>example</u>: rotation group $[\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk}\hat{L}_k$

$$g_{\alpha\beta} = i\varepsilon_{\alpha\mu\nu}\, i\varepsilon_{\beta\nu\mu} = \varepsilon_{\alpha\mu\nu}\varepsilon_{\beta\mu\nu} = 2\delta_{\alpha\beta}$$

$\det(g_{\alpha\beta}) = 8 \neq 0$; rank $= 1 \rightsquigarrow \exists\, 1$ Casimir operator $\hat{L}^2$

$$[\hat{L}^2, \hat{L}_i] = 0$$

If a Lie group is a symmetry group of a q.m. system then the eigenvalues of the Casimir operator define <u>multipletts</u>. The corresponding eigenstates form an invariant subspace; i.e. a sub-space which cannot be left by group operations

<u>example</u>: 3-dim subspace of rotational group with total angular momentum (Casimir operator) corresponding to $\ell = 1$

Y_{11}, Y_{10}, Y_{1-1}

now $\hat{L}_\pm Y_{lm} = (\hat{L}_x \pm i \hat{L}_y) Y_{lm} = \sqrt{l(l+1) - m(m\pm1)}\, Y_{l,m\pm1}$

i.e. again within subspace!

$\boxed{\text{The Hamiltonian is degenerate on the multipletts}}$

proof: $|\psi_0\rangle$ element of multiplett

$\Rightarrow \hat{U}(\alpha)|\psi_0\rangle$ element of multiplett

now ~~let~~ $\hat{H}|\psi_0\rangle = E|\psi_0\rangle$

$$\hat{H}\hat{U}|\psi_0\rangle = \hat{U}\hat{H}|\psi_0\rangle = E\,\hat{U}|\psi_0\rangle$$

remark:
- there is no general construction principle for Casimir operators
- for $SU(n)$: Casimir operators are homogeneous polynoms of the generators

$\boxed{\text{An operator } \hat{A} \text{ that commutes with all operators of a Lie group (i.e. with all generators } \hat{L}_\mu) \text{ is a function of the Casimir operators alone}}$

$\Rightarrow$ if a Lie group is a symmetry group $\Rightarrow$ H is function of Casimir operators

<u>example:</u> translational invariant system, rank 3
Casimir operators: $\vec{\hat{P}} = (\hat{P}_x, \hat{P}_y, \hat{P}_z)$

$$\hat{H} = \alpha + \vec{\beta} \cdot \vec{\hat{P}} + \gamma \vec{\hat{P}}^2 + \dots$$

time reversal symmetry (see next chapter)
$\hat{H} \to \hat{H}$ if $\vec{\hat{P}} \to -\vec{\hat{P}}$

$$\Rightarrow \hat{H} = \alpha + \gamma \vec{\hat{P}}^2 + \dots$$

<u>some important groups</u>

group	generators	rank	Casimir operators	type
translation	$\hat{P}_x, \hat{P}_y, \hat{P}_z$	3	$\hat{P}_x, \hat{P}_y, \hat{P}_z$	abelian
rotation	$\hat{L}_x, \hat{L}_y, \hat{L}_z$	1	$\vec{\hat{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$	simple ($\Rightarrow$ semisimple)
euklidian group (transl + rotation)	$\hat{L}_x, \hat{L}_y, \hat{L}_z$ $\hat{P}_x, \hat{P}_y, \hat{P}_z$	3	$\vec{\hat{P}}^2$, $\vec{\hat{P}} \cdot \vec{\hat{L}}$	not simple

IV. 3 charge conjugation, spatial inversion & time reversal, PCT theorem

(A) parity and spatial inversion

$$\hat{P} : \quad \vec{r} \longrightarrow -\vec{r} \qquad\qquad P \overset{!}{=} P^{-1}$$

scalar field
$$\boxed{\hat{P}\hat{\phi}(\vec{r},t)\hat{P}^{+} = \eta^{P}_{\phi}\hat{\phi}(-\vec{r},t)}$$

since $\hat{P}^2 = 1 \quad \Rightarrow \quad \eta^{P}_{\phi} = \pm 1$ — scalar fields are even or odd under P

vector field
$$\boxed{\hat{P}\hat{A}^{\mu}(\vec{r},t)\hat{P}^{+} = \Lambda(P)^{\mu}{}_{\nu}\hat{A}^{\nu}(-\vec{r},t)}$$

where
$$\Lambda(P)^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$
vector fields are even under P

$$\Rightarrow \quad \hat{P}\hat{E}^{\mu}(\vec{r},t)\hat{P}^{+} = -\hat{E}^{\mu}(-\vec{r},t) \qquad \hat{E}^{\mu} = -\partial_t \hat{A}^{\mu}$$
$$\hat{P}\hat{B}^{\mu}(\vec{r},t)\hat{P}^{+} = \hat{B}^{\mu}(-\vec{r},t) \qquad \hat{B}^{\mu} = (\vec{\nabla}\times\hat{\vec{A}})^{\mu}$$

how about a bi-spinor field?

Similar argument as for Lorentz invariance: we would like that in the Dirac equation all terms transform the same way under invasion

$$S^{-1}(P)\gamma^{\mu}S(P) \overset{!}{=} \Lambda^{\mu}{}_{\nu}(P)\gamma^{\nu}$$

i.e $$S^{-1}(P)\gamma^0 S(P) = \gamma^0$$

$$S^{-1}(P)\gamma^i S(P) = -\gamma^i$$

$$\Rightarrow \quad S(P) = e^{i\varphi}\gamma^0$$

now $S(P)^2 \overset{!}{=} 1 \Rightarrow e^{i\varphi} = \pm 1 = \eta^P_\psi$

$$\boxed{\hat{P}\,\hat{\psi}(\vec{r},t)\,\hat{P}^+ = \eta^P_\psi\,\gamma^0\,\hat{\psi}(-\vec{r},t)}$$

$\eta^P_\psi = \pm 1$ spinor fields are even or odd under P

action of $\hat{P}$ onto creation and annihilation operators?

$\boxed{\hat{\phi}}$ Klein Gordon

$$\hat{P}\hat{\phi}(\vec{r},t)\hat{P}^+ = \eta^P_\phi\,\hat{\phi}(-\vec{r},t)$$

$$= \eta^P_\phi \sum_{\vec{k}} \frac{1}{\sqrt{2EL^3}}\left(\hat{a}_{\vec{k}}\,e^{-i(Et+\vec{k}\cdot\vec{r})} + \hat{c}^+_{\vec{k}}\,e^{i(Et+\vec{k}\vec{r})}\right)$$

$$= \eta^P_\phi \sum_{\vec{k}} \frac{1}{\sqrt{2EL^3}}\left(\hat{a}_{-\vec{k}}\,e^{-ik_\mu x^\mu} + \hat{c}^+_{-\vec{k}}\,e^{ik_\mu x^\mu}\right)$$

$$\boxed{\begin{aligned}\hat{P}\,\hat{a}_{\vec{k}}\,\hat{P}^+ &= \eta^P_\phi\,\hat{a}_{-\vec{k}}\\ \hat{P}\,\hat{c}_{\vec{k}}\,\hat{P}^+ &= \eta^P_\phi\,\hat{c}_{-\vec{k}}\end{aligned}}$$

$\boxed{\hat{\psi}}$ Dirac apart from γ^0 similar to Klein-Gordon

$$\hat{P}\,\hat{\psi}(\vec{r},t)\,\hat{P}^+ = \eta^P_\psi\,\gamma^0\,\hat{\psi}(-\vec{r},t)$$

$$= \eta^P_\psi\,\gamma^0 \sum_{\vec{k}} \frac{1}{\sqrt{2EL^3}} \left(\hat{b}_{-\vec{k},s}\, u(-\vec{k},s)\, e^{-ik_\mu x^\mu} + \hat{d}^+_{-k s}\, v(-\vec{k},s)\, e^{ik_\mu x^\mu} \right)$$

Dirac spinors u and v obey

$$\gamma^0 u(-\vec{k},s) = u(\vec{k},s)$$

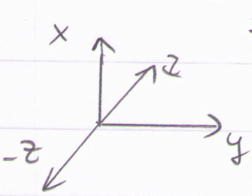
$$\gamma^0 v(-\vec{k},s) = -v(\vec{k},s)$$

$$\Rightarrow \quad \boxed{\begin{aligned} \hat{P}\,\hat{b}_{\vec{k},s}\,\hat{P}^+ &= \eta^P_\psi\,\hat{b}_{-\vec{k},s} \\ \hat{P}\,\hat{d}_{\vec{k},s}\,\hat{P}^+ &= -\eta^P_\psi\,\hat{d}_{-\vec{k},s} \end{aligned}}$$

Fermi particles and anti-particles have opposite parity

$\boxed{\hat{A}^\mu_\perp}$ transverse elm. field circular polarization

$$\hat{P}\,\vec{A}_\perp(\vec{r},t)\,\hat{P}^+ = -\vec{A}_\perp(-\vec{r},t)$$



$$= -\sum_{\vec{k},\sigma} \frac{1}{\sqrt{2\omega L^3}} \left(\vec{e}^{\,\sigma}_{-\vec{k}}\, \hat{a}_{-\vec{k},\sigma}\, e^{-ik_\mu x^\mu} + \vec{e}^{\,\sigma *}_{-\vec{k}}\, \hat{a}^+_{-\vec{k},\sigma}\, e^{ik_\mu x^\mu} \right)$$

since $\hat{P}\,\vec{e}^{\,\pm}_z\,\hat{P}^+ = \hat{P}\,\frac{1}{\sqrt{2}}(\vec{e}_x \pm i\vec{e}_y)\,\hat{P}^+ = -\frac{1}{\sqrt{2}}(\vec{e}_x \pm i\vec{e}_y)$

and $\vec{e}^{\,\pm}_{-z} = \frac{1}{\sqrt{2}}(\vec{e}_x \mp i\vec{e}_y) \quad \Rightarrow$