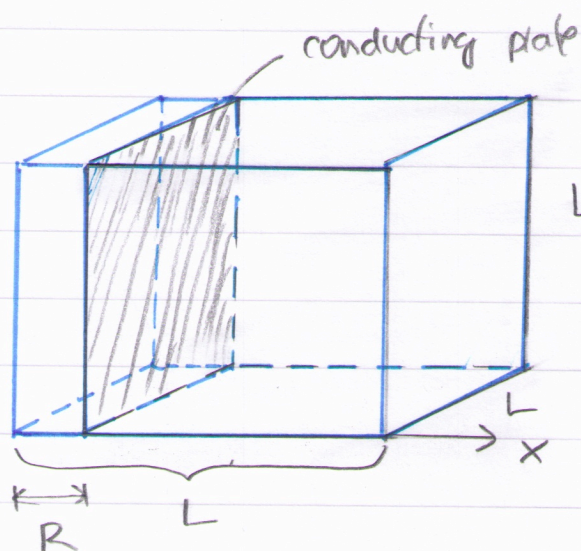


III Simple effects of field quantization

III.1 Vacuum energy of the elm. field and Casimir force

Quantizing the elm. field we have seen that a vacuum contribution appears. Although this vacuum contribution per se has no measurable consequences, changes of it by modified boundary conditions can lead to measurable forces.

quantization in L^3 with open boundary conditions



$$R \ll L$$

open BC:

$$k_n = \frac{\pi}{L} n \quad n = 0, 1, 2, 3$$

(term $n_x = n_y = n_z = 0$ excluded)

$$\Delta E = \langle 0 | \hat{H}_R | 0 \rangle + \langle 0 | \hat{H}_{L-R} | 0 \rangle - \langle 0 | \hat{H}_L | 0 \rangle$$

$$\begin{aligned} \langle 0 | \hat{H} | 0 \rangle &= \sum_{n_x n_y n_z} \omega_n = \frac{\pi}{L} \sum_{n_x n_y n_z=0}^{\infty} \sqrt{n_x^2 + n_y^2 + n_z^2} \\ &\simeq \frac{\pi}{L} 4\pi \int_0^{\infty} dn \, n^2 n = \frac{L^3}{2\pi^2} \int_0^{\infty} dk \, k^3 \xrightarrow{L \rightarrow \infty} \infty \end{aligned}$$

contribution divergent $\leadsto$ introduce smooth cut-off function $f(k/k_c)$

$$1 \xleftarrow{k \ll k_c} f\left(\frac{k}{k_c}\right) \xrightarrow{k \gg k_c} 0$$

$$\Rightarrow \langle 0 | \hat{H}_L | 0 \rangle = \frac{L^3}{2\pi^2} \int_0^\infty dk f\left(\frac{k}{k_c}\right) k^3$$

$$\langle 0 | \hat{H}_{L-R} | 0 \rangle = \frac{L^2(L-R)}{2\pi^2} \int_0^\infty dk f\left(\frac{k}{k_c}\right) k^3$$

To calculate $\langle 0 | \hat{H}_R | 0 \rangle$ we have to be more careful as in one direction $L \rightarrow \infty$ does not lead to continuous modes!

$$\langle 0 | \hat{H}_R | 0 \rangle = \sum_{n_x=0}^{\infty} \left(1 - \frac{1}{2} \delta_{n0}\right) \frac{L^2}{\pi^2} g\left(\frac{n\pi}{R}\right)$$

where the summation in y and z direction is included in

$$g(k_x) = \int_0^\infty \int_0^\infty dk_y dk_z k f\left(\frac{k}{k_c}\right) \quad k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

Note that $\left(1 - \frac{1}{2} \delta_{n0}\right)$ is because for $n=0$ there is only one polarization direction since $\vec{E}_{||} = 0$

$$\text{simp} \quad \int_0^\infty dk_x g(k_x) = \frac{4\pi}{8} \int_0^\infty dk k^2 k f\left(\frac{k}{k_c}\right)$$

$$\Rightarrow \Delta E = \frac{L^2}{\pi^2} \left\{ \sum_{n=0}^{\infty} \left(1 - \frac{1}{2} \delta_{n0}\right) g\left(\frac{n\pi}{R}\right) - \frac{R}{L} \int_0^\infty dk_x g(k_x) \right\}$$

If we approximate the first sum by an integral $\Delta E \rightarrow 0$ but we need to be more careful here:

$$k_{\perp} \equiv \sqrt{k_y^2 + k_z^2}$$

$$g(k_x) = \frac{2\pi}{4} \int_0^{\infty} dk_{\perp} k_{\perp} \sqrt{k_x^2 + k_{\perp}^2} f\left(\frac{\sqrt{k_x^2 + k_{\perp}^2}}{k_c}\right)$$

$$\text{set } k_{\perp} = \alpha \frac{\pi}{R} \quad k_x = n \frac{\pi}{R} \Rightarrow$$

$$g(k_x) = \frac{\pi}{2} \frac{\pi^3}{R^3} \int_0^{\infty} d\alpha \sqrt{n^2 + \alpha^2} \alpha f\left(\frac{\pi \sqrt{n^2 + \alpha^2}}{R k_c}\right)$$

$$\text{define } \omega \equiv n^2 + \alpha^2 \quad d\omega = 2\alpha d\alpha$$

$$g(k_x) = \frac{\pi^4}{4R^3} \underbrace{\int_{n^2}^{\infty} d\omega \sqrt{\omega} f\left(\frac{\pi \sqrt{\omega}}{R k_c}\right)}_{\equiv F(n)}$$

then

$$\Delta E = \frac{L^2 \pi^2}{4R^3} \left\{ \sum_{n=0}^{\infty} \left(1 - \frac{1}{2} \delta_{n0}\right) F(n) - \int_0^{\infty} dn F(n) \right\}$$

remember:

Euler-Mclaurin formula

$$h(x) \xrightarrow{x \rightarrow \infty} 0$$

$$\sum_{k=0}^{\infty} h(k) \simeq \frac{1}{2} h(0) + \int_0^{\infty} dk h(k)$$

$$- \sum_{n=2}^P \frac{B_n}{n!} h^{(n-1)}(0) + \text{correction}$$

$$\text{where } B_2 = \frac{1}{6}; B_3 = 0; B_4 = -\frac{1}{30}; B_5 = 0; \dots$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(1 - \frac{1}{2} \delta_{n0}\right) F(n) - \int_0^{\infty} dn F(n) = -\frac{1}{6 \cdot 2!} F'(0) + \frac{1}{30 \cdot 4!} F'''(0) + \dots$$

now $F'(n) = -2n^2 f\left(\frac{\pi n}{Rk_c}\right)$

$$F''(n) = -4n f\left(\frac{\pi n}{Rk_c}\right) - 2n^2 f'\left(\frac{\pi n}{Rk_c}\right) \frac{\pi}{Rk_c}$$

$$F'''(n) = -4 f\left(\frac{\pi n}{Rk_c}\right) - 8n f'\left(\frac{\pi n}{Rk_c}\right) \frac{\pi}{Rk_c} - 2n^2 f''\left(\frac{\pi n}{Rk_c}\right) \left(\frac{\pi}{Rk_c}\right)^2$$

$$\Rightarrow F'(0) = F''(0) = 0 \quad F'''(0) = -4 f(0) = -4$$

independent on cut-off!

$\Rightarrow$

$$\Delta E = -\frac{L^2 \pi^2}{720 R^3}$$

Casimir energy of two perfectly conducting plates at distance R ($k_c R \gg 1$)

remarks:

• $\Delta E < 0 \Rightarrow$ attractive force

$$f = \frac{F}{L^2} = -\frac{1}{L^2} \frac{\partial(\Delta E)}{\partial R} = -\frac{\pi^2}{240 R^4} \quad \left| \begin{array}{l} \text{area density} \\ \text{of force} \end{array} \right.$$

• f is independent on cut-off $f\left(\frac{k}{k_c}\right)!$

• f in SI unit $f = -\frac{\hbar c \pi^2}{240 R^4}$ very small!

first measurement: M.J. Sparnaay
Physica 24 (1958) 751-764

III.2 Decay & Scattering : general remarks

typical problem of QFT:

transition probability between asymptotic states $|\psi_\alpha\rangle, |\psi_\beta\rangle$
which are eigenstates of the free Hamiltonian, $\hat{H}_0$

$$\hat{H} = \hat{H}_0 + \hat{H}_I$$

→ interaction picture

$$|\psi(t)\rangle = T \exp \left\{ -i \int_0^t dz \hat{H}_I(z) \right\} |\psi(0)\rangle$$

$$\hat{O}(z) = e^{i\hat{H}_0 z} \hat{O} e^{-i\hat{H}_0 z}$$

$|\psi_\alpha\rangle, |\psi_\beta\rangle$ asymptotic states (eigenstates of $\hat{H}_0$)

$$S_{\beta\alpha} \equiv \frac{1}{\mathcal{N}} \langle \psi_\beta | \hat{U}_I(\infty, -\infty) | \psi_\alpha \rangle$$

given transition matrix element for $|\psi(t=-\infty)\rangle = |\psi_\alpha\rangle$
to $|\psi(t=+\infty)\rangle = |\psi_\beta\rangle$

$$\hat{U}_I(t_2, t_1) = T \exp \left\{ -i \int_{t_1}^{t_2} dz \hat{H}_I(z) \right\}$$

$\mathcal{N}$ is an appropriately chosen phase factor, which can be fixed as follows. Assume that

$$\hat{H}_I(z) \xrightarrow{|z| \rightarrow \infty} 0$$

then it should hold that the ground state of $\hat{H}$ (which at $t = \pm\infty$ is identical to $\hat{H}_0$) should go over into itself apart from a phase, i.e.

$$\langle g_{z_0} | \hat{U}_I(\infty, -\infty) | g_{z_0} \rangle \stackrel{!}{=} e^{i\phi}$$

We use this as the normalization

Scattering matrix

$$S_{\beta\alpha} = \frac{\langle \varphi_\beta | \hat{U}_I(\infty, -\infty) | \varphi_\alpha \rangle}{\langle g_{z_0} | \hat{U}_I(\infty, -\infty) | g_{z_0} \rangle}$$

Calculating this scattering matrix will be the central problem we are going to deal with!

relation to physical quantities

transition probability per time

$$\Delta W_{\beta\alpha} = \lim_{T \rightarrow \infty} \frac{|S_{\beta\alpha}(T/2, -T/2)|^2}{T}$$

We will see later that for scattering and decay processes we can write (large T)

$$S_{\beta\alpha}\left(\frac{T}{2}, -\frac{T}{2}\right) = -2\pi i \frac{T}{T^2} \frac{\sin(\Delta E T/2)}{\Delta E T} \left(\frac{1}{V L^3}\right)^m f_{\beta\alpha}$$

where for decay processes $m = p-1$
 with p being the number of particles in the final state and

and for scattering processes where the incoming states are plane waves ($m = q$)
 with q being the number of scattering particles

The sinc-term resulted from an integral like

$$d_1(T) \equiv \frac{T}{2\pi} \frac{\sin(\Delta E \frac{T}{2})}{\Delta E T} = \frac{1}{2\pi} \int_{-T/2}^{T/2} dt e^{-i(E_\alpha - E_\beta)t}$$

With this

$$\Delta W_{\beta\alpha} = \lim_{T \rightarrow \infty} \frac{d_1^2(T)}{T} (2\pi)^2 \left(\frac{1}{L^3}\right)^m |f_{\beta\alpha}|^2$$

which contains

$$d_2(T) \equiv (2\pi)^2 \frac{d_1^2(T)}{T} = 4T \frac{\sin^2(\Delta E \frac{T}{2})}{(\Delta E T)^2}$$

We note

$$\lim_{T \rightarrow \infty} d_1(T) = \delta(E_\alpha - E_\beta)$$

but also

$$\lim_{T \rightarrow \infty} d_2(T) = 2\pi \delta(E_\alpha - E_\beta)$$

We note that it is important to keep track of the order of limits. In this sense we have

$$\lim_{T \rightarrow \infty} \frac{|2\pi \delta(E_\alpha - E_\beta)|^2}{T} = 2\pi \delta(E_\alpha - E_\beta)$$

In summary we have

$$S_{\beta\alpha} = -2\pi i \delta(E_\alpha - E_\beta) \left(\frac{1}{iL^3}\right)^m f_{\beta\alpha}$$

S-matrix elements

$$\Delta W_{\beta\alpha} = 2\pi \delta(E_\alpha - E_\beta) \left(\frac{1}{iL^3}\right)^m |f_{\beta\alpha}|^2$$

transition probability per time (rate)
Fermi's golden rule

$$m = \begin{cases} P-1 & \text{decay} \\ P & \text{scattering} \end{cases}$$

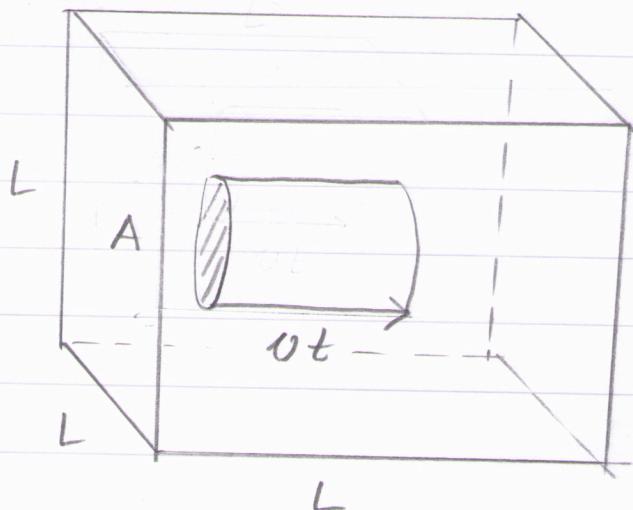
differential scattering cross section

An important experimental quantity is the differential cross section in a spatial angle $\Delta\Omega$

$$\sum_{\beta \in \Delta\Omega} \Delta W_{\beta\alpha}^S = j \Delta\sigma_{\beta\alpha}$$

i) fraction of incident particles per area A and time

$$\bar{j} = \frac{n_0}{At} = \frac{1}{L^3} A v t \frac{1}{At} = \frac{v}{L^3}$$



$$\Delta \phi_{\beta\alpha} = \frac{2\pi}{v} \sum_{\beta \in \Delta R} \delta(E_\alpha - E_\beta) \left(\frac{1}{L^3} \right)^m |f_{\beta\alpha}|^2$$

III. 3 spontaneous emission

Consider atom with two relevant states $|a\rangle$ and $|b\rangle$, which we describe as non-relativistic electron field in the fixed Coulomb potential of the atomic nucleus

$$\hat{H} = \hat{H}_{\text{elm}} + \hat{H}_{\text{atom}} + \int d^3\vec{r} \, \bar{\psi}(\vec{r}) \frac{1}{2m} (\vec{p} - e\vec{A})^2 \psi(\vec{r})$$

$\hat{H}_{\text{elm}}$ - Hamiltonian of elm. field

$\hat{H}_{\text{atom}}$ - Hamiltonian of atom (Coulomb potential of nucleus)

$$\hat{H}_I = \int d^3r \, \bar{\psi}^\dagger(\vec{r}, t) \left[-\frac{e}{m} \vec{p} \cdot \vec{A} + \frac{e^2 \vec{A}^2}{2m} \right] \psi(\vec{r}, t)$$

rem: in coulomb gauge $[\vec{p}, \vec{A}] = 0$

let $| \alpha \rangle = | a, 0 \rangle$ photon vacuum in all modes
 $| \beta \rangle = | b, 1_{n2} \rangle$ 1-photon state in mode n with polarization 2

1st order perturbation theory

$$\langle \beta | \hat{U}_I(\infty, -\infty) | \alpha \rangle = \langle \beta | T e^{-i \int_{-\infty}^{\infty} d\tau \hat{H}_I(\tau)} | \alpha \rangle$$

$$= \langle \beta | \alpha \rangle - i \int_{-\infty}^{\infty} d\tau \langle \beta | \hat{H}_I(\tau) | \alpha \rangle$$

one finds that the $\vec{A}^2$ term does not contribute and after some calculation (problem course) one finds

$$\Delta W_{ba} = \sum_{\{1n2\}} \Delta W_{\beta\alpha} = \frac{e^2}{4\pi} \left(\frac{4\omega_{ab}}{3m^2} \right) |\langle b | \vec{p} | a \rangle|^2$$

Sum over all possible final photon states

Where $\omega_{ab} = E_a - E_b$. Making use of

$$\vec{p} = -im [\vec{r}, \hat{H}_0] = \frac{m}{i} (\vec{r} \hat{H}_0 - \hat{H}_0 \vec{r})$$

$$\langle b | \vec{p} | a \rangle = \frac{m}{i} \langle b | \vec{r} | a \rangle (E_a - E_b) = \frac{m \omega_{ab}}{i} \langle b | \vec{r} | a \rangle$$

$\Rightarrow$

$$\Delta W_{ba} = \frac{\omega_{ab}^3}{3\pi} |\vec{d}_{ab}|^2$$

$$\vec{d}_{ab} = e \langle b | \vec{r} | a \rangle$$

dipole matrix
element

$\Rightarrow$ Einstein coefficient of spontaneous emission

III. 4 Lamb shift according to Bethe

With the help of microwave and laser spectroscopy W.E. Lamb and W.E. Retherford found in 1947 that there is an energy splitting between the $2S_{1/2}$ and $2P_{1/2}$ states of Hydrogen, despite the fact that in relativistic quantum mechanics (Dirac) these states should be exactly degenerate. This splitting is called Lamb shift and is a direct consequence of the quantization of the elm. field (Lamb & Retherford Phys. Rev. 72, 241 (1947)) The following calculation is due to Bethe (H.A. Bethe Phys. Rev. 72, 339 (1947)) which allegedly was done on a train ride.

starting point: non-relativistic interaction operator

$$\hat{H}_I = \int d^3r \vec{\varphi}^\dagger(\vec{r}) \left[-\frac{e}{m} \vec{p} \cdot \vec{A} + \frac{e^2 \vec{A}^2}{2m} \right] \vec{\varphi}(\vec{r})$$

calculate energy shift in second order perturbation

$$|a\rangle = |a, 0\rangle$$

$$E_a^{(0)} = E_a^{(0)}$$

$$\Delta E_\alpha = \langle \alpha | \hat{H}_I | \alpha \rangle + \sum_{\beta \neq \alpha} \frac{\langle \alpha | \hat{H}_I | \beta \rangle \langle \beta | \hat{H}_I | \alpha \rangle}{E_\alpha^{(0)} - E_\beta^{(0)}}$$

$\vec{A}^2$ yields a constant and divergent vacuum energy shift which is however the same for all states and thus will be ignored which amounts to $\vec{A}^2 \rightarrow :\vec{A}^2:$
 $\Rightarrow \langle \alpha | \hat{H}_I | \alpha \rangle = 0$

def: $:\hat{\phi}_1 \hat{\phi}_2: \equiv \hat{\phi}_1 \hat{\phi}_2 - \langle 0 | \hat{\phi}_1 \hat{\phi}_2 | 0 \rangle$
normal ordering

example boson field $:\hat{a} \hat{a}^\dagger: = \hat{a} \hat{a}^\dagger - \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle$
 $= \hat{a} \hat{a}^\dagger - 1 = \hat{a}^\dagger \hat{a}$

ignore $:\vec{A}^2:$ term only contributions $|\beta\rangle \sim |1_{\vec{b},2}\rangle$

$$|\beta\rangle = |b, 1_{\vec{b},2}\rangle$$

$$E_\beta^{(0)} = E_b^{(0)} + \omega_{\vec{b}}$$

b - all possible atomic states

We will show in the problem session that

$$\langle \beta | \hat{H}_I | \alpha \rangle = \frac{1}{L^3} e^{i(E_\beta - E_\alpha)t} f_{\beta\alpha}$$

$$f_{\beta\alpha} = \frac{e}{m} \frac{1}{\sqrt{2\omega_{\vec{b}}}} \vec{e}_{\vec{b}}^* \langle b | \vec{p} | a \rangle$$

$\Rightarrow$

$$\Delta E_\alpha = \sum_b \sum_{\vec{b},2} \frac{1}{L^3} \frac{|f_{\beta\alpha}|^2}{E_\alpha^{(0)} - E_b^{(0)} - \omega_{\vec{b}}}$$

Note that the sum also contains $b = \alpha$.

$$L \rightarrow \infty \quad (E_a = E_a^{(0)}, E_b = E_b^{(0)})$$

$$\begin{aligned} \Delta E_a &= \sum_{b,2} \int \frac{d^3 k}{2\omega_k (2\pi)^3} \frac{e^2}{m^2} \frac{|\vec{e}_{b\lambda}^* \cdot \vec{p}_{ba}|^2}{E_a - E_b - \omega_k} \\ &= \frac{1}{2\pi} \sum_b \int_0^\infty \frac{d\omega}{E_a - E_b - \omega} \underbrace{\left[\frac{e^2}{4\pi} \frac{\omega}{2\pi m^2} \sum_\lambda \int d^3 k |\vec{e}_{b\lambda}^* \cdot \vec{p}_{ba}|^2 \right]}_{\equiv \Delta \tilde{W}_{ba}(\omega)} \\ &\quad \text{divergent} \rightarrow \omega \rightarrow \omega - i\varepsilon \end{aligned}$$

It holds (problem session)

$$\frac{1}{E_a - E_b - \omega + i\varepsilon} = P \frac{1}{E_a - E_b - \omega} - i\pi \delta(E_a - E_b - \omega)$$

↑
principal part

$$\int_A^B d\omega \, P \frac{1}{\omega_0 - \omega} f(\omega) \equiv \lim_{\varepsilon \rightarrow 0} \left[\int_A^{\omega_0 - \varepsilon} d\omega \frac{f(\omega)}{\omega_0 - \omega} + \int_{\omega_0 + \varepsilon}^B d\omega \frac{f(\omega)}{\omega_0 - \omega} \right]$$

With this we find

$$\Delta E_a = \frac{1}{2\pi} \sum_b P \int_0^\infty d\omega \frac{\Delta \tilde{W}_{ba}(\omega)}{E_a - E_b - \omega} - \frac{i}{2} \sum_{\substack{b \\ E_a > E_b}} \Delta \tilde{W}_{ba}(\omega_{ab})$$

- remarks:
- the second imaginary term corresponds to the spontaneous emission and will be ignored
 - the first term is divergent! ☹

now let's calculate $\tilde{\Delta W}_{ba}(\omega)$

$$\tilde{\Delta W}_{ba}(\omega) = \frac{e^2}{4\pi} \frac{\omega}{2\pi m^2} \sum_b \int d\Omega_{\vec{b}} |\vec{e}_{k\lambda}^* \cdot \vec{P}_{ab}|^2$$

without loss of generality $\vec{P}_{ab} = |\vec{P}_{ab}| \vec{e}_z$

$$\vec{e}_{k\lambda}^* \cdot \vec{P}_{ab} = \cos\theta |\vec{P}_{ab}| \quad \text{for linear polarization}$$

$$\sum_b \int d\Omega_{\vec{b}} |\vec{e}_{k\lambda}^* \cdot \vec{P}_{ab}|^2 = 2 \cdot 2\pi \int_0^\pi d\theta \sin\theta \cos^2\theta |\vec{P}_{ab}|^2$$

$$= 4\pi \int_{-1}^1 d\cos\theta \cos^2\theta |\vec{P}_{ab}|^2 = \frac{8\pi}{3} |\vec{P}_{ab}|^2$$

$$\Rightarrow \tilde{\Delta W}_{ba}(\omega) = \frac{e^2}{4\pi} \frac{4\omega}{3m^2} |\vec{P}_{ab}|^2$$

$$\Delta E_a = \frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} \sum_b P \int_0^\infty d\omega \frac{\omega}{E_a - E_b - \omega} |\vec{P}_{ab}|^2$$

UV divergent!

The divergence of terms like this is a characteristic feature of QFT. To remove it or at least to make it less severe we use an idea of Kramers:

Mass-renormalization

Also the unbound electron interacts with the eluv. vacuum

Thus the observable energy shift is the difference of the energy shifts of bound and free electron with the same average kinetic energy

$$\Delta E^{\text{free}} = \frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} \sum_b P \int_0^\infty d\omega \frac{\omega |\vec{P}_{ba}^f|^2}{\frac{P_a^2}{2m} - \frac{P_b^2}{2m} - \omega}$$

Where $\vec{P}_{ba}^f = \int d^3r \frac{1}{L^3} e^{-i(\underbrace{\vec{P}_b + \vec{b}}_{\substack{+ \text{momentum} \\ \text{flow photon}}}) \cdot \vec{r}} (-i\vec{\nabla}) e^{i\vec{P}_a \cdot \vec{r}}$

$$= \vec{P}_a \delta(\vec{P}_a - \vec{P}_b - \vec{b})$$

(note: $|\alpha\rangle = |a, 0\rangle$ and $|\beta\rangle = |b, 1\vec{b}_z\rangle$)

due to $\delta(\vec{P}_a - \vec{P}_b - \vec{b})$ the denominator in ΔE^{free} reduces to

$$\frac{P_a^2}{2m} + \frac{P_b^2}{2m} - \omega = \frac{P_a^2}{2m} + \frac{(\vec{P}_a - \vec{b})^2}{2m} - \omega = \frac{\vec{P}_a \cdot \vec{b}}{2m} - \omega \left(1 + \frac{\omega}{2m}\right)$$

since for non-relativistic particle $|\vec{P}_a| \ll m$ ($c=1$)

$$\Delta E_a^{\text{free}} \simeq -\frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} P \int_0^\infty d\omega \frac{1}{1 + \frac{\omega}{2m}} \sum_b |\langle b | \vec{P}^3 | a \rangle|^2$$

It is interesting to note that this term has the structure

$$\begin{aligned} \Delta E_a^{\text{free}} &= -C \sum_b \langle a | \vec{P}^3 | b \rangle \langle b | \vec{P}^3 | a \rangle = \\ &= -C \langle a | \vec{P}^3^2 | a \rangle \end{aligned}$$

Thus it can be interpreted as mass renormalization

$$\frac{\vec{p}^2}{2m^*} = \frac{\vec{p}^2}{2m} - C\vec{p}^2 \quad m^* = \frac{m}{1-2mC}$$

Since the coupling to the chm. field cannot be switched off, m^* is the only observable physical quantity!

With ΔE_a^{free} we find using

$$\begin{aligned} \Delta E_a^{\text{obs}} &= \Delta E_a - \Delta E_a^{\text{free}} = \\ &= \frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} P \int_0^\infty d\omega \sum_b \left[\frac{\omega |\vec{p}_{ab}|^2}{E_a - E_b - \omega} + \frac{|\vec{p}_{ab}|^2}{1 + \frac{\omega}{2m}} \right] \end{aligned}$$

This is still divergent but if we introduce a cut-off
 $\omega_{\text{max}} = E_{\text{max}} \ll m$

$$\Delta E_a^{\text{obs}} = \frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} P \int_0^{\omega_{\text{max}}} d\omega \sum_b \frac{(E_a - E_b) |\vec{p}_{ab}|^2}{E_a - E_b - \omega}$$

We find only a logarithmic divergence with ω_{max}

$$\begin{aligned} \underline{\omega_0 > 0} \quad P \int_0^{\omega_{\text{max}}} d\omega \frac{1}{\omega_0 - \omega} &= \int_0^{\omega_0 - \epsilon} d\omega \frac{1}{\omega_0 - \omega} + \int_{\omega_0 + \epsilon}^{\omega_{\text{max}}} d\omega \frac{1}{\omega_0 - \omega} \\ &= -\ln(\omega_0 - \omega) \Big|_0^{\omega_0 - \epsilon} - \ln(\omega_0 - \omega) \Big|_{\omega_0 + \epsilon}^{\omega_{\text{max}}} \end{aligned}$$

$$= -\ln \frac{\varepsilon}{\omega_0} - \ln \frac{\omega_{\max} - \omega_0}{\varepsilon} \approx -\ln \frac{\omega_{\max}}{\omega_0}$$

$$(\omega_{\max} \gg \omega_0)$$

$$\frac{\omega_0 < 0}{P} \int_0^{\omega_{\max}} d\omega \frac{1}{\omega_0 - \omega} = -\ln \left(\frac{\omega_{\max}}{|\omega_0|} \right)$$

$$\Delta E_a^{\text{obs}} = -\frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} \sum_b (E_a - E_b) \ln \left(\frac{E_{\max}}{|E_a - E_b|} \right) |\vec{p}_{ab}|^2$$

Since the logarithm is only slowly varying it can be taken out of the sum

$$\Delta E_a^{\text{obs}} \approx -\frac{e^2}{4\pi} \frac{2}{3\pi} \frac{1}{m^2} \ln \frac{E_{\max}}{\langle |E_a - E_b| \rangle} \underbrace{\sum_b (E_a - E_b) |\vec{p}_{ab}|^2}_{\text{can be calculated explicitly!}}$$

$$\begin{aligned} \sum_b (E_a - E_b) |\vec{p}_{ab}|^2 &= \sum_b \langle a | \hat{H}_A \hat{p}^j - \hat{p}^j \hat{H}_A | b \rangle \langle b | \hat{p}^j | a \rangle \\ &= \frac{1}{2} \langle a | [[\hat{H}_A, \hat{p}^j], \hat{p}^j] | a \rangle \end{aligned}$$

Consider central potential $V(\vec{r}) = -Z \frac{e^2}{4\pi r}$

$$\begin{aligned} \Rightarrow [[\hat{H}_A, \hat{p}^j], \hat{p}^j] &= [i \partial_j V, -i \partial_j] = -\vec{\nabla}^2 V \\ &= -Ze^2 \delta^{(3)}(\vec{r}) \end{aligned}$$

$$\Delta E_a^{\text{obs}} = Z \alpha^2 \left(\frac{4}{3m^2} \right) |\psi_a(\vec{r}=0)|^2 \ln \left(\frac{E_{\text{max}}}{\langle |E_a - E_b| \rangle} \right)$$

wavefunction at origin

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137} \quad \text{Feinstruktur constant}$$

remark:

- $\Delta E \neq 0$ only for states with $\psi_a(r=0) \neq 0$
 $\rightarrow s$ states
 (for p-states higher order contributions needed, much smaller $\leadsto$ here neglected)
- $E_{\text{max}} > \langle |E_a - E_b| \rangle \Rightarrow \Delta E_a^{\text{obs}} > 0$
- $\langle |E_a - E_b| \rangle$ difficult to calculate
 $\langle |E_a - E_b| \rangle_{25} \simeq 226.3 \text{ eV}$
- $E_{\text{max}} = m \leadsto \ln \frac{m}{\langle |E_a - E_b| \rangle_{25}} \simeq 7.72 \quad (c=1)$
- Hydrogen $Z=1 \quad |\psi_{2s}(0)|^2 = \frac{1}{\pi} \frac{1}{(2a_0)^3}$

$$\Delta E_{s_{1/2} - p_{1/2}} \simeq m \alpha^5 \frac{7.72}{6\pi} \simeq 1051 \text{ MHz}$$

Experiment: 1057845(9) kHz (1986)

Theory: 1057857(14) kHz (1984)

QED extremely successful!