II.5 The Maxwell field

use Coulomb gauge! $\vec{\nabla}\cdot\vec{A}=0 \quad \vec{A}=\vec{A}_\perp$

Lagrangedensity and Hamiltonian read

$$\mathcal{L}=\frac{1}{2}\left(\vec{E}_\perp^{\,2}-\vec{B}_\perp^{\,2}\right) \qquad \vec{E}_\perp=-\partial_t\vec{A}_\perp \quad \vec{B}_\perp=\vec{\nabla}\times\vec{A}_\perp$$

$$H=\int d^3r\,\mathcal{H}=\frac{1}{2}\int d^3r\left(\vec{E}_\perp^{\,2}+\vec{B}_\perp^{\,2}\right)$$

canonical momentum $\quad \vec{\Pi}_\perp=\frac{\partial\vec{A}_\perp}{\partial t}=-\vec{E}_\perp$

Due to the Coulomb gauge the components of $\vec{A}_\perp$ and $\vec{E}_\perp$ are not independent and thus the quantization rule reads

$$(*) \qquad \boxed{\left[\vec{A}_\perp(\vec{r},t),\,\vec{E}_\perp(\vec{r}\,',t)\right]=-i\overleftrightarrow{\delta}_\perp^{(3)}(\vec{r}-\vec{r}\,')}$$

Remarks:

- $(*)$ is consistent with Coulomb gauge
- one finds (problem section)

$$\sum_{k=1}^{3}\left[A_\perp^k(\vec{r},t),\,E_\perp^k(\vec{r},t)\right]=\underset{\substack{\uparrow\\ \#\text{ of degrees of freedom}}}{-2}\,i\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

- we will see that for the Maxwell field only bosonic quantization makes sense.

consider quantization in finite volume L^3 with PBC (Note quantization in finite volume with PBC is <u>not</u>

Lorentz invariant. To get L-invariance one has to consider the limit $L \to \infty$). The fundamental field equations read

$$\Box \vec{A}(\vec{r},t) = 0 \qquad \text{with} \qquad \vec{\nabla}\cdot\vec{A}(\vec{r},t) = 0$$ i.e. only 2 degrees of freedom

Decomposition into normal modes (as for linear chain)

$$\boxed{\hat{\vec{A}}(\vec{r},t) = \sum_n \sum_{\alpha=1}^{2} \frac{1}{\sqrt{2\omega_n L^3}} \left(\vec{e}_{n\alpha}\, \hat{a}_{n\alpha}(t)\, e^{i\vec{k}_n\cdot\vec{r}} + \vec{e}_{n\alpha}^{\,*}\, \hat{a}^+_{n\alpha}(t)\, e^{-i\vec{k}_n\cdot\vec{r}} \right)}$$

where $n = (n_x, n_y, n_z)$ $\quad n_\mu = 0, \pm 1, \pm 2, \dots$

$$\vec{k}_n = (k_{nx}, k_{ny}, k_{nz}) \qquad k_{n\mu} = \frac{2\pi}{L} n_\mu \quad \Leftarrow \underline{\text{PBC}}$$

<u>Coulomb gauge</u> $\quad \vec{\nabla}\cdot\hat{\vec{A}} = 0 \qquad \boxed{\vec{k}_n \cdot \vec{e}_{n\alpha} = 0}$

$\Rightarrow$ $\vec{e}_{n\alpha}$ transversal polarization unit vectors

convention: linearly polarized basis $\qquad \vec{e}_{n\alpha} = \vec{e}_{n\alpha}^{\,*}$

$$\vec{e}_{n1} \times \vec{e}_{n2} = \frac{\vec{k}_n}{|\vec{k}_n|} \quad + \text{ cyclic permutations}$$

<u>wave equation</u> $\quad \boxed{\hat{a}_{n\alpha}(t) = \hat{a}_{n\alpha}(0)\, e^{-i\omega_n t}} \quad \boxed{|\vec{k}_n|^2 = \omega_n^2}$ (**)

In the problem courses we will prove that the field commutation relation is equivalent to the set of fundamental commutator relations between $\hat{a}_{n\alpha}$ and $\hat{a}^+_{n\alpha}$

$$\boxed{\begin{aligned}\left[\hat{a}_{n\alpha}, \hat{a}^{+}_{m\beta}\right] &= \delta_{nm}\delta_{\alpha\beta}\\ \left[\hat{a}_{n\alpha}, \hat{a}_{m\beta}\right] &= \left[\hat{a}^{+}_{n\alpha}, \hat{a}^{+}_{m\beta}\right] = 0\end{aligned}}$$

for the electric and magnetic field one finds

$$\boxed{\begin{aligned}&\vec{E}_\perp(\vec{r},t) = -\partial_t \vec{A}_\perp = \sum_n \sum_{\alpha=1}^{2} \frac{1}{\sqrt{2\omega_n L^3}} \cdot \\ &\quad i\vec{e}_{n\alpha}\,\omega_n \left(\hat{a}_{n\alpha}(t)\, e^{i\vec{k}_n\cdot\vec{r}} - \hat{a}^{+}_{n\alpha}(t)\, e^{-i\vec{k}_n\vec{r}}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}&\vec{B}_\perp(\vec{r},t) = \vec{\nabla}\times\vec{A}_\perp = \sum_n \sum_{\alpha=1}^{2} \frac{1}{\sqrt{2\omega_n L^3}} \cdot \\ &\quad i(\vec{e}_{n\alpha}\times\vec{k}_n)\left(\hat{a}_{n\alpha}(t)\, e^{i\vec{k}_n\cdot\vec{r}} - \hat{a}^{+}_{n\alpha}(t)\, e^{-i\vec{k}_n\vec{r}}\right)\end{aligned}}$$

With this we find for the Hamiltonian (yet without using commutator relations)

$$\hat{H} = \frac{1}{2}\int_{L^3} d^3r\left(\vec{E}_\perp^2 + \vec{B}_\perp^2\right) = \frac{1}{4L^3}\int_{L^3} d^3r \sum_{\substack{n,m\\ \alpha,\beta}} \frac{1}{\sqrt{\omega_n\omega_m}} *$$

$$* \Big\{ \left[\omega_n\omega_m\, \vec{e}_{n\alpha}\cdot\vec{e}_{m\beta} + (\vec{k}_n\times\vec{e}_{n\alpha})\cdot(\vec{k}_m\times\vec{e}_{m\beta})\right] \cdot$$

$$\cdot\left[\hat{a}^{+}_{n\alpha}\hat{a}_{m\beta}\, e^{-i(\vec{k}_n-\vec{k}_m)\vec{r}} + \hat{a}_{n\alpha}\hat{a}^{+}_{m\beta}\, e^{i(\vec{k}_n-\vec{k}_m)\vec{r}}\right]$$

$$- \left[\omega_n\omega_m\, \vec{e}_{n\alpha}\cdot\vec{e}_{m\beta} + (\vec{k}_n\times\vec{e}_{n\alpha})\cdot(\vec{k}_m\times\vec{e}_{m\beta})\right] \cdot$$

$$\cdot\left[\hat{a}_{n\alpha}\hat{a}_{m\beta}\, e^{i(\vec{k}_n+\vec{k}_m)\vec{r}} + \hat{a}^{+}_{n\alpha}\hat{a}^{+}_{m\beta}\, e^{-i(\vec{k}_n+\vec{k}_m)\vec{r}}\right]\Big\}$$

Making use of

$$\frac{1}{L^3}\int_{L^3} d^3r\, e^{\pm i(\vec{k}_n-\vec{k}_m)\vec{r}} = \delta_{nm}$$

$$\frac{1}{L^3}\int_{L^3} d^3r\, e^{\pm i(\vec{k}_n+\vec{k}_m)\vec{r}} = \delta_{n,-m}$$

and $\vec{k}_n = -\vec{k}_{-n}$ one finds:

$$\hat{H} = \frac{1}{4}\sum_{n\alpha\beta}\frac{1}{\omega_n}\Big\{\left[\omega_n^2\delta_{\alpha\beta}+(\vec{k}_n\times\vec{e}_{n\alpha})(\vec{k}_n\times\vec{e}_{n\beta})\right]\left[\hat{a}^+_{n\alpha}\hat{a}_{n\beta}+\hat{a}_{n\alpha}\hat{a}^+_{n\beta}\right]$$
$$-\left[\omega_n^2\vec{e}_{n\alpha}\cdot\vec{e}_{-n\beta}-(\vec{k}_n\times\vec{e}_{n\alpha})(\vec{k}_n\times\vec{e}_{-n\beta})\right]\left[\hat{a}_{n\alpha}\hat{a}_{-n\beta}+\hat{a}^+_{n\alpha}\hat{a}^+_{-n\beta}\right]\Big\}$$

now

$$(\vec{k}_n\times\vec{e}_{n\alpha})(\vec{k}_n\times\vec{e}_{-n\beta}) = k_n^2\,\vec{e}_{n\alpha}\cdot\vec{e}_{-n\beta} - \underbrace{(\vec{k}_n\cdot\vec{e}_{-n\beta})}_{=0}\underbrace{(\vec{k}_n\cdot\vec{e}_{n\alpha})}_{=0}$$
$$= \omega_n^2\,\vec{e}_{n\alpha}\cdot\vec{e}_{-n\beta}$$

and

$$(\vec{k}_n\times\vec{e}_{n\alpha})(\vec{k}_n\times\vec{e}_{n\beta}) = \omega_n^2\,\vec{e}_{n\alpha}\cdot\vec{e}_{n\beta} = \omega_n^2\delta_{\alpha\beta}$$

$$\Rightarrow\quad \hat{H} = \sum_n\sum_{\alpha=1}^{2}\frac{1}{2}\omega_n\left[\hat{a}^+_{n\alpha}\hat{a}_{n\alpha}+\hat{a}_{n\alpha}\hat{a}^+_{n\alpha}\right]$$

It is evident that anti-commutation relations would yield $\hat{H}=0$ which is obviously nonesense. Thus the elm. field must be quantized according to commutation relations

$$\boxed{\hat{H} = \sum_n\sum_{\alpha=1}^{2}\omega_n\left[\hat{a}^+_{n\alpha}\hat{a}_{n\alpha}+\frac{1}{2}\right]}$$

Hamiltonian of the free elm. field

It is a short exercise to show that the momentum of the e/m field (as defined in classical ED) reads

$$\boxed{\vec{P} = \int d^3r\, \frac{1}{2}\left(\vec{E}_\perp \times \vec{B}_\perp - \vec{B}_\perp \times \vec{E}_\perp\right) = \sum_{n,\alpha} \vec{k}_n\, \hat{a}^\dagger_{n\alpha} \hat{a}_{n\alpha} + \text{const.}}$$

I.e. the momentum per photon is $\vec{k}_n$.

Even more interesting is the operator of the angular momentum (again defined as in classical ED with proper symmetrization) with respect to $\vec{r}_0$

$$\vec{\Omega} \equiv \int d^3r\, (\vec{r}-\vec{r}_0) \times \left[\left(\vec{E}_\perp \times \vec{B}_\perp\right) - \left(\vec{B}_\perp \times \vec{E}_\perp\right)\right]$$

A little longer calculation shows that $\vec{\Omega}$ can be decomposed into a part that depends on the choice of $\vec{r}_0$ which we will define as orbital angular momentum $\vec{\Omega}_L$ and a part independent on $\vec{r}_0$ which we denote as internal angular momentum or spin $\vec{\Omega}_S$

$$\boxed{\vec{\Omega}_L \equiv i \int d^3r\, \hat{E}^k_\perp\, \vec{L}\, \hat{A}^k_\perp}$$ orbital angular momentum

where

$$\vec{L} \equiv -i\left((\vec{r}-\vec{r}_0) \times \vec{\nabla}\right)$$

$$\boxed{\begin{aligned}\vec{\Omega}_S &\equiv \int d^3r\, \hat{\vec{E}}_\perp \times \hat{\vec{A}}_\perp \\ &= \int d^3r\, \vec{e}_i\, \varepsilon_{ijk}\, \hat{E}^j_\perp \hat{A}^k_\perp\end{aligned}}$$ spin

<u>proof:</u> classical (commuting variables) $\vec{x} \equiv \vec{r} - \vec{r}_0$
(drop index "$\perp$")

$$\Omega^i = \int d^3r \, \varepsilon_{ijl} \, x^j \, \underbrace{\varepsilon_{lkm}}_{} E^k \, \underbrace{\varepsilon_{mab}}_{} \partial_a A^b$$

$$= \int d^3r \, \varepsilon_{ijl} \, x^j \left(\delta_{la}\delta_{kb} - \delta_{lb}\delta_{ka}\right) E^k \partial_a A^b$$

$$= \int d^3r \, \varepsilon_{ijl} \, x^j \left(E^b \partial_l A^b - (\vec{E}\cdot\vec{\nabla}) A^l\right)$$

partial integration of last term and use of $\vec{\nabla}\cdot\vec{E} = 0$
$\Rightarrow$

$$\Omega^i = \int d^3r \, \varepsilon_{ijl} \left(x^j E^b \partial_l A^b - E^j A^l\right)$$

depends on $\vec{r}_0$ independent on $\vec{r}_0$

lets have a closer look at $\vec{\Omega}_s$

$$\hat{\Omega}_s^i = \int d^3r \, \frac{\varepsilon_{ijk}}{2L^3} \sum_{\substack{n,m \\ \alpha\beta}} \frac{i\omega_n}{\sqrt{\omega_n \omega_m}} \left(\vec{e}_{n\alpha}\right)^j \left(\vec{e}_{m\beta}\right)^k$$

$$\Big\{ \hat{a}_{n\alpha}\hat{a}_{m\beta} \, e^{i(\vec{k}_n + \vec{k}_m)\vec{r}} - \hat{a}^+_{n\alpha}\hat{a}^+_{m\beta} \, e^{-i(\vec{k}_n + \vec{k}_m)\vec{r}}$$
$$- \hat{a}^+_{n\alpha}\hat{a}_{m\beta} \, e^{-i(\vec{k}_n - \vec{k}_m)\vec{r}} + \hat{a}_{n\alpha}\hat{a}^+_{m\beta} \, e^{i(\vec{k}_n - \vec{k}_m)\vec{r}} \Big\}$$

carrying out the $\int d^3r$ integration and using the corresponding orthogonality relations yields

$$\hat{\Omega}_S^i = i \sum_{n\alpha\beta} \frac{\varepsilon_{ijk}}{2} \Big\{ (\vec{e}_{n\alpha})^j (\vec{e}_{-n\beta})^k \left[\hat{a}_{n\alpha}\hat{a}_{-n\beta} - \hat{a}^+_{n\alpha}\hat{a}^+_{-n\beta} \right]$$
$$+ (\vec{e}_{n\alpha})^j (\vec{e}_{n\beta})^k \left[\hat{a}_{n\alpha}\hat{a}^+_{n\beta} - \hat{a}^+_{n\alpha}\hat{a}_{n\beta} \right] \Big\}$$

The first term in $\{\cdots\}$ is symmetric in $j \leftrightarrow k$ $n \leftrightarrow -n$ and thus vanishes. The second term is not symmetric if $\alpha \neq \beta$ $\Rightarrow$

$$\vec{\hat{\Omega}}_S = i \sum_{\substack{n \\ \alpha\neq\beta}} (\vec{e}_{n\alpha} \times \vec{e}_{n\beta})\, \hat{a}^+_{n\beta}\hat{a}_{n\alpha} + \underbrace{\cancel{\text{const.}}}_{\to 0}$$

Basis of linearly polarized fields $\quad \vec{e}_{n1} \times \vec{e}_{n2} = \vec{k}_n / |\vec{k}_n|$

$$\boxed{\vec{\hat{\Omega}}_S = i \sum_n \frac{\vec{k}_n}{|\vec{k}_n|} \left(\hat{a}^+_{n2}\hat{a}_{n1} - \hat{a}^+_{n1}\hat{a}_{n2} \right)}$$

introduce circular basis

$$\vec{e}_{n+} = \frac{1}{\sqrt{2}} (\vec{e}_{n1} + i\vec{e}_{n2}) \qquad \vec{e}_{n-} = \frac{1}{\sqrt{2}} (\vec{e}_{n1} - i\vec{e}_{n2})$$

$$\Rightarrow \boxed{\vec{\hat{\Omega}}_S = \sum_n \frac{\vec{k}_n}{|\vec{k}_n|} \left(\hat{a}^+_{n+}\hat{a}_{n+} - \hat{a}^+_{n-}\hat{a}_{n-} \right)}$$

$$\boxed{\text{Photon} \qquad \text{Spin-1 particle}}$$

II.6. The Dirac field

Lagrange density

$$\mathcal{L} = \bar{\psi}\left[\tfrac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right]\psi \quad \underset{\text{canonical trafo}}{\widehat{=}} \quad \mathcal{L} = \bar{\psi}\left[i\gamma_\mu \overrightarrow{\partial}_\mu - m\right]\psi$$

conjugate field (use asymmetric form of $\mathcal{L}$)

$$\bar{\pi} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^+$$

We will see later that we have to choose fermionic quantization rules for the Dirac field

$$\boxed{\begin{aligned} \{\hat{\psi}(\vec{r},t), \hat{\psi}(\vec{r}',t)\} &= \{\hat{\psi}^+(\vec{r},t), \hat{\psi}^+(\vec{r}',t)\} = 0 \\ \{\hat{\psi}_\alpha(\vec{r},t), \hat{\psi}^+_\beta(\vec{r}',t)\} &= \delta(\vec{r}-\vec{r}')\,\delta_{\alpha\beta} \end{aligned}}$$

$\alpha\beta$ – Bispinor indices

Hamiltonian

$$\boxed{\hat{H} = \int d^3r \; \hat{\psi}^+ \left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right]\hat{\psi}}$$

equation of motion

$$i\partial_t \hat{\psi} = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\hat{\psi}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

solutions of classical Dirac equation $\Rightarrow$ positive and negative energy solutions (see problem courses)

$$\psi_{\vec{p}}^{(\pm)}(\vec{r},t) = \mathcal{N}\, e^{i(\vec{k}_n\cdot\vec{r} \mp E_p t)} \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

with $E_p > 0$. PBC in L^3 $\Rightarrow$

$$\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z) \qquad n_i = 0, \pm 1, \pm 2, \ldots$$

χ and η are two-component spinors

$\boxed{\psi^{(+)}}$ $\qquad \vec{k}\cdot\vec{r} - E_p t = -p_\mu x^\mu$

insert into Dirac equation

$$E_p \begin{pmatrix} \chi \\ \eta \end{pmatrix} e^{-ip_\mu x^\mu} = \begin{pmatrix} m & -i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} e^{-ip_\mu x^\mu}$$

$\Rightarrow$

$$E_p \chi = m\chi + \vec{\sigma}\cdot\vec{p}\,\eta \qquad\qquad \vec{p} = -i\vec{\nabla}$$
$$E_p \eta = \vec{\sigma}\cdot\vec{p}\,\chi - m\eta$$

non-trivial solution only for $\det(\cdot\cdot) = 0$ $\Rightarrow$

$$\boxed{E_p^2 = p^2 + m^2}$$

relativistic energy-momentum relation

also

$$\eta = \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\,\chi$$

$\Rightarrow$

$$\psi_{\vec{p}}^{(+)}(\vec{r},t) = \mathcal{N}\, e^{-ip_\mu x^\mu} \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi \end{pmatrix}$$

χ is two-component spinor that describes the spin degree-of-freedom of a spin 1/2 particle. Eigenstates of σ_z: $\chi^{(s)}$

$$\boxed{\chi^{(\frac{1}{2})} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^{(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Normalisation condition

$$1 \overset{!}{=} \int_{L^3} d^3r\, \psi^+\psi = \mathcal{N}^2 L^3 \left[1 + \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p+m)^2}\right]$$

with $(\vec{\sigma}\cdot\vec{p})^2 = \sigma_i p_i \sigma_j p_j = \sigma_i\sigma_j p_i p_j = \frac{1}{2}\underbrace{(\sigma_i\sigma_j + \sigma_j\sigma_i)}_{\sim\delta_{ij}} p_i p_j$

$$= \sum_{i=1}^{3} p_i^2 = p^2$$

$\Rightarrow$

$$1 = \mathcal{N}^2 L^3 \left[1 + \frac{p^2}{(E_p+m)^2}\right] = \mathcal{N}_p^2 L^3 \left[1 + \frac{E_p^2 - m^2}{(E_p+m)^2}\right]$$

$$= \mathcal{N}^2 L^3 \left[1 + \frac{E_p - m}{E_p+m}\right] = \mathcal{N}_p^2 L^3 \frac{2E_p}{E_p+m}$$

$$\mathcal{N} = \frac{1}{\sqrt{2E_p L^3}} \sqrt{E_p+m}$$

This gives

$$\boxed{\psi^{(+)}_{\vec{p}s}(\vec{r},t) = \frac{1}{\sqrt{2E_p L^3}}\, u(\vec{p},s)\, e^{-ip_\mu x^\mu}}$$

with $$u(\vec{p},s) = \sqrt{E_p+m}\begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi^{(s)} \end{pmatrix} \xrightarrow[\vec{p}\to 0]{} \sqrt{E_p+m}\begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix}$$

$\boxed{\psi^{(-)}}$ similarly $E_p \to -E_p$

$$-E_p\chi = m\chi + \vec{\sigma}\cdot\vec{p}\,\eta$$
$$-E_p\eta = \vec{\sigma}\cdot\vec{p} - m\eta$$

$E_p^2 = p^2 + m^2$ as before

and

$$\chi = -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\eta$$

note $E_p + m \gg 0$ ($m - E_p$ can be small)

Spin eigenstates

convenient definition (see later)

$$\boxed{\eta^{(\frac{1}{2})} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \eta^{(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

This gives

$$\boxed{\psi^{(-)}_{-\vec{p},-s}(\vec{r},t) = \frac{1}{\sqrt{2E_p L^3}}\, v(\vec{p},s)\, e^{ip_\mu x^\mu}}$$

"$-p$" $\hat{=}$ $-\vec{p}$ and $E = -E_p$

$-ip_\mu x^\mu \to ip_\mu x^\mu$

with
$$\boxed{v(\vec{p},s)=\sqrt{E_p+m}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\eta^{(-s)}\\ \eta^{(-s)}\end{pmatrix}\xrightarrow[\vec{p}\to 0]{}\sqrt{E_p+m}\begin{pmatrix}0\\ \eta^{(-s)}\end{pmatrix}}$$

We will see in the following chapter that charge conjugation of the positive-energy solution gives the negative-energy solution :

$$\boxed{\left[\psi^{(+)}_{\vec{p},s}(\vec{r},t)\right]^{C}=\psi^{(-)}_{-\vec{p},-s}(\vec{r},t)}$$

with momentum $-\vec{p}$ and spin $-s$

Now we can decompose the Dirac field into positive- and negative-energy eigensolutions and absorb the time-dependence in the operator-valued coefficients (to be checked lateron)

$$(*)\quad\boxed{\begin{aligned}\hat{\psi}(\vec{r},t)&=\sum_{n,s}\left(\hat{b}_{n,s}\,\psi^{(+)}_{n,s}(\vec{r},t)+\hat{d}^{+}_{n,s}\,\psi^{(-)}_{-n,-s}(\vec{r},t)\right)\\&=\sum_{n,s}\left(\hat{b}_{n,s}\,e^{-iE_pt}\,\psi^{(+)}_{n,s}(\vec{r})+\hat{d}^{+}_{n,s}\,e^{iE_pt}\,\psi^{(-)}_{-n,-s}(\vec{r})\right)\end{aligned}}$$

normal mode decomposition of the Dirac field

$$\hat{b}_{ns}(t)=\hat{b}_{ns}(0)\,e^{-iE_pt}\qquad E_p=\sqrt{\vec{k}_n^{\,2}+m^2}$$
$$\hat{d}_{ns}(t)=\hat{d}_{ns}(0)\,e^{-iE_pt}$$

$\hat{b}_{ns}$: annihilates spin ½ particle with momentum $\vec{p}_n$ and spin s

$\hat{d}_{n,s}$: annihilates spin ½ anti-particle of opposite charge and with momentum $\vec{p}_n$ an spin s

Substitution of (*) in the Hamiltonian and making use of

$$(-i\vec{\alpha}\cdot\vec{\nabla}+\beta m)\,\psi^{(\pm)} = \pm E_p\,\psi^{(\pm)}$$

yields

$$\hat{H} = \int d^3r \sum_{n,s}\sum_{n',s'}\left(\hat{b}^{+}_{ns}\,\psi^{(+)+}_{ns}(\vec{r}) + \hat{d}_{ns}\,\psi^{(-)+}_{-n,-s}(\vec{r})\right)$$
$$\cdot\; E_{n'}\left(\hat{b}_{n's'}\,\psi^{(+)}_{n's'}(\vec{r}) - \hat{d}^{+}_{n's'}\,\psi^{(-)}_{-n,-s}(\vec{r})\right)$$

using the orthonormality

$$\int d^3r\;\psi^{(\pm)+}_{m,s}(\vec{r})\,\psi^{(\pm)}_{m',s'}(\vec{r}) = \delta_{mm'}\,\delta_{ss'}$$

$$\int d^3r\;\psi^{(+)+}_{m,s}(\vec{r})\,\psi^{(-)}_{m',s'}(\vec{r}) = 0$$

yields

$$\hat{H} = \sum_{n,s} E_n\left(\hat{b}^{+}_{ns}\,\hat{b}_{ns} - \hat{d}_{ns}\,\hat{d}^{+}_{ns}\right)$$

using bosonic quantization yields

$$\hat{H}_{Bose} = \sum_{n,s} E_n \left(\hat{b}^+_{ns} \hat{b}_{ns} - \hat{d}^+_{ns} \hat{d}_{ns} \right) + const$$

not positive definite ↯

despite the fact that we had associated positive energies E_p to both

⇒ fermionic quantization rules

$$\hat{d}_{ns} \hat{d}^+_{ns} = -\hat{d}^+_{ns} \hat{d}_{ns} + 1$$

$$\boxed{\hat{H}_{Fermi} = \sum_{n,s} E_n \left(\hat{b}^+_{ns} \hat{b}_{ns} + \hat{d}^+_{ns} \hat{d}_{ns} \right) + const}$$

particle — anti-particle

$$\{ \hat{b}_{ns}, \hat{b}^+_{n's'} \} = \delta_{nn'} \delta_{ss'}$$

$$\{ \hat{d}_{ns}, \hat{d}^+_{n's'} \} = \delta_{nn'} \delta_{ss'}$$

$$\{ \hat{b}_{ns}, \hat{b}_{n's'} \} = \{ \hat{d}_{ns}, \hat{d}_{n's'} \} = \{ \hat{b}_{ns}, \hat{d}^{(+)}_{n's'} \} = 0$$

The Dirac field describes spin 1/2 particles and anti-particles of opposite charge and must be quantized as a fermionic field