II.4 The charged Klein-Gordon field

The simplest relativistic field is the charged Klein-Gordon field. In order to allow for a non-vanishing current the Klein-Gordon field must be complex valued.

$$\mathcal{L} = \frac{\partial \phi^*}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu} - m^2 \phi^* \phi$$

conjugate momenta

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^* = \dot{\phi}^* \qquad \pi = \dot{\phi}$$

$\Rightarrow$

$$\mathcal{H} = \pi^* \partial_0 \phi + \partial_0 \phi^* \pi - \mathcal{L}$$

$$= \pi^* \pi + \vec{\nabla} \phi^* \vec{\nabla} \phi + m^2 \phi^* \phi$$

Hamilton operator

$$\hat{H} = \int d^3r \left(\hat{\pi}^\dagger \hat{\pi} + \vec{\nabla} \hat{\phi}^\dagger \nabla \hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} \right)$$

We will see now that for the Klein-Gordon field which is a scalar field, i.e. with spin 0, only commutator relations make sense. The Euler Lagrange equations are the Klein-Gordon equations

$$(\Box + m^2)\, \phi = 0$$

From QM II we know that two types of solutions with positive and negative energy exist.

$$\phi_n^{(\pm)}(\vec{r},t) = \mathcal{N} e^{i(k_n \vec{r} \mp E_n t)}$$

with $E_n = \sqrt{m^2+k_n^2} > 0$. With periodic boundary conditions in a volume L^3 one finds

$$\vec{k}_n = \frac{2\pi}{L}(n_x, n_y, n_z) \qquad n_i = 0, \pm 1, \pm 2, \dots$$

$\Rightarrow$ ansatz

$$\boxed{\begin{aligned} \hat{\phi}(\vec{r},t) &= \sum_n \left(\hat{a}_n \phi_n^{(+)}(\vec{r}) + \hat{c}_n^{+} \phi_{-n}^{(-)}(\vec{r}) \right) \\ \hat{\phi}^{+}(\vec{r},t) &= \sum_n \left(\hat{a}_n^{+} \phi_n^{(+)*}(\vec{r}) + \hat{c}_n \phi_{-n}^{(-)*}(\vec{r}) \right) \end{aligned}}$$

interpretation:

$\hat{a}_n$ destroys particle with momentum p_n and positive charge

$\hat{c}_n^{+}$ creates particle with momentum $-p_n$ and negative charge (= creates antiparticle with momentum p_n and neg. charge)

rem.: (i) $\hat{a}_n$ is connected to $\hat{c}_n^{+}$ such that $\hat{\phi}$ reduces the total charge

(ii) $\hat{c}_n^{+}$ is connected with $\phi_{-n}^{(-)}$ since states with negative energy and negative momentum correspond to anti-particles with positive energy and momentum orthonormality (can easily be checked)

$$i\int d^3r\, \phi_n^{(+)*} \overleftrightarrow{\frac{\partial}{\partial t}} \phi_m^{(+)} = 2\delta_{nm} N^2 L^3 E_n$$

$$\Rightarrow \text{choose} \quad N = \frac{1}{\sqrt{2E_n L^3}}$$

$$i\int d^3r\, \phi_n^{(-)*} \overleftrightarrow{\frac{\partial}{\partial t}} \phi_m^{(-)} = -2\delta_{nm} N^2 L^3 E_n = -\delta_{nm}$$

$$i\int d^3r\, \phi_n^{(-)*} \overleftrightarrow{\frac{\partial}{\partial t}} \phi_m^{(+)} = 0$$

with this we can rewrite:

$$\hat{H} = \int d^3r \left(\partial_t \hat{\phi}^+ \partial_t \hat{\phi} + \vec{\nabla}\hat{\phi}^+ \vec{\nabla}\hat{\phi} + m^2 \hat{\phi}^+ \hat{\phi} \right)$$

$$\underset{\text{part. int.}}{=} \int d^3r \left(\partial_t \hat{\phi}^+ \partial_t \hat{\phi} - \hat{\phi}^+ \underbrace{\left(\vec{\nabla}^2 \hat{\phi} - m^2 \hat{\phi} \right)}_{= \partial_t^2 \hat{\phi}} \right)$$

$$= -\int d^3r\, \hat{\phi}^+ \overleftrightarrow{\frac{\partial}{\partial t}} \left(\frac{\partial}{\partial t} \hat{\phi} \right)$$

plug in mode decomposition:

$$\hat{H} = \int d^3r \sum_{n,m} \left(\hat{a}_n^+ \phi_n^{(+)*} + \hat{c}_n \phi_{-n}^{(-)*} \right) i \overleftrightarrow{\frac{\partial}{\partial t}} E_m \cdot$$

$$\cdot \left(\hat{a}_m \phi_m^{(+)} - \hat{c}_m^+ \phi_{-m}^{(-)} \right)$$

$$\hat{H} = \sum_n E_n \left(\hat{a}_n^+ \hat{a}_n + \hat{c}_n \hat{c}_n^+ \right)$$

assuming commutator relations we find

$$\hat{H} = \underbrace{\sum_n E_n \left(\hat{a}_n^+ \hat{a}_n + \hat{c}_n^+ \hat{c}_n \right)}_{\text{positive definite}} + \sum_n E_n$$

assuming anticommutator relations we find

$$\hat{H} = \underbrace{\sum_n E_n \left(\hat{a}_n^+ \hat{a}_n - \hat{c}_n^+ \hat{c}_n \right)}_{\text{not definite } \lightning} + \sum_n E_n$$

$\Rightarrow$ need commutator relations

$$\boxed{\text{spin } 0 \quad \Rightarrow \quad \text{bosons}}$$

fundamental commutator rules

$$\boxed{\begin{aligned} &[\hat{a}_n, \hat{a}_m^+] = [\hat{c}_n, \hat{c}_m^+] = \delta_{nm} \\ &[\hat{a}_n, \hat{a}_m] = [\hat{a}_n^+, \hat{a}_m^+] = [\hat{a}_n, \hat{c}_m] = [\hat{a}_n^+, \hat{c}_m^+] \\ &\quad = [\hat{c}_n, \hat{c}_m] = [\hat{c}_n^+, \hat{c}_m^+] = 0 \end{aligned}}$$

from this we find for the fields

$$[\hat{\phi}(\vec{r},t),\hat{\pi}(\vec{r}',t)] = \sum_{n,m}\Big[\Big(\hat{a}_n\phi_n^{(+)}(\vec{r}) + \hat{c}_n^{+}\phi_{-n}^{(-)}(\vec{r})\Big),$$

$$iE_m\Big(\hat{a}_m^{+}\phi_m^{(+)*}(\vec{r}') - \hat{c}_m\phi_{-m}^{(-)*}(\vec{r}')\Big)\Big]$$

$$= \sum_{n,m}\underbrace{[\hat{a}_n,\hat{a}_m^{+}]}_{=\delta_{n,m}}\, iE_m\,\phi_n^{(+)}(\vec{r})\,\phi_m^{(+)*}(\vec{r}')$$

$$-\sum_{n,m}\underbrace{[\hat{c}_n^{+},\hat{c}_m]}_{-\delta_{nm}}\, iE_m\,\phi_{-n}^{(-)}(\vec{r})\,\phi_{-m}^{(-)*}(\vec{r}')$$

$$= \sum_m iE_m\Big(\phi_m^{(+)}(\vec{r})\,\phi_m^{(+)*}(\vec{r}') + \phi_{-m}^{(-)}(\vec{r})\,\phi_{-m}^{(-)*}(\vec{r}')\Big)$$

now from the Klein Gordon theory one knows that the following completeness relations hold

$$\sum_n E_n\,\phi_n^{(+)*}(\vec{r})\,\phi_n^{(+)}(\vec{r}') = \frac{1}{2}\,\delta(\vec{r}-\vec{r}') \quad \text{in } L^3$$

$$\sum_n E_n\,\phi_{-n}^{(-)}(\vec{r})\,\phi_{-n}^{(-)*}(\vec{r}') = \frac{1}{2}\,\delta(\vec{r}-\vec{r}') \quad \text{in } L^3$$

$$\Rightarrow \quad [\hat{\phi}(\vec{r},t),\hat{\pi}(\vec{r}',t)] = i\delta(\vec{r}-\vec{r}') \quad \square$$

With the Klein-Gordon field we have seen the first example for a fundamental theorem of relativistic quantum field theory:

Spin-Statistics Theorem

fields with integer spin $\Longleftrightarrow$ bosons

fields with half integer spin $\Longleftrightarrow$ fermions

commutation relations and causality

The commutation relations imply that fields $\hat{\phi}$ ($\hat{\phi}^+$) and their conjugates $\hat{\pi}$ ($\hat{\pi}^+$) at the <u>same instant of time</u> can be simultaneously measured at <u>different points in space</u>; i.e. a measurement of the field at

$$(\vec{r}, t)$$

cannot influence the measurement at

$$(\vec{r}', t) \qquad (x-x')^2 = \underbrace{(t-t)^2}_{=0} - (r-r')^2 < 0$$

Since the theory is Lorentz invariant this implies:

> measurements of fields or any local observables at spacelike separated points cannot influence each other
>
> $\Rightarrow$ micro causality