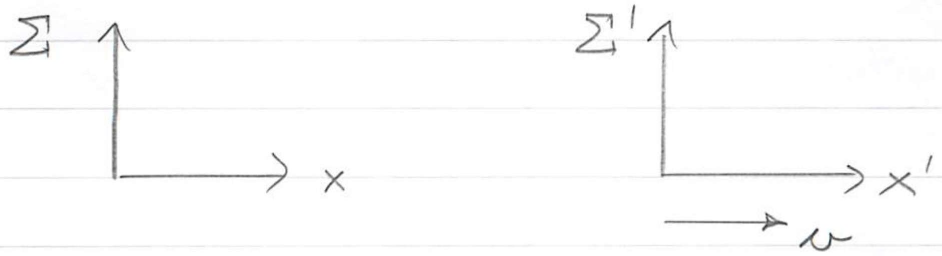
I. Elements of classical field theory

I.1. a reminder: special relativity (c=1)

postulates of SR

I.	laws of nature have the same form in all inertial frames
II.	the vacuum speed of light is the same in all inertial frames

(II) implies that transformation between inertial frames is no longer Galilei trafo but Lorentz trafo: which mixes space and time



$$\vec{r}, t \longrightarrow x^\mu \hat{=} (t, \vec{r}) \qquad \mu = 0,1,2,3$$

$$\Sigma \to \Sigma': \qquad x^\mu \to x'^\mu = L^{\mu'}{}_\nu x^\nu$$

for $v \parallel$ x-axis: "boost" (c=1)

$$\cosh \xi = \frac{1}{\sqrt{1-v^2}} \qquad \sinh \xi = \frac{v}{\sqrt{1-v^2}}$$

L-Trafo:

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\xi & -\sinh\xi & & \\ -\sinh\xi & \cosh\xi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\det(L) = 1$$

$$L^{0}{}_{0} \geq 1 \quad \text{orthochrone}$$

more general trafo: add rotation

time reversal $T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

parity reversal $P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

▸ Minkowski space 4-dim

$x^{\mu} \hat{=} (t, \vec{r})$ • contravariant space-time coordinate

$p^{\mu} \hat{=} (E, \vec{p})$ • 4-momentum

metric tensor

$$g_{\mu\nu} \hat{=} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

scalar product (sum convention) $xy \equiv g_{\mu\nu} x^{\mu} x^{\nu}$

L-Trafo conserves *this* scalar product

$$x'^{\mu} = L^{\mu'}_{\ \nu} x^{\nu}$$

$$x'^{\mu} x'_{\mu} = \underbrace{L^{\mu'}_{\ \nu} L_{\mu'}^{\ \lambda}}_{=\delta_{\nu\lambda}} x^{\nu} x_{\lambda} = \delta_{\nu\lambda} x^{\nu} x_{\lambda} = x^{\nu} x_{\nu}$$

but norm of a L-vector no longer positive definite

$$xx = x_0^2 - \vec{x} \cdot \vec{x}$$

Separation between space-time points no longer positive definite $\quad s = x - y$

$$s^2 = (x-y)(x-y) = (t_x - t_y)^2 - (\vec{x} - \vec{y})^2$$

$s^2 > 0$	$s^2 = 0$	$s^2 < 0$
time-like separated	light-like separated	space-like separated

if $t_x > t_y$ $(x_0 > y_0)$
then $t'_x > t'_y$ $(x'_0 > y'_0)$
in all inertial frames

time order conserved
spatial order not
(causal!)

if $x_1 > y_1$
then $x'_1 > y'_1$
in all inertial frames

Space order conserved
time order not
(a-causal!)

introduce co-variant vector (components)

$$x_\mu \equiv g_{\mu\nu} x^\nu = (t, -\vec{r})$$

4-gradient

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

geometric objects

(i) t^μ Lorentz vector (tensor of rank 1) iff it transforms like space-time vector under L-trafo's between inertial frames:

$$x^\mu \to x'^\mu = L^{\mu'}{}_\nu x^\nu \quad \Rightarrow \quad t'^\mu = L^{\mu'}{}_\nu x^\nu$$

(ii) g Lorentz scalar invariant under L-trafo

important examples:
- charge e
- scalar product $x^\mu y_\mu$
- 4-divergence $\partial^\mu a_\mu$

(iii) $F^{\mu\nu}$ Lorentz tensor (of rank 2) iff

$$F'^{\mu\nu} = L^{\mu'}{}_\lambda L^{\nu'}{}_\sigma F^{\lambda\sigma}$$

I.2 Examples of elementary field theories

A. non-relativistic quantum mechanics: Schrödinger field

- complex, single-component field

$$\boxed{i\frac{\partial}{\partial t}\psi(\vec r,t) = -\frac{1}{2m}\Delta\psi(\vec r,t) + V(\vec r)\,\psi(\vec r,t)}$$

conservation law for positive definite density
$\rho = \psi^*\psi \geq 0$ (as we will see later this is a consequence of the continuous symmetry $\psi \to \psi e^{i\alpha}$)

$$\frac{\partial}{\partial t}\rho = -\vec\nabla\cdot\vec j \qquad \vec j = \frac{1}{2mi}\left(\psi^*\vec\nabla\psi - \psi\vec\nabla\psi^*\right)$$

B. Maxwell field (rationalized Gauß units, no "4π")

4-Potential $\quad A^\mu = (\phi,\vec A)$
$\qquad\qquad\quad A_\mu = (\phi,-\vec A)$

field tensor $\quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

$F^{\mu\nu} = -F^{\nu\mu}$ anti-symmetric $\Rightarrow$ 6 components real-valued

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

<u>dual field tensor</u>

$$\mathcal{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\sigma\lambda} F_{\sigma\lambda}$$

$\varepsilon^{\mu\nu\sigma\lambda}$ 4-dim. Levi-Civita

$$\boxed{\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\mu \mathcal{F}^{\mu\nu} = 0}$$

Maxwell equations

<u>gauge invariance</u> $A_\mu \to \tilde{A}_\mu = A_\mu - \partial_\mu \Lambda$

Λ - Lorentz scalar

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu = \partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\mu \partial^\nu \Lambda - \partial^\nu \partial^\mu \Lambda$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

<u>Lorentz gauge</u>

$$\partial_\mu A^\mu = \partial^\mu A_\mu = 0$$

$$\Rightarrow \quad \boxed{\Box A^\mu = \partial_\nu \partial^\nu A^\mu = 0}$$

<u>Maxwell-field with external sources</u>

4-current $j^\mu = (\rho, \vec{j})$ $\quad j_\mu = (\rho, -\vec{j})$

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0}$$

inhomogeneous MWE homogeneous MWE

consistency requirement $\partial_\nu j^\nu = 0$. (since $\underbrace{\partial_\nu \partial_\mu}_{\text{symmetric}} \underbrace{F^{\mu\nu}}_{\text{anti-symmetric}} = 0$)

$\hat{=}$ conservation of charge

then $\boxed{\Box A^\mu = j^\mu}$

C. Dirac field

ψ bispinor 4-component complex field

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

γ^μ : 4×4 matrices (<u>not</u> Lorentz vectors)

$$\gamma^\mu = (\beta, \beta\alpha_i)$$

where $\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$ $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

σ_i - Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \gamma^i = \beta \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

the following anti-commutator ("$\{.,.\}$") rule holds

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

furthermore

$$\gamma^{0\dagger} = \gamma^0 \quad \text{but} \quad \gamma^{i\dagger} = -\gamma^i$$

in all cases

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$$

▶ transformation of bispinor under L-trafo?

m · L-scalar $\qquad \partial_\mu$ L-vector

$$x^\mu \to x'^\mu = L^{\mu}{}_{\nu} x^\nu \qquad \psi(x) \to \psi'(x') = S\,\psi(x')$$

$$S = ?$$

Dirac equation form invariant if

$$S^{-1} \gamma^\mu S \overset{!}{=} L^\mu{}_\nu \gamma^\nu$$

rotation about k-axis	$S = e^{i\frac{\theta}{2}\Sigma^k} = \cos\frac{\theta}{2} + i\Sigma^k \sin\frac{\theta}{2}$
boost along k-axis	$S = e^{-\frac{\varphi}{2}\alpha^k} = \cosh\frac{\varphi}{2} + \alpha^k \sinh\frac{\varphi}{2}$

where $\Sigma^k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$ $\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$

θ - rotation angle about k-axis

$\cosh \varphi = \frac{1}{\sqrt{1-v^2}}$ boost in k-direction

▶ Scalars, L-vectors etc.

$\psi^+ \psi$ is not a L-scalar but $\bar{\psi}\psi$ where

$$\bar{\psi} = \psi^+ \gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

is the adjoint Dirac bi-spinor

scalar $\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - (|\psi_3|^2 + |\psi_4|^2)$

4-vector $\bar{\psi}\gamma^\mu \psi$

anti-symmetric tensor $\bar{\psi}\sigma^{\mu\nu}\psi$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$\sigma^{0j} = i\alpha^j$ $\sigma^{ij} = \Sigma^k$ i, j, k cyclic pseudo-scalar $\bar{\psi}\gamma^5\psi$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

▶ Dirac field coupled to external em. field

$$\boxed{\left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m\right]\psi = 0}$$

I.3 Lagrange-Hamilton formalism for fields

(A) a reminder: mechanics of point masses

$q_i, \dot{q}_i$ canonical coordinates & velocities

$L(q_i, \dot{q}_i, t)$ Lagrangian

Hamiltonian principle

$$\boxed{\delta \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t) = 0 \quad \text{with } \delta q_i(t_{1,2}) = 0}$$

$$\delta L = \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

partial integration

$$0 \overset{!}{=} \int_{t_1}^{t_2} dt\, \delta L = \sum_j \int dt \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right) \delta q_j$$

$$+ \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2} \Big\} = 0$$

independence of δq_j $\Rightarrow$

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0}$$

Euler-Lagrange equations

generalized momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian and canonical equations

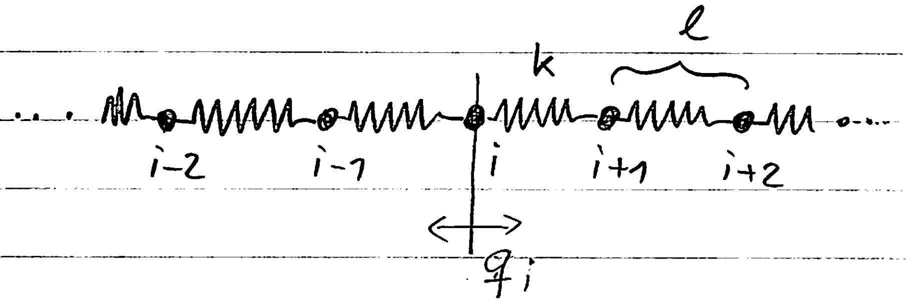
$$\boxed{H = \sum_j \dot{q}_j p_j - L} \qquad \boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad \dot{q}_j = \frac{\partial H}{\partial p_j}}$$

Poisson brackets $\quad \{f,g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$

$$\boxed{\dot{p}_j = \{p_j, H\}} \qquad \boxed{\dot{q}_j = \{q_j, H\}}$$

$$\frac{d}{dt} f = \{f, H\} + \frac{\partial f}{\partial t}$$

(B) the one-dimensional harmonic chain in an external force field



+ external force

periodic boundary conditions $\quad q_N = q_0$

$\Rightarrow$ N coupled harmonic oscillators with spring constant k and in external (harmonic) force field

$$T = \frac{m}{2} \sum_{j=0}^{N-1} \dot{q}_j^2$$

$$V = \underbrace{\frac{k}{2} \sum_{j=0}^{N-1} (q_{j+1} - q_j)^2}_{\text{springs}} + \underbrace{\frac{g}{2} \sum_{j=0}^{N-1} q_j^2}_{\text{external potential}}$$

Lagrangian

$$L = \sum_{j=0}^{N-1} \left[\frac{m}{2} \dot{q}_j^2 - \frac{k}{2} (q_{j+1} - q_j)^2 - \frac{g}{2} q_j^2 \right]$$

continuum limit

$$\ell \to 0,\ N \to \infty,\ m \to 0,\ k \to \infty$$

$$L = N\ell = \text{const} \qquad \mu = \frac{m}{\ell} = \text{const} \qquad \varkappa = k\ell = \text{const}$$

$$\gamma = \frac{g}{\ell} = \text{const}$$

$$q_j(t) = q(z_j, t) \longrightarrow q(z,t)$$

$$\sum_{j=0}^{N-1} \ell \longrightarrow \int_0^L dz$$

this yields

$$L = \int_0^L dz \left[\frac{\mu}{2} \left(\frac{\partial q}{\partial t} \right)^2 - \frac{\kappa}{2} \left(\frac{\partial q}{\partial z} \right)^2 - \frac{\mu'}{2} q^2 \right]$$

we see that the Lagrangian has the form

$$\boxed{L = \int_0^L dz\, \mathcal{L}\left(q(z,t), \partial_t q(z,t), \partial_z q(z,t); t \right)}$$

$\mathcal{L}$ – <u>Lagrange density</u>

similarly one finds

$$\boxed{H = \int_0^L dz\, \mathcal{H}}$$

$\mathcal{H}$ – <u>Hamiltonian density</u>

principle of minimum action

$$\delta \int_{t_1}^{t_2} dt\, L = \delta \underbrace{\int_{t_1}^{t_2} dt \int_0^L dz}_{\text{equal treatment of space \& time!}} \mathcal{L} = 0$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial q_j} = -K\left[-(q_{j+1}-q_j) + (q_j - q_{j-1})\right] - g q_j$$

$$= \varkappa \left[\frac{q_{j+1}-q_j}{\ell} - \frac{q_j - q_{j-1}}{\ell}\right] - \gamma \ell \, q_j$$

$$\xrightarrow[\ell \to 0]{} \ell \varkappa \frac{1}{\ell}\left(\left.\frac{\partial q}{\partial z}\right|_{z+\frac{\ell}{2}} - \left.\frac{\partial q}{\partial z}\right|_{z-\frac{\ell}{2}}\right) - \gamma \ell q$$

$$= \ell\left(\varkappa \frac{\partial^2 q}{\partial z^2} - \gamma q\right) = -\ell \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial z}\right)} + \ell \frac{\partial \mathcal{L}}{\partial q}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = m\ddot{q}_j = \mu \ell \frac{\partial^2 q}{\partial t^2} = \ell \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial t}\right)}$$

$\Rightarrow$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \longrightarrow \ell\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial t}\right)} - \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial z}\right)}\right]$$

$$\boxed{\underbrace{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial t}\right)} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial z}\right)}}_{\text{symmetric in space \& time}} - \frac{\partial \mathcal{L}}{\partial q} = 0}$$

where $\mathcal{L} = \mathcal{L}\left[q(z,t), \frac{\partial q(z,t)}{\partial t}, \frac{\partial q(z,t)}{\partial z}\right]$

(c) generalization: Euler-Lagrange equations for fields; Hamilton formalism for fields

What we have found for the harmonic chain can be generalized to arbitrary fields

$$L = \int dV \, \mathcal{L} \qquad \mathcal{L} = \mathcal{L}\left[\phi, \frac{\partial}{\partial x^\mu}\phi\right]$$

Lagrangian is a volume integral of the Lagrange density which depends on the field and all its partial derivatives

action principle (covariant!) $\qquad 0 = \delta \int_{t_1}^{t_2} dt \int dV \, \mathcal{L}\left[\phi, \frac{\partial}{\partial x^\mu}\phi\right]$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

Euler-Lagrange equations $\hat{=}$ field equations

canonical momentum

$$\dot{q}_j \rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \boxed{\pi(x^\mu) \equiv \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial q}{\partial t}\right)}}$$

problem: not L-invariant

Hamilton density

$$\boxed{\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L}}$$

remark: there is a covariant generalization of $\mathcal{H}$ but it is of not much help here, - we just have to live with this deficiancy

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} (\partial^\beta \phi) - g^{\alpha\beta}\mathcal{L} \qquad T^{00} = \mathcal{H}$$

canonical energy-momentum tensor

Note: in $\mathcal{L} \to \mathcal{H}$ we have only eliminated $\partial_0 \phi$ in favor of π; thus $\mathcal{H}$ still depends on the spatial derivatives $\partial_j \phi$

$$\mathcal{H} = \mathcal{H}[\phi, \pi, \partial_j \phi ; x^\mu] \qquad j = 1,2,3$$

The dependence of $\mathcal{H}$ on $\partial_j \phi$ will make the canonical formalism a bit more involved. Let's go through it:

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L}$$

$$\Rightarrow \quad \boxed{\partial_0 \phi = \frac{\partial}{\partial t}\phi = \frac{\partial \mathcal{H}}{\partial \pi}} \qquad \text{1st Hamilton equation}$$

Euler-Lagrange

$$\frac{\partial \pi}{\partial t} = \frac{\partial}{\partial x^0} \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^j} \frac{\partial \mathcal{L}}{\partial(\partial_j \phi)}$$

$$\boxed{\frac{\partial \pi}{\partial t} = -\left[\frac{\partial \mathcal{H}}{\partial \phi} - \frac{\partial}{\partial x^j} \frac{\partial \mathcal{H}}{\partial(\partial_j \phi)}\right]} \qquad \text{2nd Hamilton eq.}$$

$\uparrow$ additional terms

Def.: <u>functional derivative</u>

let $F = F[\phi(x), \phi'(x)] = \int dx \; f(\phi(x), \phi'(x))$

$$\frac{\delta F}{\delta \phi(x_0)} \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int dx \left\{ f\left[\phi(x) + \varepsilon\, \delta(x-x_0), \frac{d}{dx}\left(\phi(x) + \varepsilon\, \delta(x-x_0)\right)\right] - f\left[\phi(x), \frac{d}{dx}\phi(x)\right] \right\}$$

$$= \lim_{\varepsilon \to 0} \int dx \left[\frac{\partial f}{\partial \varphi} \delta(x-x_0) + \frac{\partial f}{\partial \varphi'} \frac{d}{dx} \delta(x-x_0) \right]$$

$$\underset{\text{part. Integr.}}{=} \int dx \left(\frac{\partial f}{\partial \varphi(x)} - \frac{d}{dx} \frac{\partial f}{\partial \varphi'(x)} \right) \delta(x-x_0) + \underbrace{\text{surface terms}}_{=0}$$

$$\boxed{\frac{\delta F}{\delta \phi(y)} \equiv \left(\frac{\partial f}{\partial \varphi} - \frac{d}{dx} \frac{\partial f}{\partial \varphi'} \right)_{x=y}}$$

With this we can write the Hamilton equations in a form identical to those of mechanics

$$\boxed{\frac{\partial}{\partial t} \phi(\vec{x},t) = \frac{\delta H}{\delta \pi(\vec{x},t)}} \qquad \boxed{\frac{\partial}{\partial t} \pi(\vec{x},t) = -\frac{\delta H}{\delta \phi(\vec{x},t)}}$$

Now we have to generalize the Poisson bracket to fields:

Def: Poisson brackets

let $F(t) = \int d^3x \; f(\phi, \pi, \partial_j \phi, x^\mu)$

$G(t) = \int d^3x \; g(\phi, \pi, \partial_j \phi, x^\mu)$

note that we can use a small trick to write the fields $\phi(\vec{x},t)$ and $\pi(\vec{x},t)$ as functionals

$$\phi(\vec{y},t) = \int d^3x \; \phi(\vec{x},t)\, \delta(\vec{x}-\vec{y})$$

$$\pi(\vec{y},t) = \int d^3x \; \pi(\vec{x},t)\, \delta(\vec{x}-\vec{y})$$

$$\boxed{\{F, G\} \equiv \int d^3x \; \frac{\delta F}{\delta \phi(\vec{x})} \frac{\delta G}{\delta \pi(\vec{x})} - \frac{\delta F}{\delta \pi(\vec{x})} \frac{\delta G}{\delta \phi(\vec{x})}}$$ Poisson bracket

with this we find

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} \phi(\vec{x},t) &= \{\phi(\vec{x},t), H\} \\ \frac{\partial}{\partial t} \pi(\vec{x},t) &= \{\pi(\vec{x},t), H\} \end{aligned}}$$

canonical equations of classical field theory

proof:
$$\frac{\delta \phi(\vec{x},t)}{\delta \phi(\vec{y},t)} = \Bigg(\underbrace{\frac{\partial \phi(x)}{\partial \phi(x)}}_{=1} - \frac{d}{dx^j} \underbrace{\frac{\partial \phi(x)}{\partial(\partial_j \phi(x))}}_{=0} \Bigg) \delta(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y})$$

i.e.

$$\{\phi(\vec{y},t), H\} = \int d^3x \left(\underbrace{\frac{\delta\phi(\vec{y},t)}{\delta\phi(\vec{x},t)}}_{\delta(\vec{x}-\vec{y})} \frac{\delta H}{\delta\pi(\vec{x},t)} - \underbrace{\frac{\delta\phi(\vec{y},t)}{\delta\pi(\vec{x},t)}}_{0} \frac{\delta H}{\delta\phi(\vec{x},t)} \right)$$

$$= \frac{\delta H}{\delta\pi(\vec{y},t)} = \frac{\partial\phi(\vec{y},t)}{\partial t} \qquad \square$$

furthermore we find the <u>fundamental Poisson brackets</u>

$$\boxed{\begin{gathered} \{\phi(\vec{y},t), \phi(\vec{z},t)\} = \{\pi(\vec{y},t), \pi(\vec{z},t)\} = 0 \\ \{\phi(\vec{y},t), \pi(\vec{z},t)\} = \delta^{(3)}(\vec{y}-\vec{z}) \end{gathered}}$$

<u>proof:</u>

$$\{\phi(\vec{y}), \pi(\vec{z})\} = \int d^3x \left(\frac{\delta\phi(\vec{y})}{\delta\phi(\vec{x})} \frac{\delta\pi(\vec{z})}{\delta\pi(\vec{x})} - \underbrace{\frac{\delta\phi(\vec{y})}{\delta\pi(\vec{x})}}_{0} \underbrace{\frac{\delta\pi(\vec{z})}{\delta\phi(\vec{x})}}_{0} \right)$$

$$= \int d^3x \, \delta^{(3)}(\vec{x}-\vec{y}) \, \delta^{(3)}(\vec{x}-\vec{z}) = \delta^{(3)}(\vec{y}-\vec{z})$$

$\square$