I.5 Noether theorem for fields: continuous symmetries and conservation laws

We know from classical mechanics that continuous symmetries of the Lagrangian are related to conservation laws. We will now see that the same is true for fields, however in a more general form:

Noether theorem

For each continuous transformation of coordinates x^μ and fields ϕ^i which leaves the action in any finite volume invariant there is a function of the fields and their derivatives which is a conserved quantity.

Let ε^i be a set of infinitesimal parameter that characterizes the transformation, i.e.

$$x^\mu \longrightarrow x'^\mu = x^\mu + \Lambda^\mu_i(x)\,\varepsilon^i$$

$$\Psi_\alpha \longrightarrow \Psi'_\alpha(x') = \Psi_\alpha(x) + \Omega_{\alpha i}(x)\,\varepsilon^i$$

and
$$S_V = \int_V d^4x\,\mathcal{L}(x) = S'_V = \int_{V'} d^4x'\,\mathcal{L}'(x')$$

invariance of action

The index "i" numbers the different symmetry transformations.

$$\boxed{\partial_\mu O^\mu_i = 0} \quad \text{in } V$$

local conservation law

$i = 1,2,3,\dots N$ conservation laws

where

$$\boxed{O^\mu_i = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)}\left[\frac{\partial \psi_\alpha}{\partial x^\nu}\lambda^\nu_i - \Omega_{\alpha i}\right] - \mathcal{L}\lambda^\mu_i}$$

the conservation law implies a conserved Noether charge

$$\boxed{Q_i \equiv \int d^3r \; O^0_i(\vec{r},t) = \text{const}_t}$$

proof: $\quad \mathcal{L}(x) \longrightarrow \mathcal{L}'(x') \quad$ (only one ε!)

$$S' = \int_{V'} d^4x' \mathcal{L}'(x') \overset{!}{=} S = \int_V d^4x \, \mathcal{L}(x)$$

$$0 = S' - S = \int_{V'} d^4x' \left(\mathcal{L}'(x') - \mathcal{L}(x')\right) + \int_{V'} d^4x' \mathcal{L}(x') - \int_V d^4x \, \mathcal{L}(x)$$

$$\underset{\substack{\text{1st order} \\ \text{in } \varepsilon}}{=} \int_V d^4x \left(\mathcal{L}'(x) - \mathcal{L}(x)\right) + \int_V d^4x \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\left(\mathcal{L}(x) + \frac{\partial \mathcal{L}(x)}{\partial x^\mu}\frac{\partial x^\mu}{\partial \varepsilon}\varepsilon\right) - \int_V d^4x \, \mathcal{L}(x)$$

Here $\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$ is the Jacobi-determinant needed to transform the integral $\int_{V'} d^4x' \longrightarrow \int_V d^4x \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$

Expanding to first order in ε

- $$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det\left(1 + \frac{\partial \lambda^\mu}{\partial x^\nu}\varepsilon\right) \cong 1 + \frac{\partial \lambda^\mu}{\partial x^\mu}\varepsilon$$

- $$\delta\mathcal{L}(x) \equiv \mathcal{L}'(x) - \mathcal{L}(x) = \underline{\frac{\partial \mathcal{L}}{\partial \varphi_\alpha}\delta\varphi_\alpha} + \underline{\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_\alpha}{\partial x^\mu}\right)}\delta\left(\frac{\partial \varphi_\alpha}{\partial x^\mu}\right)}$$

- $$\delta\varphi_\alpha = \varphi_\alpha'(x) - \varphi_\alpha(x) = \underbrace{\varphi_\alpha'(x) - \varphi_\alpha'(x')}_{-\frac{\partial \varphi_\alpha}{\partial x^\mu}\delta x^\mu} + \underbrace{\varphi_\alpha'(x') - \varphi_\alpha(x)}_{=\Omega_\alpha(x)\varepsilon}$$

$$\delta x^\mu = \lambda^\mu \varepsilon$$

i.e.

$$\delta\varphi_\alpha = \left(-\frac{\partial \varphi_\alpha}{\partial x^\mu}\lambda^\mu + \Omega_\alpha\right)\varepsilon$$

similarly

$$\delta\left(\frac{\partial \varphi_\alpha}{\partial x^\nu}\right) = \frac{\partial}{\partial x^\nu}\left(-\frac{\partial \varphi_\alpha}{\partial x^\mu}\lambda^\mu + \Omega_\alpha\right)\varepsilon$$

this yields

$$0 = \int_V d^4x \left\{ \underline{\frac{\partial \mathcal{L}}{\partial \varphi_\alpha}\left(\Omega_\alpha - \frac{\partial \varphi_\alpha}{\partial x^\nu}\lambda^\nu\right)} + \underline{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)}\frac{\partial}{\partial x^\nu}\left(\Omega_\alpha - \frac{\partial \varphi_\alpha}{\partial x^\nu}\lambda^\nu\right)} \right.$$

$$\left. + \underline{\frac{\partial \mathcal{L}}{\partial x^\mu}\lambda^\mu} + \underline{\frac{\partial \lambda^\mu}{\partial x^\mu}\mathcal{L}} \right\}\varepsilon$$

use of the Euler-Lagrange equations allows to eliminate $\partial\mathcal{L}/\partial\varphi_\alpha$

$$\frac{\partial \mathcal{L}}{\partial \psi_\alpha} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)}$$

$\Rightarrow$

$$0 = \int d^4x \, \frac{\partial}{\partial x^\mu} \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} \left(\Omega_\alpha - \frac{\partial \psi_\alpha}{\partial x^\nu} \lambda^\nu \right) + \lambda^\mu \mathcal{L} \right\}}_{= -\mathcal{O}^\mu} \varepsilon$$

$\Rightarrow$ $$\partial_\mu \mathcal{O}^\mu = 0 \quad \square$$

example: invariance of Dirac theory under phase transformations

$$x \rightarrow x' = x$$

$$\psi(x) \rightarrow \psi'(x) = e^{-iq\varepsilon} \psi(x) \simeq \psi(x) - iq\varepsilon \psi(x)$$

$\Rightarrow \quad \lambda^\mu \equiv 0 \qquad \Omega = -iq\,\psi(x)$

conserved current

$$\mathcal{O}^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Omega = iq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi$$

Dirac $\quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i \bar{\psi} \gamma^\mu \quad \boxed{\mathcal{O}^\mu = -q \bar{\psi} \gamma^\mu \psi = -j^\mu_D}$

The continuous phase symmetry of Lagrangedensities of complex valued fields such as the Dirac field implies the <u>conservation of charge</u>

$$\boxed{\partial_\mu j^\mu_D = 0}$$

I.6 invariance under local phase transformations

We have seen that the Dirac theory as well as other field theories of complex fields is invariant under a <u>global</u> phase transformation (global $U(1)$ symmetry)

$$\psi(x) \longrightarrow \psi'(x) = e^{iq\theta}\psi(x)$$

What happens if the phase transformation is generalized to a local one? $\quad \theta \to \theta(x^\mu)$

Dirac: $\quad \psi \to \psi'(x) = e^{iq\theta(x)}\psi(x) \qquad \bar\psi \to \bar\psi'(x) = e^{-iq\theta(x)}\bar\psi(x)$

$$\mathcal{L}_D \to \mathcal{L}_D' = \frac{1}{2}\bar\psi'(i\gamma^\mu \overrightarrow{\partial_\mu} - m)\psi' + \frac{1}{2}\bar\psi'(-i\gamma^\mu \overleftarrow{\partial_\mu} - m)\psi'$$

$$= \mathcal{L}_D + q\bar\psi\gamma^\mu\psi\,\partial_\mu\theta(x) \neq \mathcal{L}_D$$

$\Rightarrow$ Lagrange density does <u>not</u> remain invariant under a <u>local</u> phase transformation.

Interestingly invariance can be restored if we add a coupling to a vector field A_μ

$$\mathcal{L}_D \rightarrow \mathcal{L} = \mathcal{L}_D - j^\mu A_\mu$$

and if this vector field transforms simultaneously like

$$\boxed{A_\mu \longrightarrow A'_\mu = A_\mu(x) + \partial_\mu \theta(x)} \qquad (*)$$

which is nothing else than a gauge transformation with gauge $\theta(x)$

$(*)$ together with

$$\boxed{\psi \longrightarrow \psi'_{(x)} = e^{iq\theta(x)} \psi_{(x)}} \qquad (**)$$

constitutes a joint local gauge-phase-transformation.

Under such a transformation one has

$$\mathcal{L}' = \mathcal{L}'_D - j'^\mu A'_\mu = \mathcal{L}_D - j^\mu A_\mu = \mathcal{L}$$

If one assumes that the vector field A_μ which is also called a gauge field is a dynamical quantity with a (free) Lagrangian

$$\mathcal{L}_F = \alpha F_{\mu\nu} F^{\mu\nu}$$

one has in this way constructed a minimal version

of an interacting field theory which is invariant under local gauge-phase transformations.
These types of theories play an important role in quantum field theory and are called

$$\boxed{\text{gauge field theories}}$$

If Ψ is the Dirac field and A^μ the gauge potential of electromagnetism the requirement of local gauge invariance (invariance under local gauge-phase-transformations) yields the minimum coupling Lagrangian of QED

$$\boxed{\begin{aligned}\mathcal{L}_{MWD} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\Psi}(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m)\Psi \\ - q\bar{\Psi}\gamma^\mu\Psi A_\mu\end{aligned}}$$

Note that the value of the charge q is however undetermined.

One recognizes that the minimum coupling term can be generated from the Dirac Lagrangian with the substitution

$$\boxed{\begin{aligned}\overrightarrow{\partial}_\mu &\longrightarrow \overrightarrow{D}_\mu \equiv \overrightarrow{\partial}_\mu + iqA_\mu \\ \overleftarrow{\partial}_\mu &\longrightarrow \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - iqA_\mu\end{aligned}}$$

principle of minimum coupling

<u>covariant derivatives</u>