I.4. Important examples for field Lagrangians

How should a Lagrange-density of a relativistic field theory look like?

- Lorentz invariance $\Rightarrow$ $\mathcal{L}$ is Lorentz scalar and independent on x^μ
- linear field theory $\Rightarrow$ $\mathcal{L}$ is at most bilinear or quadratic in ϕ and $\partial_\mu\phi$
- field equations at most 2nd order DE $\Rightarrow$ $\mathcal{L}$ only function of ϕ and $\partial_\mu\phi$
- local field theory $\Rightarrow$ $\mathcal{L} = \mathcal{L}[\phi(x), \partial_\mu\phi(x)]$ but not of fields at different space-time points
- dynamical quantities real $\Rightarrow$ $\mathcal{L}$ real

(A) Schrödinger field

non-relativistic, complex field, only first derivative in time

$$\boxed{\mathcal{L} = -\frac{i}{2}(\partial_t\psi^*)\psi + \frac{i}{2}\psi^*(\partial_t\psi) - \frac{1}{2m}\vec{\nabla}\psi^*\vec{\nabla}\psi}$$

derivation of field equations and Hamilton density
→ problem course

<u>note:</u> $\mathcal{L}$ is invariant under a continuous transformation

$$\psi \rightarrow \psi e^{i\varphi}$$

We will see later that this has an important consequence. There will be a conserved quantity that allows to interpret $|\psi|^2$ as probability density. More on this later.

(B) Klein-Gordon field

scalar, real field; relativistic theory $\phi(x^\mu)$

only bilinear Lorentzscalars $\partial_\nu\phi\partial^\nu\phi$ and ϕ^2

$$\boxed{\mathcal{L}_{KG} = \frac{1}{2}\partial_\nu\phi\partial^\nu\phi - \frac{m^2}{2}\phi^2}$$

Euler Lagrange (note $\partial_\nu\phi$ and $\partial^\nu\phi$ <u>not</u> independent)

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial x^\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = -m^2\phi - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}\phi$$

$$\Rightarrow \quad \boxed{(\partial_\nu\partial^\nu + m^2)\phi = (\Box + m^2)\phi = 0}$$

Klein-Gordon equation

canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = \partial_0\phi = \dot{\phi}$$

$$\Rightarrow \quad \mathcal{H}_{KG} = \dot{\phi}\pi - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\partial_j\partial^j\phi + \frac{m^2}{2}\phi^2$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2$$

$$\boxed{H_{KG} = \frac{1}{2}\int dV\left(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)}$$

Hamiltonian of Klein-Gordon field

(c) electromagnetic field

6 real fields, L-invariant ; fieldtensor $F^{\mu\nu}$

Lorentzscalar that is quadratic in $F^{\mu\nu} \Rightarrow F_{\mu\nu}F^{\mu\nu}$

$$\boxed{\mathcal{L}_{ED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{B}^2}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

canonical field variable: $\quad A^\nu$ or A_ν

Euler-Lagrange ($\partial_\mu A_\nu$ not independent on $\partial^\mu A^\nu$)

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = 0 + 4 \frac{\partial}{\partial x^\nu} \frac{1}{4} F^{\nu\mu} = 0$$

$$\boxed{\partial_\nu F^{\nu\mu} = 0}$$

inhomogeneous MWE without sources

or:

$$\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = 0$$

$$\Box A^\mu - \partial^\mu \partial_\nu A^\nu = 0$$

Lorentz gauge $\partial_\nu A^\nu = 0$ $\Rightarrow \Box A^\mu = 0$

add external sources j_μ $\Rightarrow$ a simple L-scalar is $j_\mu A^\mu$

$$\Rightarrow \boxed{\mathcal{L}_{ED} \to \mathcal{L}_{ED} - j_\mu A^\mu = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu}$$

Euler Lagrange get additional term

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -j^\mu$$

thus

$$\boxed{\partial_\nu F^{\nu\mu} = j^\mu}$$

inhomogeneous MWE with sources

the homogeneous MWE follow from a relation which is always true $\rightarrow$ problem courses

In order to have a consistent theory we need to have

$$\partial_\mu j^\mu = \underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} \underbrace{F^{\nu\mu}}_{\text{antisymmetric}} = 0$$

This is of course nothing else than charge conservation which must hold for the external sources.

gauge transformations

the field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
is invariant under the continuous gauge transformation

$$A^\mu \to \tilde{A}^\mu = A^\mu + \partial^\mu \Lambda(x)$$

since $\tilde{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + \underbrace{\partial^\mu \partial^\nu \Lambda - \partial^\nu \partial^\mu \Lambda}_{=0}$

The Lagrange density of the em. field coupled to external sources is <u>not</u> gauge invariant. But the Lagrangian is gauge invariant up to a canonical transformation

$$L \to \tilde{L} = \int dV \left(\mathcal{L} - j_\mu \partial^\mu \Lambda \right)$$

$$= \int dV \left(\mathcal{L} - \partial^\mu (j_\mu \Lambda) + \Lambda \underbrace{\partial^\mu j_\mu}_{=0} \right)$$

$$= \int dV \left[\mathcal{L} - \partial^k (j_k \Lambda) - \partial^0 (j_0 \Lambda) \right]$$

volume integral over 3-divergence $\Rightarrow$ surface term (Gauß law) $\Rightarrow 0$

$$L' = L - \underbrace{\frac{d}{dt} \int dV \, j_0 \Lambda(x)}_{\text{adding a total time derivative is a canonical trafo that does not change the physics}}$$

canonical momenta of $A^\mu = (A^0, A^1, A^2, A^3)$

$\Rightarrow$ here we note a problem

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = 0 \quad !$$

i.e. $\nexists$ canonical conjugate field to A^0. The poisson bracket between A^0 and π^0 always vanishes!

In order to fix this problem we have to <u>fix the gauge</u>

e.g. $\quad \partial_\mu A^\mu \overset{!}{=} 0 \quad$ Lorentz gauge

or $\quad \nabla \cdot \vec{A} = \partial_j A^j = 0 \quad$ Coulomb gauge

There are different ways to do this:

(i) <u>elimination of A^0; Coulomb gauge</u>

This is the conceptually simpler approach but comes at the price of giving up explicit covariance

$$\vec{\nabla}\cdot\vec{A}=0 \qquad \text{i.e. } \vec{A}=\vec{A}_\perp$$

With this we can formally eliminate A^0 from the theory

$$\partial_\mu F^{\mu 0} = j^0 = \rho$$

$$\partial_\mu\left[\partial^\mu A^0-\partial^0 A^\mu\right] = -\Delta A^0 - \partial_0\vec{\nabla}\cdot\vec{A} = -\Delta A^0 = \rho$$

only $\mu=1,2,3$ contribute

solution $$A^0(\vec{r},t)=\frac{1}{4\pi}\int d^3r'\,\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$ Coulomb law

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-j_\nu A^\nu$$

$$=-\frac{1}{2}\partial_\mu A^0\left(\partial^\mu A^0-\partial^0A^\mu\right)+\frac{1}{2}\partial_\mu A^k\left(\partial^\mu A^k-\partial^k A^\mu\right)$$

$$-\rho A^0 - j_k A^k$$

$$=-\frac{1}{2}\partial_k A^0\left(\partial^k A^0-\partial^0 A^k\right)+\frac{1}{2}\partial_0 A^k\left(\partial^0 A^k-\partial^k A^0\right)$$

$$-\frac{1}{2}\partial_j A^k\left(\partial_j A^k-\partial_k A^j\right) \quad -\rho A^0+\vec{j}\cdot\vec{A}$$

lower index $\Rightarrow$ sign

lets simplify the terms

(i) $\partial_j A^k (\partial_j A^k - \partial_k A^j) = \vec{B}^2$ since $B_i = \varepsilon_{ijk} \partial_j A^k$

(ii) $\partial_k A^0 \partial^k A^0 = \underbrace{\cancel{\partial_k (A^0 \partial^k A^0)}}_{\substack{\text{3-Divergence} \\ \text{vanishes in } L = \int d^3r \mathcal{L}}} + \underbrace{A^0 \underbrace{\partial_k \partial^k A^0}_{= \Delta A^0}}_{= -A^0 \varrho}$

(iii) $\partial_k A^0 \partial_0 A^k = \underbrace{\cancel{\partial_0 (\partial_k A^0 \cdot A^k)}}_{\substack{\text{total time derivative in } L. \\ \text{can be eliminated by} \\ \text{canonical trafo}}} - A^k \partial_0 \partial_k A^0$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \varrho A_0 - \frac{1}{2} A^k \partial_0 \partial_k A^0 + \frac{1}{2} \underbrace{\partial_0 A^k \partial^0 A^k}_{= \vec{E}_\perp^2}$$

$$- \frac{1}{2} A^k \partial_0 \partial_k A^0$$

$$- \frac{1}{2} \vec{B}^2 - \varrho A_0 + \vec{j} \vec{A}$$

since $\vec{E} = -\frac{\partial}{\partial t} \vec{A} - \vec{\nabla} A^0 = \underbrace{-\partial_0 \vec{A}}_{= \vec{E}_\perp} - \underbrace{\vec{\nabla} A^0}_{= \vec{E}_\parallel}$

(remember $\vec{A} = \vec{A}_\perp$)

so

$$\mathcal{L} = \frac{1}{2} (\vec{E}_\perp^2 - \vec{B}^2) - \frac{1}{2} \varrho A_0 - \underbrace{(j^k - \partial_0 \partial_k A^0) A_k}_{= ?}$$

$$j^k - \partial_0 \partial_k A^0$$

now $\quad \vec{j}_\perp = \vec{j} - \partial_t \vec{\nabla}\phi$:

since $\quad \vec{\nabla}\cdot\vec{j}_\perp = \vec{\nabla}\cdot\vec{j} - \partial_0 \vec{\nabla}\cdot\vec{\nabla}\phi$

$$= \vec{\nabla}\cdot\vec{j} - \partial_t \Delta\phi$$

$$= \vec{\nabla}\cdot\vec{j} + \partial_t \rho = 0 \qquad \square$$

$\Rightarrow$

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}_\perp^{\,2} - \vec{B}_\perp^{\,2}\right) - \frac{1}{2}\rho A_0 + \vec{j}_\perp\cdot\vec{A}_\perp$$

neither $\vec{E}_\perp$ nor $\vec{B}_\perp$ depend on A_0. If we finally eliminate A_0 in $\frac{1}{2}\rho A_0$ by Coulomb's law we arrive at

$$\boxed{L = \int d^3r\, \mathcal{L} = \int d^3r \left\{ \frac{1}{2}\left(\vec{E}_\perp^{\,2} - \vec{B}_\perp^{\,2}\right) + \vec{j}_\perp\cdot\vec{A}_\perp \right\} - \frac{1}{8\pi}\int d^3r \int d^3r' \, \frac{\rho(\vec{r},t)\,\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}}$$

Lagrange function of the elm. field with external sources in (non Lorentz invariant) Coulomb gauge

Hamilton formalism:

field: $\quad \boxed{A_\perp^i(\vec{r},t) = A^i(\vec{r},t)}$

conjugate field: $$\boxed{\pi^i(\vec{r},t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\perp^i)} = -\frac{\partial A_\perp^i}{\partial t} = -E_\perp^i}$$

<u>fundamental poisson brackets</u>

$$\boxed{\begin{aligned}\left\{A_\perp^i(\vec{r},t),\, \pi^j(\vec{r}',t)\right\} &= \left\{A_\perp^i(\vec{r},t),\, -E_\perp^j(\vec{r}',t)\right\} \\ &= \left(\overleftrightarrow{\delta}_\perp^{(3)}(\vec{r}-\vec{r}')\right)_{ij}\end{aligned}}$$

due to the transversal character of $\vec{A}_\perp$ and $\vec{E}_\perp$ the following relations must be fulfilled

$$0 = \vec{\nabla}_{\vec{r}} \cdot \left\{\vec{A}_\perp(\vec{r},t),\, \vec{E}_\perp(\vec{r}',t)\right\} = \vec{\nabla}_{\vec{r}'} \cdot \left\{\vec{A}_\perp(\vec{r},t), \vec{E}_\perp(\vec{r}',t)\right\}$$

$\Rightarrow$ $\delta^{(3)}(\vec{r},\vec{r}')$ must be replaced by its transverse counterpart, such that

$$\vec{\nabla}_{\vec{r}} \cdot \overleftrightarrow{\delta}_\perp^{(3)}(\vec{r},\vec{r}') = \vec{\nabla}_{\vec{r}'} \cdot \overleftrightarrow{\delta}_\perp^{(3)}(\vec{r},\vec{r}') = 0$$

<u>interlude: transversal and longitudinal vector fields</u>

it is always possible to decompose $\vec{F} = \vec{F}_\perp + \vec{F}_\parallel$

where $\vec{\nabla} \cdot \vec{F}_\perp = 0$ and $\vec{\nabla} \times \vec{F}_\parallel = 0$

construction of $\vec{F}_\perp$ and $\vec{F}_\parallel$?

we know

$$\delta^{(3)}(\vec{r}-\vec{r}') = -\frac{1}{4\pi}\Delta\frac{1}{|\vec{r}-\vec{r}'|}$$

$\Rightarrow$

$$\vec{F}(\vec{r}) = \int d^3\vec{r}'\ \delta^{(3)}(\vec{r}-\vec{r}')\ \vec{F}(\vec{r}')$$

$$= -\frac{1}{4\pi}\int d^3r'\ \Delta\frac{1}{|\vec{r}-\vec{r}'|}\ \vec{F}(\vec{r}')$$

$$= \frac{1}{4\pi}\int d^3r'\ \vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$$

$\Rightarrow$

$$\vec{F}_{\parallel}(\vec{r}) = \frac{1}{4\pi}\int d^3r'\ \vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) \qquad \vec{\nabla}\times\vec{F}_{\parallel}=0$$

$$\boxed{\overset{\leftrightarrow}{\delta}{}^{(3)}_{\parallel}(\vec{r}-\vec{r}') = \frac{1}{4\pi}\int d^3r'\ \vec{\nabla}\circ\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}}$$

$$\boxed{\overset{\leftrightarrow}{\delta}{}^{(3)}_{\perp}(\vec{r}-\vec{r}') = \delta^{(3)}(\vec{r}-\vec{r}') - \overset{\leftrightarrow}{\delta}{}^{(3)}_{\parallel}(\vec{r}-\vec{r}')}$$

Fourier

$$\overset{\leftrightarrow}{\delta}{}^{(3)}_{\parallel}(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\ \frac{\vec{k}\circ\vec{k}}{k^2}\ e^{i\vec{k}\cdot\vec{r}}$$

$$\overset{\leftrightarrow}{\delta}{}^{(3)}_{\perp}(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\left(\mathbb{1}-\frac{\vec{k}\circ\vec{k}}{k^2}\right)e^{i\vec{k}\vec{r}}$$

Hamiltonian of the elm. field in Coulomb gauge and in external fields

$$\boxed{\begin{aligned} H &= \int d^3r \left(\vec{\pi} \cdot \frac{\partial \vec{A}}{\partial t} - \mathcal{L} \right) \\ &= \int d^3r \left\{ \frac{1}{2} \left(\vec{E}_\perp^2 + \vec{B}_\perp^2 \right) - \vec{j}_\perp \cdot \vec{A}_\perp \right\} \\ &\quad + \frac{1}{8\pi} \int d^3r \int d^3r' \quad \frac{\rho(\vec{r},t)\,\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} \end{aligned}}$$

without sources

$$\boxed{H = \int d^3r \; \frac{1}{2} \left(\vec{E}_\perp^2 + \vec{B}_\perp^2 \right)}$$

An alternative approach to circumvent the problem of π^0 is the so-called Gupta-Bleuler approach

<u>(ii) gauge-fixing term in Lagrange density for Lorentz gauge</u>

If one chooses the Lorentz gauge

$$\partial_\mu A^\mu = 0$$

One can add a term

$$\mathcal{L}_g = -\frac{1}{2\alpha} \left(\partial_\mu A^\mu \right) \left(\partial_\nu A^\nu \right)$$

to the Lagrangian that does not affect the physics as long as the Lorentz gauge is used later on. Now one can argue the other way around: Let's not make the Lorentz gauge a priori and

$$\mathcal{L}_{EM} \rightarrow \mathcal{L}_{EM} + \mathcal{L}_g = \tilde{\mathcal{L}}_{EM}$$

$\tilde{\mathcal{L}}_{EM}$ contains $\partial_0 A^0$ and thus π^0 can be defined. This will allow to quantize all 4 field components. We can then however not apply the Lorentz gauge on the level of field operators but must require that for all relevant states $|\psi\rangle$ holds

$$\partial_\mu \hat{A}^\mu |\psi\rangle = 0 \quad \Leftarrow \quad \cancel{\partial_\mu \hat{A}^\mu = 0}$$

(D) Dirac field

Lagrange density should contain first derivatives only in a linear way

Lorentz scalars that are linear in derivatives are

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \quad \text{and} \quad (\partial_\mu \bar{\psi})\gamma^\mu \psi$$

Mass term $\qquad m\bar{\psi}\psi$

$\Rightarrow$

$$\boxed{\mathcal{L}_D = \bar{\psi}\left[\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right]\psi}$$

Lagrange density of Dirac field

where $\overleftrightarrow{\partial}_\mu \equiv \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$

i.e. $\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi = \bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi$

Since ψ is a complex-valued object ψ and $\bar{\psi}$ need to be treated independently (alternatively one can consider real and imaginary parts separately). Thus the Euler Lagrange equations yield (see problem course)

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}_D}{\left(\partial \frac{\partial \bar{\psi}}{\partial x^\mu}\right)} - \frac{\partial \mathcal{L}_D}{\partial \bar{\psi}} = -\frac{i}{2}\gamma^\mu \partial_\mu \psi - \frac{i}{2}\gamma^\mu \partial_\mu \psi + m\psi = -(i\gamma^\mu \partial_\mu - m)\psi = 0$$

i.e. the Dirac equation for ψ. Similarly one finds the Dirac equation for $\bar{\psi}$

<u>conjugate field</u>

$$\boxed{\begin{aligned} \pi &\equiv \frac{\partial \mathcal{L}_D}{\partial(\partial_0 \bar{\psi})} = -\frac{i}{2}\gamma^0 \psi \\ \bar{\pi} &\equiv \frac{\partial \mathcal{L}_D}{\partial(\partial_0 \psi)} = \frac{i}{2}\bar{\psi}\gamma^0 = \frac{i}{2}\psi^+ \end{aligned}}$$

fields and conjugate fields are the same!

remark: π is a column vector, $\bar{\pi}$ is a row vector convention: use only ψ and ψ^+ as conjugates

fundamental Poisson brackets

$$\boxed{\{\psi(\vec{r},t),\ \psi^+(\vec{r}\,',t)\}_P = -i\,\delta^{(3)}(\vec{r}-\vec{r}\,')}$$

note absence of 1/2 due to $\psi \sim \pi$ and $\psi^+ \sim \bar{\pi}$
doubling

Hamiltonian density

$$\mathcal{H}_D = \bar{\pi}\,\frac{\partial \psi}{\partial t} + \frac{\partial \bar{\psi}}{\partial t}\,\pi - \mathcal{L} = \frac{i}{2}\,\bar{\psi}\gamma^0 \frac{\overleftrightarrow{\partial}}{\partial t}\psi - \mathcal{L}_D$$

row column row column

$$\boxed{\mathcal{H}_D = \bar{\psi}\left[-\frac{i}{2}\gamma^k \frac{\overleftrightarrow{\partial}}{\partial x^k} + m\right]\psi = \psi^+\left[-\frac{i}{2}\alpha_k \frac{\overleftrightarrow{\partial}}{\partial x^k} + \beta m\right]\psi}$$

α_k, β Dirac matrices

$$\boxed{H_D = \int d^3r\,\mathcal{H}_D \underset{\text{part. Int.}}{=} \int d^3r\ \psi^+\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right]\psi}$$

note: $\mathcal{L}_D$ is invariant under a global phase transformation

$$\boxed{\psi \to \psi e^{i\theta}}$$