With this we find

$$\overline{X(t)} = \overline{X(0)}\, e^{-kt} \tag{368}$$

$$\overline{\Delta X(t)^2} = \overline{\Delta X(0)^2}\, e^{-2kt} + \frac{D}{2k}\left(1 - e^{-2kt}\right) \tag{369}$$

I.e. the mean value decays to zero and the fluctuations approach the steady state value

$$\boxed{\left.\overline{\Delta X(t)^2}\right|_{t\to\infty} = \frac{D}{2k}} \tag{370}$$

# VII.2 Relation between Fokker-Planck equation and Ito SDE

## (A) Single variable

Let $X(t)$ a stochastic variable with initial condition

$$X(t): \quad X(t_0) = X_0$$

Then the average of any function of $x(t)$ can be written as

$$\overline{f(x(t))} = \int_{-\infty}^{\infty} dx\, f(x)\, p(x,t\,|\,x_0,t_0) \tag{371}$$

where $p(x,t|x_0,t_0)$ is the conditional probability to find the value $x$ at time $t$, provided it had been $x_0$ at time $t_0$

Using Ito's formula (356) we find the EOM of $\overline{f(x(t))}$ using $\overline{\xi} = 0$

$$\frac{d\,\overline{f(x(t))}}{dt} = \int_{-\infty}^{\infty} dx\, f(x)\, \frac{\partial}{\partial t} p(x,t\,|\,x_0,t_0)$$

$$\overset{(356)}{=} \overline{A(x(t),t)\frac{\partial f}{\partial x}} + \overline{\frac{1}{2} B(x(t),t)^2 \frac{\partial^2 f}{\partial x^2}} \tag{372}$$

$$= \int_{-\infty}^{\infty} dx \left[A(x,t)\frac{\partial f}{\partial x} + \frac{1}{2} B(x,t)^2 \frac{\partial^2 f}{\partial x^2}\right] p(x,t\,|\,x_0,t_0)$$

We now integrate by parts assuming

vanishing boundary terms

$$p(\pm\infty, t\,|\,x_0,t_0) = 0 \tag{373}$$

$$= \int_{-\infty}^{\infty} dx\, f(x) \left[ -\frac{\partial}{\partial x}\Big(A(x,t)\, p\Big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(B^2(x,t)\, p\Big) \right]$$

Since this must be true for any function $f(x)$ we arrive at

$$\frac{dx(t)}{dt} = A(x(t),t) + B(x(t),t)\, \xi(t)$$

Ito SDE

with $x(t_0) = x_0$

$$\Updownarrow$$

$$\frac{\partial}{\partial t} p(x,t|x_0,t_0) = -\frac{\partial}{\partial x} A(x,t)\, p(x,t|x_0,t_0) + \frac{1}{2}\frac{\partial^2}{\partial x^2} B(x,t)^2\, p(x,t|x_0,t_0)$$

(374)

Fokker-Planck eq.

where

(375) $$\overline{f(x(t))} = \int_{-\infty}^{\infty} dx\, f(x)\, p(x,t|x_0,t_0)$$

Eq. (374) relates a FPE for a quasi-probability to an Ito SDE which can be numerically simulated if $B^2$ is positive.

## (B) Positivity of diffusion

We have seen that the diffusion coefficient of the FPE is related to the noise term of the Ito SDE via

(376) $$D(x) = B(x)^2$$

Thus a real-valued $B(x)$ is only obtained if

(377) $$D(x) \geq 0$$ positive diffusion

$$\boxed{\text{Thus a mapping from a FPE to an Ito stochastic DE is only possible for a positive diffusion.}}$$

This is consistent with what we have seen e.g. for the Ornstein-Uhlenbeck process for which we found the solution (369)

$$\overline{\Delta x^2(t)} = \overline{\Delta x^2(0)}\, e^{-2kt} + \frac{D}{2k}\left(1 - e^{-2kt}\right)$$

For $D<0$ we would get for large enough times $t$

$$\overline{\Delta x^2(t)} < 0 \quad \lightning$$

One could be tempted to map the FPE for the real variable $x$ to a Ito SDE of a complex stochastic variable $y(t)$, such that for

(378) $$D = -|D| \stackrel{?}{\Longrightarrow} B = i\sqrt{|D|}$$

and

(379) $$\frac{dy}{dt} = A(y) + i\sqrt{|D|}\,\xi(t)$$

with $y(0) = x(0) = y^*(0)$ i.e. $\mathrm{Im}[y(0)] = 0$.
This is however inconsistent as then

(380) $$\frac{dy^*}{dt} = A(y) - i\sqrt{|D|}\,\xi(t)$$

and

(381) $$\frac{d}{dt}\mathrm{Im}[y] = \sqrt{|D|}\,\xi(t)$$

so the imaginary part of $y(t)$ would build up fluctuations.

## (C) Multi-variable case

For all interesting application the FPE, resp. the quasi-probability $P$ has multiple variables

So let us consider a set of n SDEs for the variables

$$\underline{x}(t) \equiv (x_1(t), x_2(t), \dots, x_n(t))^T$$

which can be written in vector form

(382) $$\boxed{d\underline{x}(t) = \underline{A}(x,t)dt + \underline{\underline{B}}(x,t)\,d\underline{W}}$$

where

(383) $$\underline{A} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}$$

are the drift vector, B is an $n \times n$ matrix

and

(384) $$d\underline{W} = \begin{pmatrix} dW_1 \\ dW_2 \\ \vdots \\ dW_n \end{pmatrix}$$

is a n-dimensional vector of independent stochastic increments (n-variable Wiener process)

Then we have to apply the many-variable version of Ito's equation which we obtain from

$$
(385) \quad \begin{aligned}
dW_i dW_j &= \delta_{ij}\, dt \\
(dW_i)^{n+2} &= 0 \qquad n > 0 \\
dW_i\, dt &= 0 \\
dt^{n+1} &= 0 \qquad n > 0
\end{aligned}
$$

i.e. treating $dW_i$ as independent infinitesimal object of degree 1/2. Following the same lines as in VII.1B we arrive at

$$
(386) \quad \boxed{\begin{aligned}
df(\underline{x}) = \Big[ &\sum_j A_j(\underline{x},t) \frac{\partial f}{\partial x_j} \\
&+ \frac{1}{2} \sum_{j,e} \left( \underline{\underline{B}}(\underline{x},t)\, \underline{\underline{B}}^T(\underline{x},t) \right)_{je} \frac{\partial^2 f}{\partial x_j \partial x_e} \Big] dt \\
&+ \sum_{j,e} B_{je}(\underline{x},t) \frac{\partial f}{\partial x_j}\, dW_e
\end{aligned}}
$$

This then gives the following correspondence between a multi-variable FPE and a set of Ito SDE:

$$d\underline{x}(t) = \underline{A}(\underline{x}(t),t)dt + \underline{\underline{B}}(\underline{x}(t),t)\, d\underline{W}(t) \qquad \text{Ito SDE's}$$

$$\text{with} \quad \underline{x}(0) = \underline{x}_0$$

$$\frac{\partial p(\underline{x},t|\underline{x}_0,0)}{\partial t} = -\sum_j \frac{\partial}{\partial x_j} A_j(\underline{x},t)\, p(\underline{x},t|\underline{x}_0,0) + \frac{1}{2}\sum_{j,l} \frac{\partial^2}{\partial x_j \partial x_l} \left( \underline{\underline{B}}(\underline{x},t)\, \underline{\underline{B}}^T(\underline{x},t) \right)_{jl} p(\underline{x},t|\underline{x}_0,0) \tag{387}$$

multivariable FPE

We note the relation between the diffusion matrix $\underline{\underline{D}}(\underline{x},t)$ and the matrix $\underline{\underline{B}}$ in the SDE

$$\boxed{\underline{\underline{D}}(\underline{x},t) = \underline{\underline{B}}(\underline{x},t)\, \underline{\underline{B}}^T(\underline{x},t)} \tag{388}$$

Thus a mapping from a FPE to a set of coupled SDE's is only possible if a matrix $\underline{\underline{B}}$ with real eigenvalues exists that fulfills (388)

# (D) equivalent SDE's

We realize that all systems of SDE's with

$$\underline{\underline{B}} \longrightarrow \underline{\underline{B}}' = \underline{\underline{B}}\,\underline{\underline{S}} \tag{389}$$

where $\underline{\underline{S}}$ is an orthogonal matrix, i.e.

$$\underline{\underline{S}}^T \underline{\underline{S}} = 1 \tag{390}$$

are equivalent to the same FPE.

This means that

$$d\underline{x}(t) = \underline{A}\,dt + \underline{\underline{B}}\,d\underline{W} \tag{391}$$

and

$$d\underline{x}(t) = \underline{A}\,dt + \underline{\underline{B}}\,\underline{\underline{S}}\,d\underline{W} \tag{392}$$

are equivalent.

Thus

$$\boxed{d\underline{V} = \underline{\underline{S}}\,d\underline{W}} \tag{393}$$

gives an alternative n-variable Wiener process

$$dV_i\,dV_j = \delta_{ij}\,dt \qquad \text{etc.}$$

Thus we can make a symmetric transformation

(394) $$\boxed{\underline{y}(t) = \underline{\underline{S}}\,\underline{x}(t)}$$

$$d\underline{y}(t) = \underline{\underline{S}}\,\underline{A}\,dt + \underline{\underline{S}}\,\underline{\underline{B}}\,\underline{\underline{S}}^T\,\underline{\underline{S}}\,d\underline{W}$$

(395) $$\boxed{d\underline{y}(t) = \underline{A}'dt + \underline{\underline{B}}'d\underline{V}}$$

where

(396) $$\boxed{\underline{A}' = \underline{\underline{S}}\,\underline{A} \quad \text{and} \quad \underline{\underline{B}}' = \underline{\underline{S}}\,\underline{\underline{B}}\,\underline{\underline{S}}^T}$$

and $d\underline{V}$ is formally identical to $d\underline{W}$

## application: cartesian to polar coordinates

In quantum optics the positive (negative) frequency component of the electric field are often described in terms of real and imaginary parts.

E.g.

(397)
$$dE_1(t) = -\gamma E_1(t)\,dt + \varepsilon\, dW_1(t)$$
$$dE_2(t) = -\gamma E_2(t)\,dt + \varepsilon\, dW_2(t)$$

where $dW_1$ and $dW_2$ are independent. It is often of interest to work with polar coordinates amplitude and phase

(398) $$E_1(t) = a(t)\cos\phi(t) \qquad E_2(t) = a(t)\sin\phi(t)$$

We set

(399) $$\mu(t) = \ln a(t)$$

i.e.

(400) $$\mu(t) + i\phi(t) = \ln\left[E_1(t) + iE_2(t)\right]$$

Using Ito calculus we find (expand to 2nd order)

$$d(\mu(t) + i\phi(t)) = \frac{d(E_1+iE_2)}{E_1+iE_2} - \frac{[d(E_1+iE_2)]^2}{2(E_1+iE_2)^2}$$

(401) $$= -\frac{\gamma(E_1+iE_2)}{E_1+iE_2}dt + \frac{\varepsilon[dW_1+idW_2]}{E_1+iE_2} - \frac{\varepsilon^2[dW_1+idW_2]^2}{2(E_1+iE_2)^2}$$

Noting that $dW_1^2 = dt$ ; $dW_2^2 = dt$ ; $dW_1 dW_2 = 0$ we get

(402) $$d(\mu(t) + i\phi(t)) = -\gamma dt + \varepsilon e^{(-\mu(t) - i\phi(t))}(dW_1 + idW_2)$$

Inverting (399) yields

$$da(t) = \left(-\gamma a(t) + \frac{\varepsilon^2}{2a(t)}\right)dt$$
(403)
$$+ \varepsilon\left(\cos\phi(t)\, dW_1 + \sin\phi(t)\, dW_2\right)$$

and

(404) $$d\phi(t) = \frac{\varepsilon}{a(t)}\left(-\sin\phi(t)\, dW_1 + \cos\phi(t)\, dW_2\right)$$

We note that we now can use a symmetric trafo

(405) $$\begin{bmatrix} dW_a \\ dW_\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}$$

to obtain

$$d\phi(t) = \frac{\varepsilon}{a(t)}\, dW_\phi$$
(406)
$$da(t) = \left[-\gamma a(t) + \frac{\varepsilon^2}{2a(t)}\right]dt + \varepsilon\, dW_a$$

for the phase and amplitude.