# VII. Damping and Decoherence: Diffusion and equivalence to stochastic DE

Let us now consider an interacting Bose system with damping and/or decoherence, e.g.

$$H = \hbar\omega \hat{a}^{\dagger}\hat{a} + \hbar\frac{\eta}{2}\hat{a}^{\dagger 2}\hat{a}^{2}$$

(318)
$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \frac{\gamma}{2}(\bar{n}+1)\{2a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a^{\dagger}a\} + \frac{\gamma}{2}\bar{n}\{2a^{\dagger}\rho a - aa^{\dagger}\rho - \rho aa^{\dagger}\}$$

With the translation rules for the Wigner function (for example) we find the generalized FPE

(319)
$$\frac{\partial W}{\partial t} = \left[\frac{\partial}{\partial\alpha}\left(\frac{\gamma}{2} + i\omega + i\eta(|\alpha|^2-1)\right)\alpha + \text{c.c.}\right]W + \gamma\left(\bar{n}+\frac{1}{2}\right)\frac{\partial^2}{\partial\alpha\partial\alpha^*}W - i\frac{\eta}{4}\cancel{\frac{\partial^2}{\partial\alpha\partial\alpha^*}\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^*\right)}W$$

In TWA the third-order term is neglected.

What is left is now however a partial differential equation with 2nd order derivatives, i.e. a free

**Fokker-Planck equation with positive diffusion**

note that with $\alpha = x + iy$ $\quad \alpha^* = x - iy$

$$\frac{\partial}{\partial \alpha} = \frac{\partial x}{\partial \alpha}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \alpha}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \tag{320}$$

$$\frac{\partial}{\partial \alpha^*} = \frac{\partial x}{\partial \alpha^*}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \alpha^*}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

so

$$\frac{\partial^2}{\partial \alpha \partial \alpha^*} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{321}$$

and $\quad D_{xx} = D_{yy} = 1 \qquad D_{xy} = D_{yx} = 0$

so

$$\underline{\underline{D}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{diffusion matrix} \tag{322}$$

## VII.1 Ito and Stratonovich stochastic differential equations

We will now see that a FPE with nonvanishing but positive definite diffusion matrix can be mapped to stochastic ordinary differential equations

Consider a classical variable $x_\mu(t)$ obeying

$$(323) \qquad \boxed{\frac{dx_\mu(t)}{dt} = A_\mu[x_\nu] + B_\mu[x_\nu]\,\xi_\mu(t)}$$

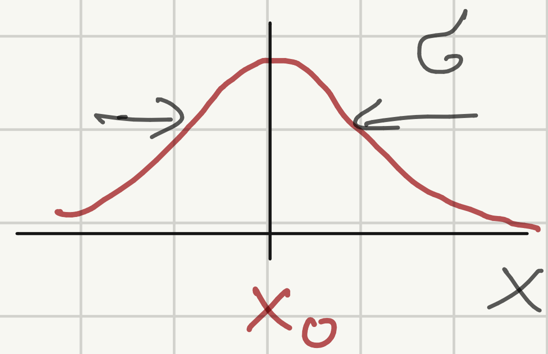
with $\xi_\mu(t)$ being a stochastic function of time which is Gaussian and Markovsch, i.e.

$$(324) \qquad \boxed{\overline{\xi_\mu(t)} = 0 \qquad \overline{\xi_\mu(t)\,\xi_\nu(t')} = D_{\mu\nu}\,\delta(t-t')}$$

## Gaussian random variables

let $x$ be a random variable with probability $p(x)$

$$(325) \qquad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$




$$(326) \qquad \overline{x} = \int dx\, p(x)\, x = x_0$$

$$(327) \qquad \overline{(x-x_0)^2} = \int dx\, p(x)\,(x-x_0)^2 = \sigma^2$$

All higher moments can be expressed in terms of $x_0$ and $\sigma^2$, i.e. expressed in terms of the lowest moments.

Def: Cumulants $\langle\langle x^n \rangle\rangle$

$$\boxed{\begin{aligned}\chi(y) &\equiv \overline{\exp\{yx\}}^{\,x} = \int dx\, p(x)\, e^{yx} \\ &= \exp\left\{\sum_{n=1}^{\infty} \frac{y^n}{n!} \langle\langle x^n \rangle\rangle\right\}\end{aligned}} \tag{328}$$

Cumulant generating function

$$\langle\langle x^n \rangle\rangle = \frac{\partial^n}{\partial y^n} \ln \chi(y)\Big|_{y=0} \tag{329}$$

For a Gaussian distribution holds

$$\begin{aligned} \langle\langle x \rangle\rangle &= \bar{x} = x_0 \\ \langle\langle x^2 \rangle\rangle &= \overline{(x-x_0)^2} = \overline{x^2} - \bar{x}^2 = \sigma^2 \\ \langle\langle x^3 \rangle\rangle &= \overline{x^3} - 3\,\overline{x^2}\,\bar{x} + 2\bar{x}^3 = 0 \end{aligned} \tag{330}$$

and so $\boxed{\langle\langle x^n \rangle\rangle \equiv 0} \qquad n > 2$ (331)

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The problem with eq. (323) is that if $\xi_\mu(t)$ is a Markovian ($\delta$-correlated) noise $x_\mu(t)$ is not differentiable! So we have to interpret it as an integral equation

$$x_\mu(t) = x_\mu(t_0) + \int_{t_0}^{t} d\tau\, A_\mu(x_\nu(\tau)) + \int_{t_0}^{t} d\tau\, \xi_\mu(\tau)\, B_\mu(x_\nu(\tau)) \tag{332}$$

We define

$$(333) \qquad dt\,\xi_\mu(\tau) \equiv dW_\mu(\tau)$$

stochastic increment

▲ How do we have to understand the integral

$$\int_{t_0}^{t} dW(\tau)\, B[x(\tau)] \;?$$

In the usual sense of calculus we write

$$(334) \qquad \int_{t_0}^{t} dW(\tau)\, B[x(\tau)] \equiv \lim_{n\to\infty} \sum_{i=1}^{n} B(\tau_i)\left[W(t_i)-W(t_{i-1})\right]\Delta\tau$$

where $t_0 \le t_1 \le t_2 \le \cdots \le t_n = t \qquad \Delta\tau = t_i - t_{i-1}$

and $\tau_i$ is taken from the interval

$$(335) \qquad \tau_i \in [t_{i-1}, t_i] \qquad \text{which } \tau\,?$$

Here we have to consider

$$(336) \qquad \Delta W_i \equiv \left(W(t_i) - W(t_{i-1})\right)\Delta\tau$$

as independent Gaußian statistical variables
i.e.

$$(337) \qquad \boxed{\overline{\Delta W_i} \equiv 0 \qquad \overline{\Delta W_i \Delta W_j} = \delta_{ij}\,\Delta\tau} \; *$$

* since $dW = dt\,\xi(t)$

$$\overline{dW(t_1)dW(t_2)} = \overline{\xi(t_1)\xi(t_2)dt_1dt_2} = \delta(t_1-t_2)dt_1dt_2$$

continuous time $\rightarrow$ discrete time:

$$\delta(t_1-t_2) \rightarrow \frac{\delta_{12}}{\Delta t} \qquad dt \rightarrow \Delta t$$

Now $B(\tau_i) = B[x(\tau_i)]$ and $x(\tau_i)$ with $\tau_i \in [t_{i-1}, t_i]$ can be statistically dependent on $\Delta W_i$. Two definitions

(338) Stratonovich $$\boxed{B(\tau_i) = \frac{B(t_i)+B(t_{i-1})}{2}}$$

- here $B(\tau_i)$ is statistically dependent on $\Delta W_i$

(339) Ito $$\boxed{B(\tau_i) = B(t_{i-1})}$$

- here $B(\tau_i)$ is independent on $\Delta W_i$

The Ito choice (339) is the physically reasonable one since in evaluating the stochastic integral we should assume that

$B(\tau)$ is a non-anticipating function i.e. it should not depend on $W(\tau)$ in the future. Unfortunately for an Ito SDE ordinary calculus does not apply !

## (A) Ito calculus

To illustrate the Ito stochastic integral let us calculate

$$I = \int_{t_0}^{t} W(t')\, dW(t') \tag{340}$$

in ordinary calculus we would obtain

$$\frac{1}{2}\left(W(t)^2 - W(t_0)^2\right)$$

Lets see $\quad (W(t_i) = W_i)$

$$I = \lim_{n\to\infty} S_n$$

$$S_n = \sum_{i=1}^{n} W_{i-1}\left(W_i - W_{i-1}\right) = \sum_{i=1}^{n} W_{i-1}\, \Delta W_i$$

$$= \frac{1}{2}\sum_{i=1}^{n} (W_{i-1} + \Delta W_i)^2 - (W_{i-1})^2 - (\Delta W_i)^2$$

$$= \frac{1}{2}\left(W(t) - W(t_0)\right)^2 - \frac{1}{2}\sum_{i=1}^{n} (\Delta W_i)^2$$

after taking the average the last term becomes

$$\frac{1}{2}\sum_{i=1}^{n}\Delta W_i^2 = \frac{1}{2}\sum_{i=1}^{n}\Delta T_i = \frac{1}{2}(t-t_0)$$

so

$$(341)\quad \boxed{I=\int_{t_0}^{t} dW(t')\,W(t') = \frac{1}{2}\left(W(t)^2-W(t_0)^2\right)-\frac{1}{2}(t-t_0)}$$

rem.: in the Stratonovich calculus we would obtain the ordinary result

$$(342)\quad I_S=\int_{t_0}^{t} dW(t')\,W(t') = \frac{1}{2}\left(W(t)^2-W(t_0)^2\right)$$

## ▶ non-anticipating functions and powers of dW

Def: A function $f(t)$ is called non-anticipating if it is statistically independent on $W(s)-W(t)$ for all $s,t$ $s>t$. We always have this!

If we consider only integrals with non-anticipating functions we can identify

$$(343)\quad \boxed{\begin{aligned}(dW)^2 &\equiv dt\\ (dW)^{2+n} &\equiv 0\end{aligned}}$$

▶ integration of polynomials

$$d(W(t))^n = \left(W(t) + dW(t)\right)^n - W(t)^n$$

$$= \sum_{r=1}^{n} \binom{n}{r} W(t)^{n-r} \underbrace{(dW(t))^r}_{\text{zero for } r>2}$$

$$= n(W(t))^{n-1} dW(t) + \frac{n(n-1)}{2} W(t)^{n-2} dt$$

i.e.

(344) $$\boxed{d(W^n) = n W^{n-1} dW + \frac{n(n-1)}{2} W^{n-2} dt}$$

so that from the first term on the r.h.s we find

$$\int_{t_0}^{t} W(t')^n dW(t') = \int_{t_0}^{t} \frac{d(W^{n+1})}{n+1} - \frac{n}{2} \int_{t_0}^{t} W^{n-1} dt'$$

(345) $$\boxed{\int_{t_0}^{t} W(t')^n dW(t') = \frac{1}{n+1}\left[W(t)^{n+1} - W(t_0)^{n+1}\right] - \frac{n}{2} \int_{t_0}^{t} W(t')^{n-1} dt'}$$

## expectation values

For a non-anticipating function (and we only have to consider those!) it holds

(346) $$\boxed{\int_{t_0}^{t} \overline{f(t')\,dW(t')} = 0}$$

uncorrelated

With $dW = \xi\, dt$ this amounts to the rule

(347) $$\boxed{\text{"}\ \overline{f(t')\,\xi(t')} = 0\ \text{"}}$$

which comes in handy for calculations

## correlation formula

(348) $$\boxed{\overline{\int_{t_0}^{t} f_1(t')\,dW(t') \int_{t_0}^{t} f_2(t'')\,dW(t'')} = \int_{t_0}^{t} \overline{f_1(t')\,f_2(t')}\,dt'}$$

again using $dW = \xi(t)\,dt$ this can be written in the less strict sense as

(349) $$\boxed{\text{"}\;\overline{\xi(t')\,\xi(t'')} = \delta(t'-t'')\;\text{"}}$$

since

$$\overline{f_1(t')\,\xi(t')\,f_2(t'')\,\xi(t'')}$$

(350) $$= f_1(t')f_2(t'')\;\overline{\xi(t')\,\xi(t'')}$$

$$= f_1(t')f_2(t'')\,\delta(t'-t'')$$

## ▶ Ito's formula

consider an arbitrary function $f(W(t),t)$

(351) $$df(W(t),t) = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial f}{\partial W}dW + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}(dW)^2 + \frac{\partial^2 f}{\partial W\partial t}\,dW dt + \cdots$$

using the Ito rules

$$dt^2 = 0$$

(352) $$dW^2 = dt$$

$$dW dt = 0$$

we arrive at Ito's formula

$$(353) \quad \boxed{df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW}$$

## (B) Ito SDE

We have seen in Sec. (A) that the physical interpretation of a Langevin equation

$$(354) \quad \frac{dx}{dt} = A(x,t) + B(x,t)\,\xi(t)$$

is in terms of an Ito integral equation

$$(355) \quad x(t) - x(0) = \int_0^t dt'\, A(x(t'),t') + \int_{t_0}^t dW(t')\, B(x(t'),t')$$

If we consider a function of $x(t)$ we find similar to Ito's formula (353)

$$df(x(t)) = f(x(t) + dx(t)) - f(x(t))$$

$$= \frac{\partial f}{\partial x}dx(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx(t))^2 + \cdots$$

$$= \frac{\partial f}{\partial x}\left[A dt + B dW\right] + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}B^2 dW^2$$

i.e

$$\boxed{\frac{df}{dt} = A[x(t),t]\frac{\partial f}{\partial x} + \frac{1}{2}B^2[x(t),t]\frac{\partial^2 f}{\partial x^2} + B[x(t),t]\frac{\partial f}{\partial x}\xi(t)} \tag{356}$$

## (C) Relation between Ito and Stratonovich SDE

In Stratonovich formalism the usual calculus holds, i.e. it is helpful to have a relation between Ito and Stratonovich.

$$\frac{dx}{dt} = A[x,t] + B[x,t]\xi(t) \quad \text{Ito}$$

?

$$\frac{dx}{dt} = \alpha[x,t] + \beta[x,t]\xi_s(t)$$

Stratonovich

One can show that the equivalence requires

$$\alpha[x,t] = A[x,t] - \frac{1}{2}B[x,t]\frac{\partial}{\partial x}B[x,t] \tag{357}$$

$$\beta[x,t] = B[x,t]$$

i.e

$$\boxed{\begin{aligned} dx &= A(x)\,dt + B(x)\,dW && \text{Ito} \\ & && \Updownarrow \\ dx &= \left(A(x) - \frac{1}{2} B(x) \frac{\partial B(x)}{\partial x}\right) dt + B(x)\,dW && \text{Stratonovich} \end{aligned}} \tag{358}$$

## (D) Some important examples

- coefficients without x dependence

$$dx(t) = \underbrace{A(t)}_{\text{deterministic}}\,dt + B(t)\,dW(t) \tag{359}$$

This is a purely additive noise

solution

$$X(t) = X(t_0) + \int_{t_0}^{t} d\tau\, A(\tau) + \int_{t_0}^{t} dW(\tau)\, B(\tau)$$

$$\overline{X(t)} = \overline{X(t_0)} + \int_{t_0}^{t} d\tau\, A(\tau) \tag{360}$$

$$\overline{\left(X(t)-\overline{X(t)}\right)\left(X(t')-\overline{X(t')}\right)} = \overline{\Delta X(t_0)^2}$$

$$+ \overline{\int_{t_0}^{t} dW(\tau)\, B(\tau) \int_{t_0}^{t'} dW(\tau')\, B(\tau')}$$

(361)

$$= \overline{\Delta X(t_0)^2} + \int_{t_0}^{t} d\tau \int_{t_0}^{t'} d\tau'\, B(\tau) B(\tau') \underbrace{\overline{\xi(\tau)\xi(\tau')}}_{= \delta(\tau-\tau')}$$

$$= \overline{\Delta X(t_0)^2} + \int_{0}^{\min(t,t')} d\tau\, B(\tau)^2$$

- oscillator with fluctuating frequency

(362) $$\frac{dz}{dt} = i\left(\omega + \sqrt{2\gamma}\, \xi_S(t)\right) z \qquad z \in \mathbb{C}$$

noisy frequency

which is physically justified if interpreted as a Stratonovich SDE. Translating this into an Ito equation yields

(363) $$\frac{dz}{dt} = (i\omega - \gamma) z + i\sqrt{2\gamma}\, \xi(t) z$$

this gives a damping of the amplitude

(364) $$\overline{z(t)} = \overline{z(0)} \exp\left((i\omega - \gamma)t\right)$$

and

(365) $$\overline{z(t)z(0)} = \overline{z(0)^2} \exp\left((i\omega-\gamma)t\right)$$

- Ornstein - Uhlenbeck process

An important stochastic process is the Ornstein - Uhlenbeck process, described by

(366) $$\frac{dx}{dt} = -kx + \sqrt{D}\,\xi(t)$$

This can be solved easily with the substitution

$$y = x e^{kt}$$

$$\frac{dy}{dt} = \frac{dx}{dt} e^{kt} + k x e^{kt} = \sqrt{D} e^{kt} \xi(t)$$

Solution

$$y(t) = y(t_0) + \int_{t_0}^{t} d\tau \sqrt{D} e^{k\tau} \xi(\tau)$$

i.e.

(367) $$x(t) = x(0) e^{-kt} + \sqrt{D} \int_{t_0}^{t} e^{-k(t-t')} dW(t')$$

With this we find

$$\boxed{\begin{aligned} &(368) \quad \overline{X(t)} = \overline{X(0)}\, e^{-kt} \\ &(369) \quad \overline{\Delta X(t)^2} = \overline{\Delta X(0)^2}\, e^{-2kt} + \frac{D}{2k}\left(1 - e^{-2kt}\right) \end{aligned}}$$

I.e. the mean value decays to zero and the fluctuations approach the steady state value

$$\boxed{\left.\overline{\Delta X(t)^2}\right|_{t\to\infty} = \frac{D}{2k}} \tag{370}$$

## VII.2 Relation between Fokker-Planck equation and Ito SDE

Let $x(t)$ a stochastic variable with initial condition

$$x(t): \quad x(t_0) = x_0$$

Then the average of any function of $x(t)$ can be written as

$$\overline{f(x(t))} = \int_{-\infty}^{\infty} dx\, f(x)\, p(x,t\,|\,x_0,t_0) \tag{371}$$