VI. Truncated Wigner approximation and equivalence to ordinary differential equations

VI.1. General approach

We have seen in the last section that for a unitary system with interaction Hamiltonians that are at most bi-quadratic the EOM of the Wigner function has no second derivatives but third-order ones. Using a scaling argument (237) for highly occupied modes the higher-order derivatives can be neglected.

Then starting from the initial condition

(239) $$W(\{\alpha_j\}, t=t_0) = W_0(\{\alpha_j\})$$

one has to solve the partial differential equation

(240) $$\frac{\partial W}{\partial t} \approx -\sum_j \left(\frac{\partial}{\partial \alpha_j} A_j + \frac{\partial}{\partial \alpha_j^*} A_j^* \right) W$$

of first order. Here $A_j = A_j[\alpha_\ell, \alpha_\ell^*]$ are the drift coefficients

- Observables can then be calculated

(241) $$\left\langle f_s(\vec{a}_j, \vec{a}_j^{+}) \right\rangle = \prod_j \int d^2\alpha_j \, f_s(\alpha_j, \alpha_j^*) \, W(\alpha_j, t)$$

Solving eq. (240) is equivalent to solving ordinary differential equations. This can be seen as follows:

(i) consider a classical variable $X(t)$ with equation of motion

(242) $$\boxed{\frac{d}{dt} X(t) = A(X(t))}$$

and random initial condition

(243) $$\boxed{X(t_0) = X_0 \quad \text{with probability } W_0(X_0)}$$

(ii) introduce conditional probability $p(x, t \mid x_0, t_0)$ with

$$p(x, t_0 \mid x_0, t_0) = \delta(x - x_0)$$

such that the average of a function $f(x(t))$ can be written as

(244) $$\overline{f(x(t))} = \int dx_0 \int dx \, f(x) \, p(x, t \mid x_0, t_0) \, W_0(x_0)$$

(iii) Then the following EOM for $p(x, t \mid x_0, t_0)$ results

$$\frac{d}{dt} \overline{f(x(t))} = \overline{\frac{\partial f}{\partial x} \frac{d x(t)}{dt}}$$

(245)
$$= \int dx_0 \int dx \, \frac{\partial f}{\partial x} A(x) \, p(x,t|x_0,t_0) \, W_0(x_0)$$

$$= \int dx_0 \int dx \, f(x) \left(-\frac{\partial}{\partial x} A(x) P(x,t|x_0,t_0) W_0(x_0) \right)$$

part. Int.

and using (244)

(246)
$$\frac{d}{dt} \overline{f(x(t))} = \int dx_0 \int dx \, f(x) \, \dot{p}(x,t|x_0,t_0) \, W_0(x_0)$$

comparing the two results yields

(247)
$$\boxed{\frac{d}{dt} W(x,t) = -\frac{\partial}{\partial x} A(x) W(x,t)}$$

where

(248)
$$W(x,t) = \int dx_0 \, p(x,t|x_0,t_0) \, W_0(x_0)$$

i.e. $W(x,t_0) = W_0(x)$

Symmetrically ordered expectation values of bosonic operators can be calculated in Truncated Wigner approximation by solving the ordinary differential equations

(249)
$$\frac{d\alpha_j}{dt} = A_j(\{\alpha_\ell, \alpha_\ell^*\}).$$

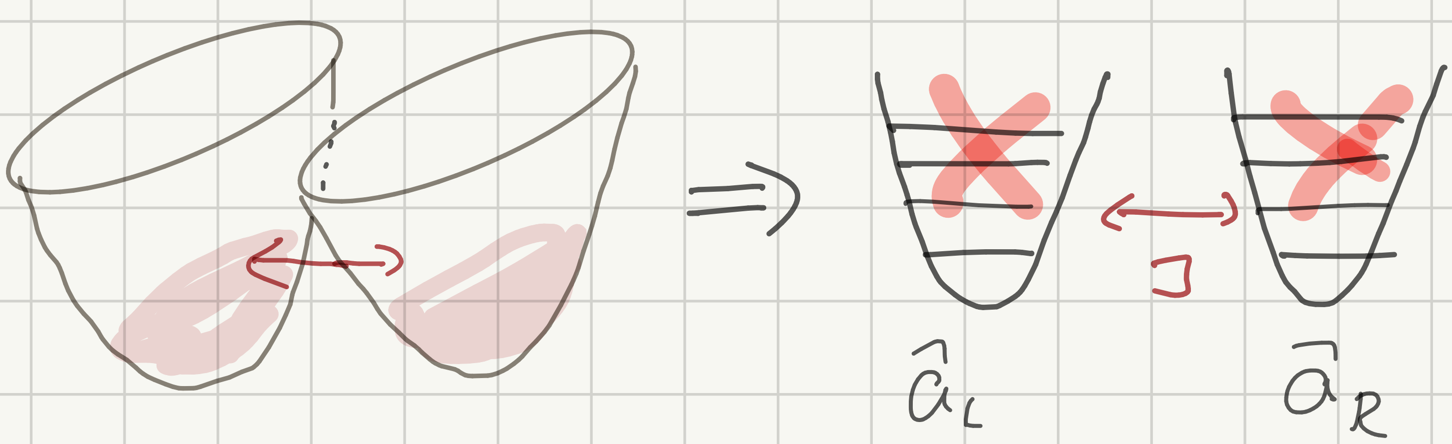
with initial condition $\alpha_j(t_0) = \alpha_{j0}$ and averaging the α_{j0} over the initial Wigner distribution, i.e.

$$\langle f_{symm}(\hat{a}_j, \hat{a}_j^+) \rangle = \prod_j \int d^2\alpha_{j0}\, f_s(\alpha_j(t), \alpha_j^*(t))\, W(\alpha_{j0}) \tag{250}$$

VI.2 A simple example: Two coupled single-mode condensates

Let us start with a simple example of two coupled modes of bosons with high number of particles. This is an approximation to a BEC in a double-well potential where higher order spatial modes in the two wells are ignored

Phys. Rev. A 55, 4318 (1997); Phys. Rev. A 68 033609 (2003).



$$\boxed{H = -J(\hat{a}_L^+ \hat{a}_R + h.c.) + \frac{U(t)}{2} \sum_{\alpha=L,R} \hat{n}_\alpha(\hat{n}_\alpha - 1)} \tag{251}$$

(A) Mean-field approach

Let us start from full Heisenberg equations:

(252)
$$\dot{\hat{a}}_L = -iJ\hat{a}_R - iU(t)\,\hat{a}_L^\dagger \hat{a}_L \hat{a}_L$$
$$\dot{\hat{a}}_R = -iJ\hat{a}_L - iU(t)\,\hat{a}_R^\dagger \hat{a}_R \hat{a}_R$$

mean field approximation $\hat{a} \to \langle \hat{a} \rangle \equiv \alpha$

(253)
$$\dot{\alpha}_L = -iJ\alpha_R - iU(t)\,|\alpha_L|^2\alpha_L$$
$$\dot{\alpha}_R = -iJ\alpha_L - iU(t)\,|\alpha_R|^2\alpha_R$$

parameterization: N total number of particle

(254)
$$\tilde{\alpha}_L \equiv \frac{\alpha_L}{N} \qquad \tilde{\alpha}_R \equiv \frac{\alpha_R}{N} \qquad \boxed{\lambda(t) \equiv \frac{U(t)\,N}{J}}$$

$$\tau \equiv Jt$$

(255)
$$\frac{d}{d\tau}\tilde{\alpha}_L = -i\tilde{\alpha}_R - i\lambda(t)\,|\tilde{\alpha}_L|^2\tilde{\alpha}_L$$
$$\frac{d}{d\tau}\tilde{\alpha}_R = -i\tilde{\alpha}_L - i\lambda(t)\,|\tilde{\alpha}_R|^2\tilde{\alpha}_R$$

ansatz

(256)
$$\tilde{\alpha}_{L,R} = \sqrt{1 \mp n}\; e^{i\theta \mp i\phi/2}$$

$\Rightarrow$ eqs. (255) become (Phys. Rev. A 66 053607 (2002))

(257)
$$\frac{d^2 n}{d\tau^2} + 4n + 4\lambda n\sqrt{1-n^2}\,\cos\phi = 0$$

and

$$\frac{d}{dn}\cos\phi = \frac{n}{1-n^2}\cos\phi + \frac{\lambda n}{\sqrt{1-n^2}} \tag{258}$$

which can be reduced to a single equation

$$\frac{d^2 n}{dt^2} + 4n + 4\lambda n\left(\cos\phi_0 + \frac{\lambda n^2}{2}\right) = 0 \tag{259}$$

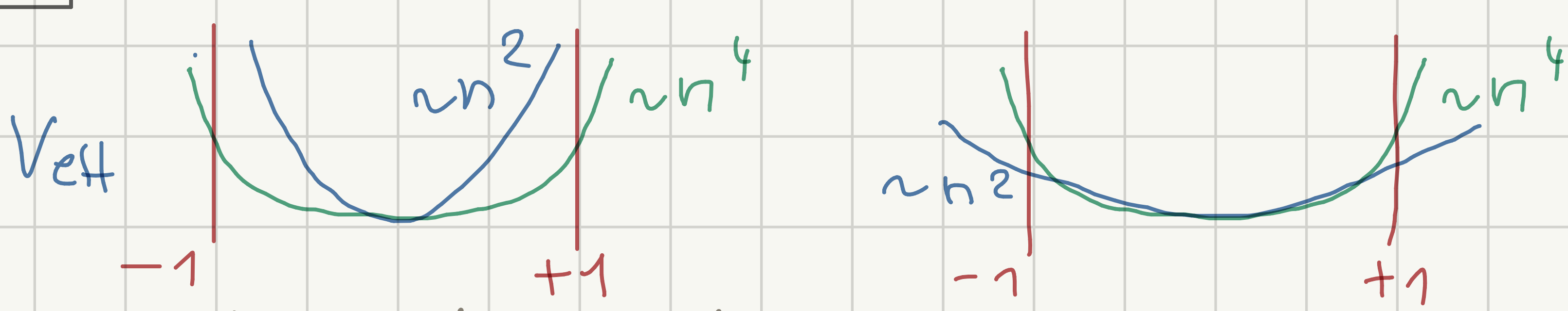
where $n(0) = n_0$ $\quad \frac{dn(0)}{dt} = 2\sin\phi_0$

Eq. (259) describes the motion of a classical particle at position n with unit mass in a effective potential

$$\boxed{V_{eff}(n) = 2n^2\left(1+\lambda\cos\phi_0\right) + \frac{\lambda^2 n^4}{2}} \tag{260}$$

stationary solutions $\qquad \sin\phi_0 = 0$

$\boxed{\lambda \le 1}$ $\qquad \phi = \phi_0 = 0 \qquad \phi = \phi_0 = \pi$



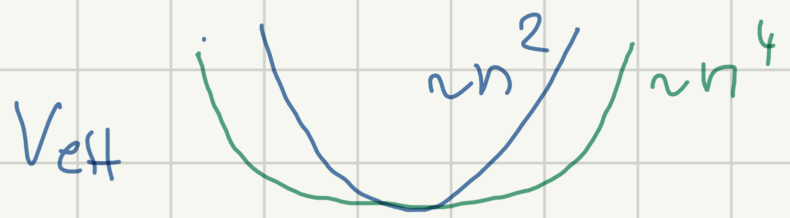
$\lambda = \frac{UN}{J}$

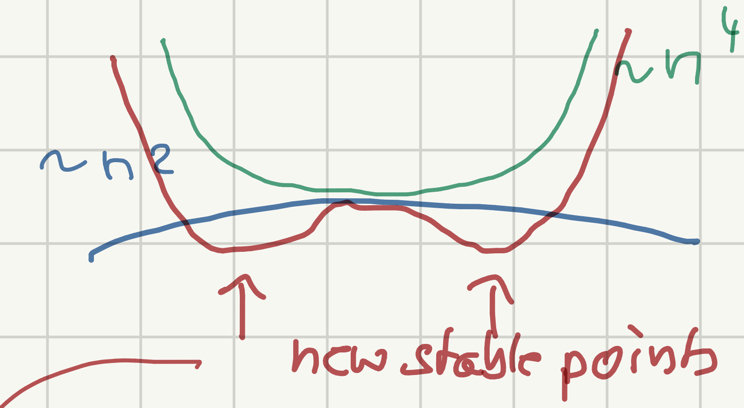
for the dynamics starting e.g with $n_0 = -1$

$\boxed{\lambda > 1}$

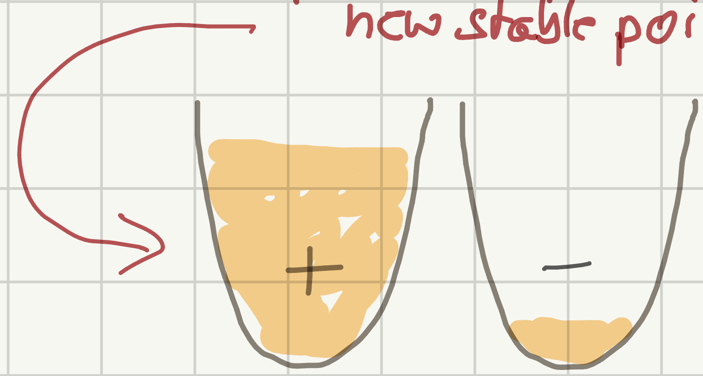
$\phi = \phi_0 = 0$
stable

$\phi = \phi_0 = \pi$
unstable





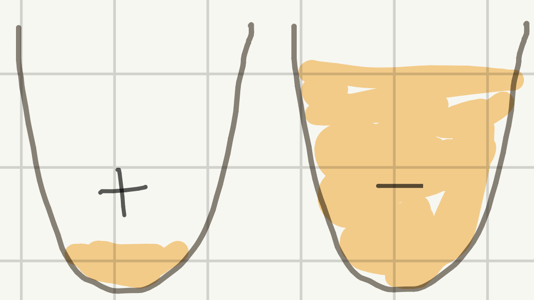
$\phi = 0$



$\phi = \pi$

Self trapping

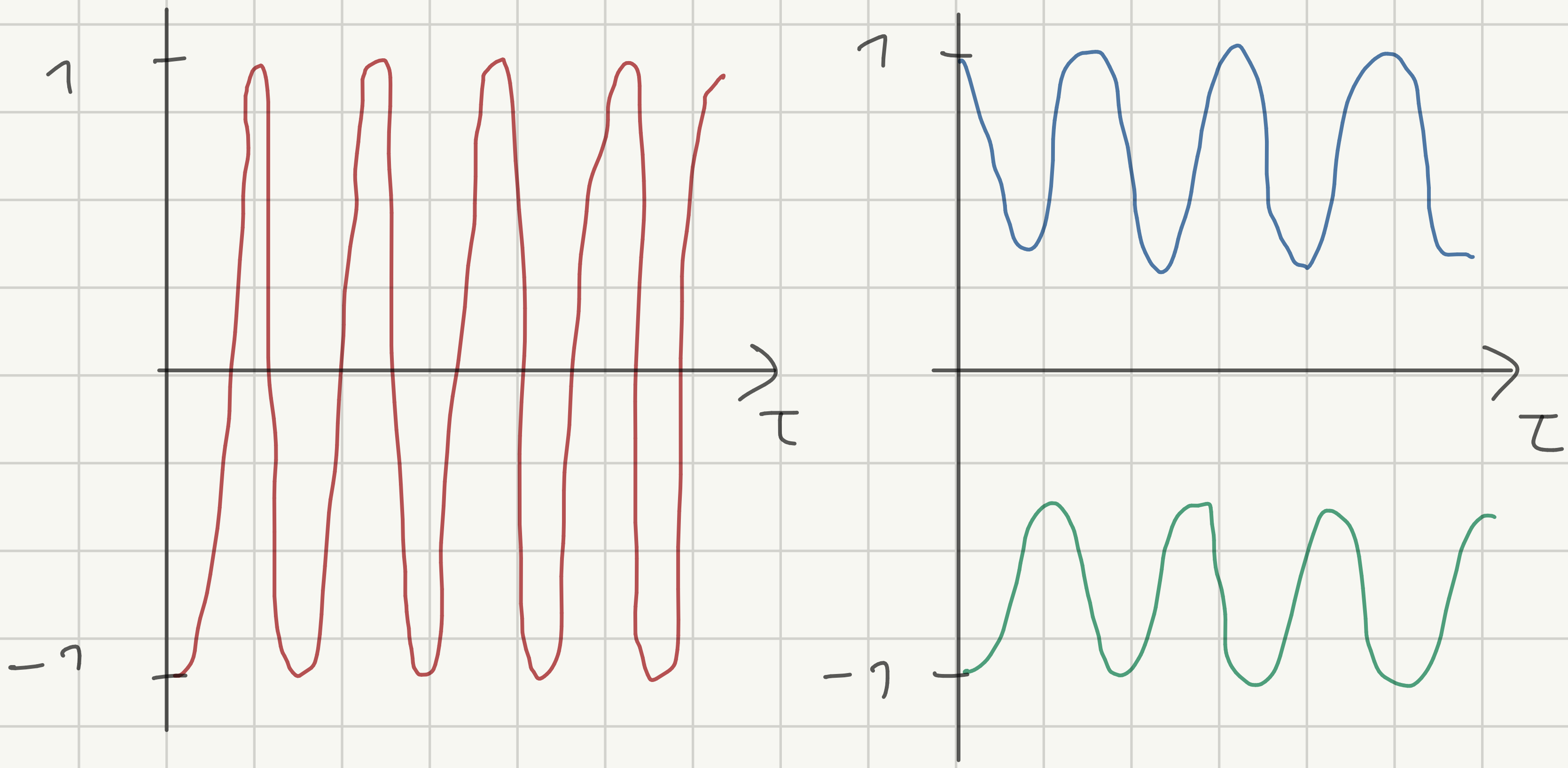
or



for $\phi = \phi_0 = \pi$
new stable points

$$n_{ss} = \pm\sqrt{\frac{2(\lambda - 1)}{\lambda^2}} \tag{261}$$

for the dynamics one would find with $n(0) = -1$



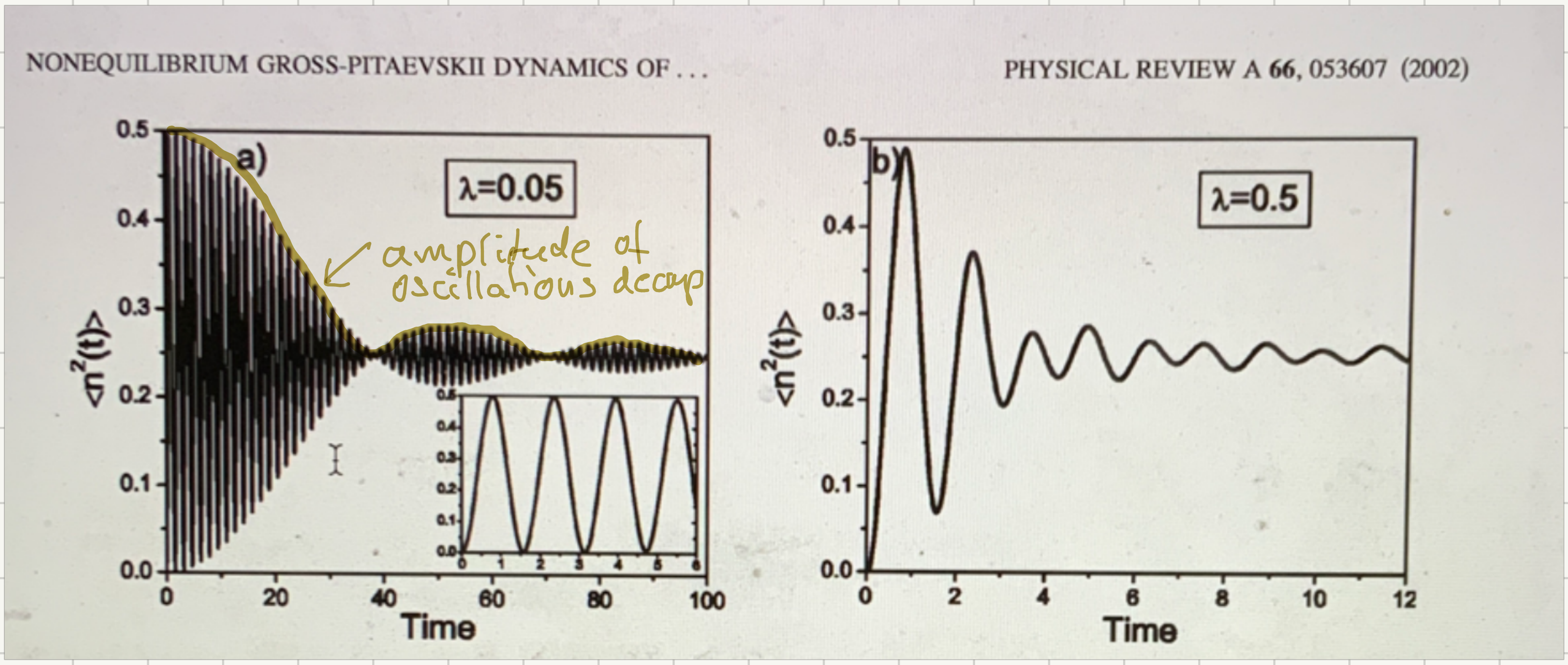
The mean field approximation yields unstable stationary solutions. If such a solution is chosen as initial state there will be no dynamics. Clearly here it is necessary to take quantum fluctuations into account.

Fixing the initial difference phase could however be considered a particularly bad choice. If we would start from two independently prepared condensates one should rather take an average over all ϕ_0. So lets consider this case and consider the first-order correlation between the modes

$$\mathrm{Re}\left[\langle \hat{a}_L^\dagger \hat{a}_R \rangle\right]$$

mean field

(262) $$\langle \alpha_L^* \alpha_R + \alpha_R^* \alpha_L \rangle_{\phi_0} = \frac{\lambda^4}{4} \langle n^2(t) \rangle_{\phi_0}$$



If one compara the amplitude of the oscillations (yellow line) with an exact quantum simulation one finds substantial deviations in the long-time behaviour in particular for small times.

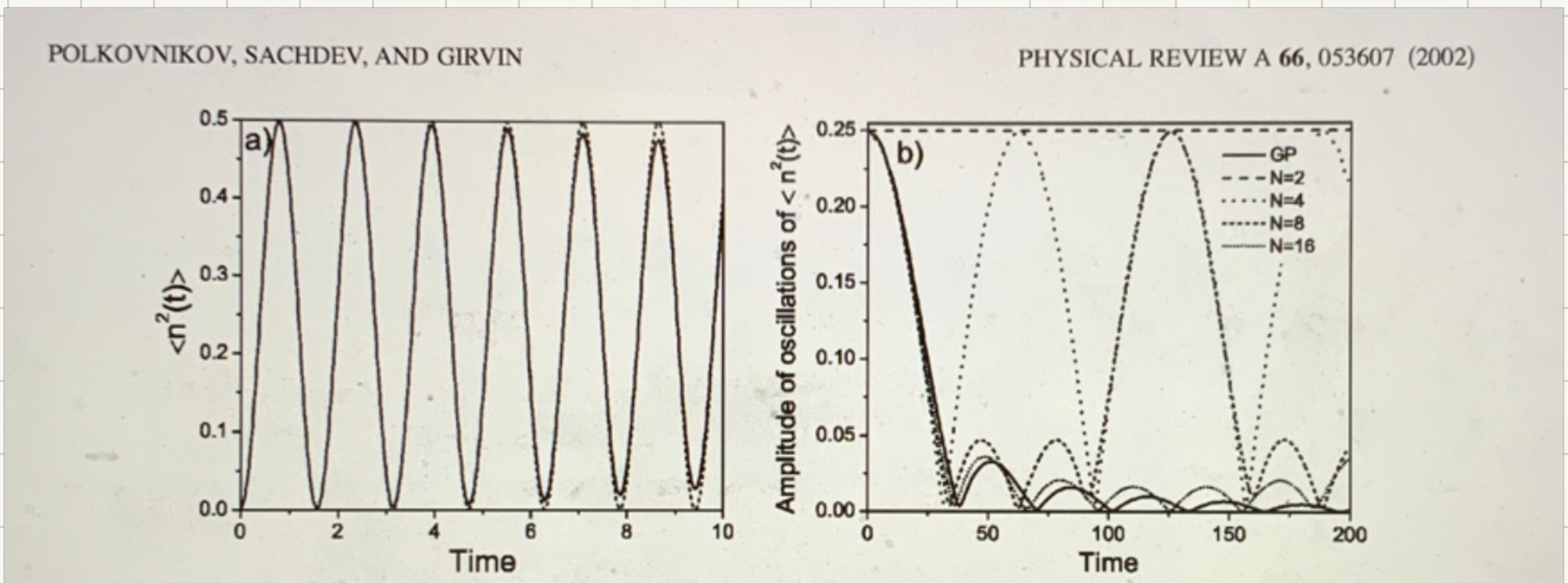


FIG. 2. (a) Semiclassical (solid line) and quantum variance of n as a function of time for the weak-coupling case $\lambda=0.05$. Dashed line corresponds to the total number of particles $N=2$, dotted line to $N=4$. Solid and dotted lines are indistinguishable on this plot. (b) Amplitude of the oscillations of the variance of n versus time.

let us now consider the TWA

(B) Truncated Wigner approximation

Wigner amplitude equations (see Sec. V.3)

$$\frac{d}{dt}\alpha_L = -i\alpha_R - i\lambda\left(|\alpha_L|^2-1\right)\alpha_L \qquad (263)$$

$$\frac{d}{dt}\alpha_R = -i\alpha_L - i\lambda\left(|\alpha_R|^2-1\right)\alpha_R$$

The terms "-1" in $\left(|\alpha_{L/R}|^2-1\right)$ lead to a trivial oscillatory term and can be ignored.

initial state average

An important part of the TWA is the averaging over the initial Wigner distribution. Here there are different approaches

(i) Monte Carlo sampling

If the initial Wigner function is positive definite

$$W(\alpha_j, t=0) = W_0(\alpha_j) \geq 0 \qquad (264)$$

than W_0 can be considered as probability distribution which can be sampled by a Monte Carlo technique.

(ii) Integration

If the initial Wigner function is not positive one has to integrate over the α_{j0}. This is clearly possible only if the number of modes is very small.

(iii) Adiabatic preparation

If the initial state with non-positive Wigner function is the ground state $|\Psi_0\rangle$ of a gapped Hamiltonian $H(\lambda)$ for, say, $\lambda=1$ and if this Hamiltonian has a ground state with a positive Wigner function for some other value of λ, say, $\lambda=0$, one can simulate the adiabatic preparation

$$|\Psi_0(\lambda=0)\rangle \xrightarrow[H(\lambda(t))]{} |\Psi_0(\lambda=1)\rangle \tag{265}$$

- If we consider as initial state the product of two number states with N particles in each well

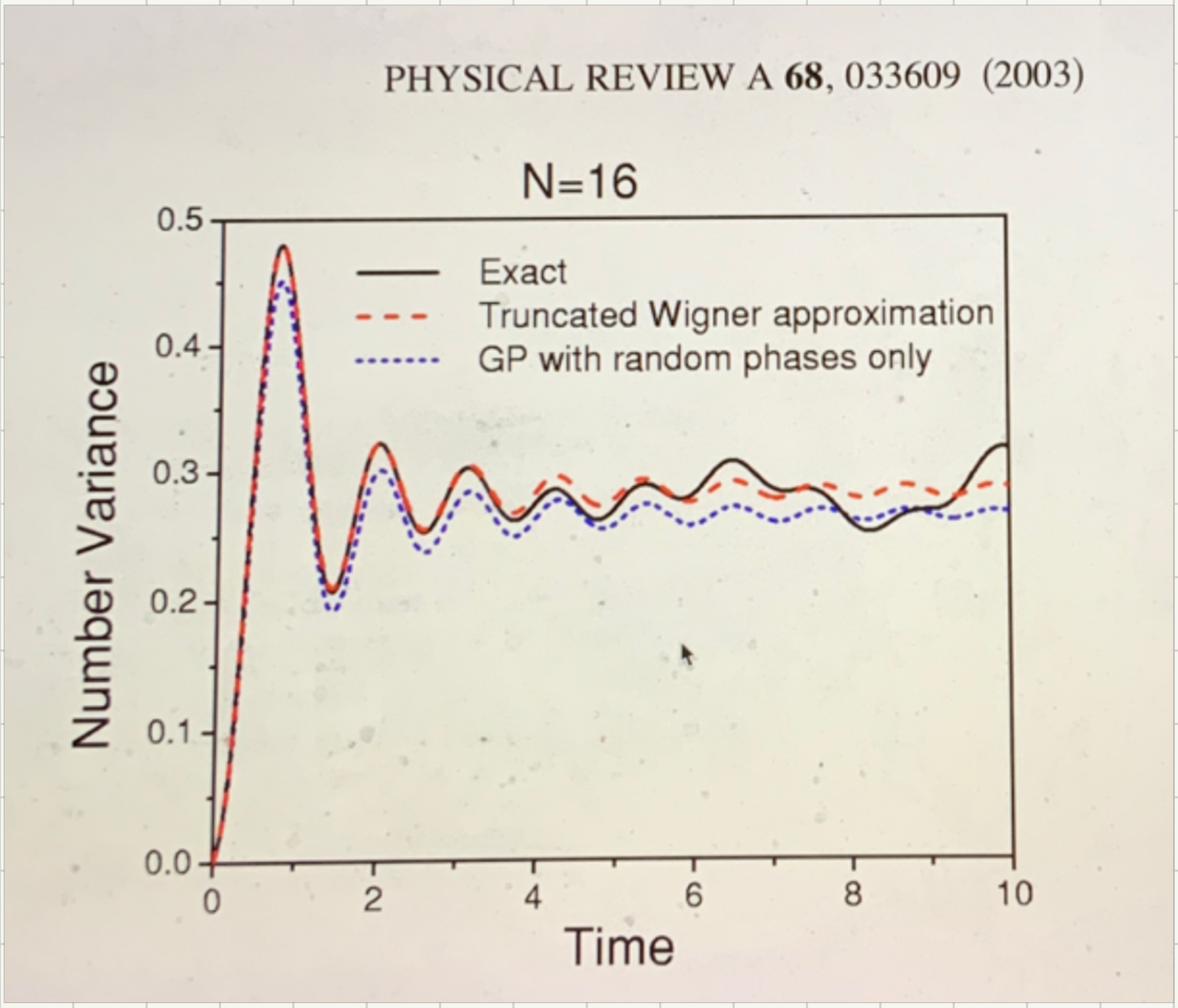
$$|\Psi(t=0)\rangle = |N\rangle_L |N\rangle_R \tag{266}$$

then

$$W_0(\alpha_{L_0}) = W_0(\alpha_{R_0}) = 2e^{-2n} L_N(4n) \tag{267}$$

where $n=|\alpha_{L_0}|^2$ or $n=|\alpha_{R_0}|^2$

W_0 is not positive and thus one has to numerically solve eqs. (263) and subsequently numerically integrate over W_0.



GP corresponds to the mean-field result. One recognizes a rather good agreement for a number of tunneling times.

VI.3 Truncated Wigner approximation to trapped, multi-mode Bose gases

We now want to discuss the application of the TWA to multi-mode ultra-cold bose gases at finite T.