V. Equations of motion of phase space quasidistributions

The question we want to address in the following: how to translate the Lindblad-Liouville equation for ρ

$$\dot{\rho} = -\frac{i}{\hbar}\left[H, \rho\right] + \sum_\mu \frac{\Gamma_\mu}{2}\left(2\hat{L}_\mu \rho \hat{L}_\mu^+ - \hat{L}_\mu^+ \hat{L}_\mu \rho - \rho \hat{L}_\mu^+ \hat{L}_\mu\right)$$

$$\downarrow$$

$$\frac{\partial P}{\partial t} \quad \text{or} \quad \frac{\partial Q}{\partial t} \quad \text{or} \quad \frac{\partial W}{\partial t}$$

Before we do this lets remind ourselves how such equations look like in classical phase space

V.1. Dynamics in classical phase-space

Classical dynamics in phase space spanned by generalized coordinates q_j and momenta p_j (without dissipation) is governed by canonical equations

$$(202) \quad \frac{dq_j}{dt} = \{q_j, H\} = \frac{\partial H}{\partial p_j} \qquad \frac{dp_j}{dt} = \{p_j, H\} = -\frac{\partial H}{\partial q_j}$$

where

(203) $$\{a,b\} = \sum_j \frac{\partial a}{\partial q_j}\frac{\partial b}{\partial p_j} - \frac{\partial a}{\partial p_j}\frac{\partial b}{\partial q_j}$$

is the Poisson bracket

Any function $f(q,p)$ evolves according to

(204) $$\frac{d}{dt} f(q,p) = \frac{\partial f}{\partial t} + \{f, H\}$$

which follows from

$$\{f,H\} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}$$

$$= \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{q}$$

Conservation of probability requires that the total derivative of the phase-space density $\rho(q,p)$ vanishes

(205) $$\frac{d\rho}{dt} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\{\rho, H\} = \{H, \rho\}}$$

which can be written in the form

(206) $$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p}}$$

first-order partial derivatives

So we expect to obtain partial differential equations for the quasi-probability distributions $Q, P, W \ldots$

V.2 Equation of motion for P, Q, W

To obtain an equation of motion for the phase-space quasi distributions we start from the Liouville-Lindblad equation in the Schrödinger picture

$$(207) \quad \dot{\rho} = -\frac{i}{\hbar}\left[H, \rho\right] + \sum_\mu \frac{\Gamma_\mu}{2}\left(2\hat{L}_\mu \rho \hat{L}_\mu^\dagger - \hat{L}_\mu^\dagger \hat{L}_\mu \rho - \rho \hat{L}_\mu^\dagger \hat{L}_\mu\right)$$

(A) Glauber-Sudarshan P function

$$\rho(t) = \int d^2\alpha \; P(\alpha, t)\, |\alpha\rangle\langle\alpha|$$

On the r.h.s. of (207) we will have to evaluate expressions like

$$\hat{a}\rho \,, \quad \hat{a}^\dagger\rho \quad \text{or} \quad \rho\hat{a} \,, \quad \rho\hat{a}^\dagger \quad \text{etc.}$$

Making use of the property of coherent states eq. (90) or Bargmann states, eq. (89)

$$(208) \qquad \hat{a}^\dagger \|\alpha\rangle = \frac{\partial}{\partial\alpha}\|\alpha\rangle \qquad |\alpha\rangle = \|\alpha\rangle e^{-|\alpha|^2/2}$$

$$\hat{a}^\dagger \rho = \int d^2\alpha\, P(\alpha,t)\, e^{-|\alpha|^2}\, \hat{a}^\dagger \| \alpha\rangle\langle\alpha\|$$

$$(209) \qquad = \int d^2\alpha\, P(\alpha,t)\, e^{-|\alpha|^2} \left(\frac{\partial}{\partial\alpha} \|\alpha\rangle\right)\langle\alpha\|$$

$$\underset{\text{partial int.}}{=} \int d^2\alpha\, \|\alpha\rangle\langle\alpha\|\, e^{-|\alpha|^2} \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha,t)$$

$$= \int d^2\alpha\, |\alpha\rangle\langle\alpha| \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha,t)$$

Similarly

$$\rho\hat{a} = \int d^2\alpha\, P(\alpha,t)\, e^{-|\alpha|^2} \|\alpha\rangle\langle\alpha\|\hat{a}$$

$$(210) \qquad = \int d^2\alpha\, \|\alpha\rangle\langle\alpha\|\, e^{-|\alpha|^2} \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha,t)$$

$$= \int d^2\alpha\, |\alpha\rangle\langle\alpha| \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha,t)$$

This yields the following replacement rules

$$(211) \qquad \boxed{\begin{aligned} \hat{a}\rho &\longleftrightarrow \alpha P(\alpha) \\ \hat{a}^\dagger\rho &\longleftrightarrow \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha) \\ \rho\hat{a} &\longleftrightarrow \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha) \\ \rho\hat{a}^\dagger &\longleftrightarrow \alpha^* P(\alpha) \end{aligned}}$$

Expressions with multiple operators have to be treated successively

$$\hat{a}^{+2}\rho = \hat{a}^{+}(\hat{a}^{+}\rho)$$

$$= \hat{a}^{+}\int d^2\alpha \; ||\alpha\rangle\langle\alpha|| \, e^{-|\alpha|^2}\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha)$$

(212) $$= \int d^2\alpha \left(\frac{\partial}{\partial\alpha}||\alpha\rangle\right)\langle\alpha|| \; e^{-|\alpha|^2}\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha)$$

$$\underset{\text{part. Int.}}{=} \int d^2\alpha \; ||\alpha\rangle\langle\alpha|| \, e^{-|\alpha|^2}\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)^2 P(\alpha)$$

$$= \int d^2\alpha \; |\alpha\rangle\langle\alpha| \left(\alpha^* - \frac{\partial}{\partial\alpha}\right)^2 P(\alpha)$$

Similarly

(213) $$\hat{a}^{+}\rho\hat{a} = \int d^2\alpha \; |\alpha\rangle\langle\alpha| \left(\alpha^* - \frac{\partial}{\partial\alpha}\right)\left(\alpha - \frac{\partial}{\partial\alpha^*}\right)P$$

$$= \int d^2\alpha \; |\alpha\rangle\langle\alpha| \left(\alpha - \frac{\partial}{\partial\alpha^*}\right)\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P$$

One recognizes that the order of the operators does not matter since

$$\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)\left(\alpha - \frac{\partial}{\partial\alpha^*}\right) = |\alpha|^2 - \alpha^*\frac{\partial}{\partial\alpha^*} - \frac{\partial}{\partial\alpha}\alpha + \frac{\partial^2}{\partial\alpha\partial\alpha^*}$$

$$= |\alpha|^2 - 1 - \alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha} + \frac{\partial^2}{\partial\alpha\partial\alpha^*}$$

and

$$\left(\alpha-\frac{\partial}{\partial\alpha^*}\right)\left(\alpha^*-\frac{\partial}{\partial\alpha}\right)=|\alpha|^2-\frac{\partial}{\partial\alpha^*}\alpha^*-\alpha\frac{\partial}{\partial\alpha}+\frac{\partial^2}{\partial\alpha\partial\alpha^*}$$

$$=|\alpha|^2-1-\alpha^*\frac{\partial}{\partial\alpha^*}-\alpha\frac{\partial}{\partial\alpha}+\frac{\partial^2}{\partial\alpha\partial\alpha^*}$$

so

(214) $$\left[\alpha^*-\frac{\partial}{\partial\alpha},\alpha-\frac{\partial}{\partial\alpha^*}\right]=0$$

To see how equations of motion for $P(t)$ follow let us consider as a specific example the damped harmonic oscillator ($\hbar=1$)

(215) $$\begin{aligned}\dot{\rho}=&-i\omega[a^\dagger a,\rho]\\&+\frac{\gamma}{2}(\bar{n}+1)(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a)\\&+\frac{\gamma}{2}\bar{n}(2a^\dagger\rho a-aa^\dagger\rho-\rho aa^\dagger)\end{aligned}$$

$\Rightarrow$

(216) $$\begin{aligned}\int d^2\alpha\,|\alpha\rangle\langle\alpha|\,\dot{P}(\alpha,t)=\int d^2\alpha\,|\alpha\rangle\langle\alpha|\Big\{&\left(\frac{\gamma}{2}+i\omega\right)\frac{\partial}{\partial\alpha}\alpha\\&+\left(\frac{\gamma}{2}-i\omega\right)\frac{\partial}{\partial\alpha^*}\alpha^*+\gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Big\}P(\alpha,t)\end{aligned}$$

where we have to remember that α,α^* are treated as independent variables (equivalent to treat real and imaginary part of α)

Due to the completeness of coherent states if follows (217)

$$\boxed{\frac{\partial P}{\partial t} = \left[\left(\frac{\gamma}{2}+i\omega\right)\frac{\partial}{\partial\alpha}\alpha + \left(\frac{\gamma}{2}-i\omega\right)\frac{\partial}{\partial\alpha^*}\alpha^* + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P}$$

drift term as in deterministic mechanics

diffusion term

this is a Fokker-Planck equation

remarks

(i) there are only derivatives of P

(ii) interaction terms in $\hat{H}$ which are at most bi-quadratic e.g. $\hat{a}^{\dagger 2}\hat{a}^2$ i.e correspond to two-particle interactions give at most second derivatives of P

(iii) the EOM for P is only a Fokker-Planck equation if the diffusion term is positive (here this is simple as $\gamma\bar{n} \geq 0$)

Eq. (217) can be solved by separation

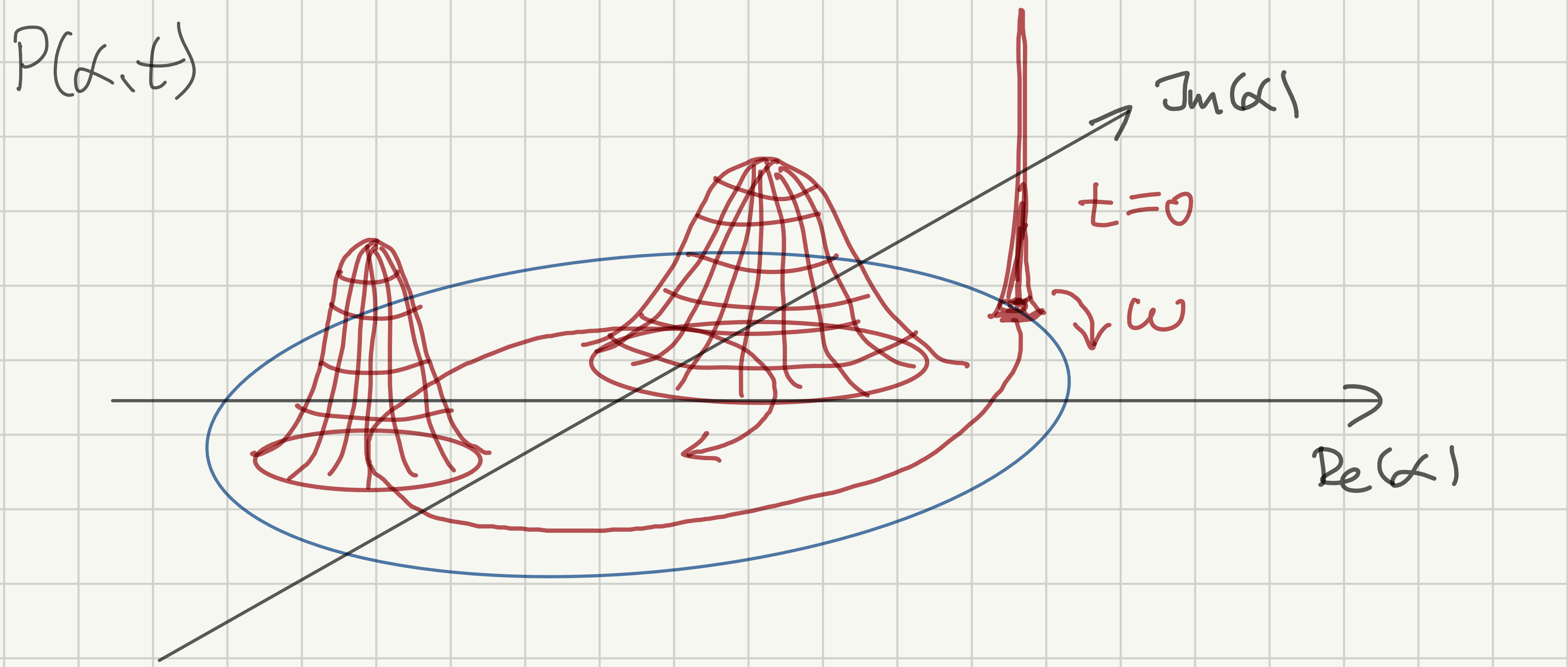
(218) $$P(\alpha,t) = X(x,t)\,Y(y,t)$$

$$x = \mathrm{Re}(\alpha) \quad y = \mathrm{Im}(\alpha)$$

The solution is for an initial coherent state

$$P(\alpha, t=0) = \delta^{(2)}(\alpha - \alpha_0) \tag{219}$$

$$P(\alpha,t) = \frac{1}{\pi \bar{n}(1-e^{-\gamma t})} \exp\left\{ - \frac{|\alpha - \alpha_0 e^{-i\omega t - \gamma/2 t}|^2}{\bar{n}(1-e^{-\gamma t})} \right\} \tag{220}$$



Gaußian with width $\sqrt{\bar{n}(1-e^{-\gamma t})}$

and average amplitude $\alpha(t) = \alpha_0 e^{-i\omega t - \frac{\gamma}{2}t}$

(B) Q-function

In order to derive an EOM for $Q(\alpha,t)$ we use

$$Q(\alpha,t) = \langle \alpha | \rho(t) | \alpha \rangle \tag{221}$$

Here we can proceed analogously to P

(222) $\langle\alpha|\hat{a}^\dagger \rho|\alpha\rangle = \alpha^* Q(\alpha)$

$$\langle\alpha|\hat{a}\rho|\alpha\rangle = \langle\alpha\|\hat{a}\rho\|\alpha\rangle e^{-|\alpha|^2}$$

(223) $$= \frac{\partial}{\partial\alpha^*}\left(\langle\alpha\|\right)\rho\|\alpha\rangle e^{-|\alpha|^2}$$

$$= \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) Q(\alpha)$$

and for multiple operators

$$\langle\alpha|\hat{a}^2\rho|\alpha\rangle = \langle\alpha\|\hat{a}^2\rho\|\alpha\rangle e^{-|\alpha|^2}$$

(224) $$= \frac{\partial}{\partial\alpha^*}\left(\langle\alpha\|\right)\hat{a}\rho\|\alpha\rangle e^{-|\alpha|^2}$$

$$= \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)^2 Q(\alpha) \quad \text{etc.}$$

for expressions of the form

$$\langle\alpha|\hat{a}\rho\hat{a}^\dagger|\alpha\rangle = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)\left(\alpha^* + \frac{\partial}{\partial\alpha}\right) Q(\alpha)$$

(225) $$= \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)\left(\alpha + \frac{\partial}{\partial\alpha^*}\right) Q(\alpha)$$

so again the order of operations is not relevant

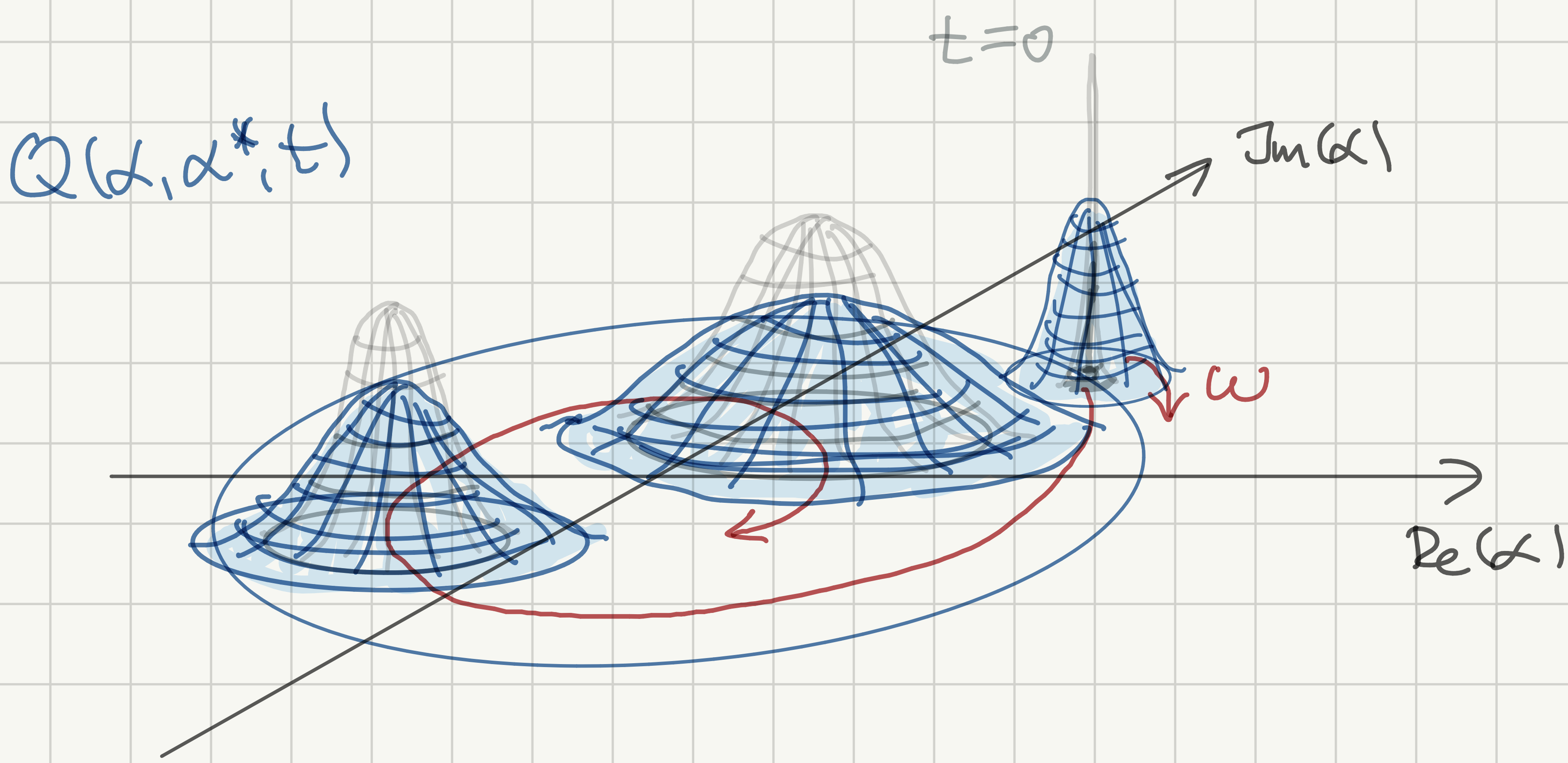
This then gives the translation rule for the Husimi or Q-function

(226)
$$\boxed{\begin{aligned} \hat{a}\rho &\longleftrightarrow \left(\alpha+\frac{\partial}{\partial\alpha^*}\right)Q(\alpha) \\ \hat{a}^\dagger\rho &\longleftrightarrow \alpha^* Q(\alpha) \\ \rho\hat{a} &\longleftrightarrow \alpha Q(\alpha) \\ \rho\hat{a}^\dagger &\longleftrightarrow \left(\alpha^*+\frac{\partial}{\partial\alpha}\right)Q(\alpha) \end{aligned}}$$

Let us consider again the damped harmonic oscillator, eq. (215). For this we find

(225)
$$\boxed{\frac{\partial Q}{\partial t}=\left[\left(\frac{\gamma}{2}+i\omega\right)\frac{\partial}{\partial\alpha}\alpha+\left(\frac{\gamma}{2}-i\omega\right)\frac{\partial}{\partial\alpha^*}\alpha^*+\gamma(\bar{n}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]Q}$$

almost the same FPE as for P with $\bar{n}\to\bar{n}+1$



remarks:

(i) again only derivations of Q

(ii) interaction terms in $\hat{H}$ which are at most bi-quadratic (2-particle interactions) give at most 2nd-order derivatives of Q

(C) Wigner-Weyl distribution

The EOMs for the Wigner distribution can be obtained from (226) and the relation (167)

$$Q(\alpha) = \frac{2}{\pi}\int d^2\beta\, e^{-2|\alpha-\beta|^2} W(\beta)$$

$$\hat{a}\varsigma \to \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) Q(\alpha) = \frac{2}{\pi}\int d^2\beta \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) e^{-2|\alpha-\beta|^2} W(\beta)$$

with

(226) $$\frac{\partial}{\partial\alpha^*} e^{-2|\alpha-\beta|^2} = \frac{\partial}{\partial\alpha^*} e^{-2(\alpha^*-\beta^*)(\alpha-\beta)}$$

$$= -2(\alpha-\beta)\, e^{-2(\alpha^*-\beta^*)(\alpha-\beta)} = -\frac{\partial}{\partial\beta^*} e^{-2|\alpha-\beta|^2}$$

and

(227) $$\alpha\, e^{-2|\alpha-\beta|^2} = \left(\beta + \frac{1}{2}\frac{\partial}{\partial\beta^*}\right) e^{-2|\alpha-\beta|^2}$$

$$\Rightarrow \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) Q = \frac{2}{\pi}\int d^2\beta \left(\beta + \frac{1}{2}\frac{\partial}{\partial\beta^*} - \frac{\partial}{\partial\beta^*}\right) e^{-2|\alpha-\beta|^2} W(\beta)$$

$$= \frac{2}{\pi}\int d^2\beta \left(\beta - \frac{1}{2}\frac{\partial}{\partial\beta^*}\right) e^{-2|\alpha-\beta|^2} W(\beta)$$

partial integration

(228) $$= \frac{2}{\pi}\int d^2\beta \, e^{-2|\alpha-\beta|^2}\left(\beta + \frac{1}{2}\frac{\partial}{\partial\beta^*}\right)W(\beta)$$

This then gives the translation rules for the Wigner function

(229)
$$\boxed{\begin{aligned}
\hat{a}\rho &\longleftrightarrow \left(\alpha + \frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha)\\
\hat{a}^\dagger\rho &\longleftrightarrow \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial\alpha}\right)W(\alpha)\\
\rho\hat{a} &\longleftrightarrow \left(\alpha - \frac{1}{2}\frac{\partial}{\partial\alpha^*}\right)W(\alpha)\\
\rho\hat{a}^\dagger &\longleftrightarrow \left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial\alpha}\right)W(\alpha)
\end{aligned}}$$

So here all operator expressions yield derivatives but only with a prefactor 1/2

For the example of the damped harmonic oscillator eq. (215) one finds

(230) $$\boxed{\frac{\partial W}{\partial t} = \left[\left(\frac{\gamma}{2}+i\omega\right)\frac{\partial}{\partial\alpha} + \left(\frac{\gamma}{2}-i\omega\right)\frac{\partial}{\partial\alpha^*} + \gamma\left(\bar{n}+\frac{1}{2}\right)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]W}$$

again the same FPE with $\bar{n}+\frac{1}{2}$, so

$$P: \bar{n} \; ; \; W: \bar{n}+\frac{1}{2} \; ; \; Q: \bar{n}+1$$

remarks

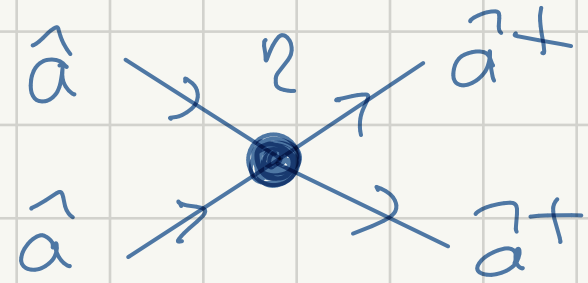
(i) again only derivatives of W

(ii) in contrast to P and Q even interaction Hamiltonians that are at most bi-quadratic lead to higher-order derivatives!

V.3 Interactions

In the previous section we have discussed an example of a non-interacting system, which can be solved in many different ways. Now lets discuss what happens if we add interactions. As generic example of a two-particle interaction let us consider the Kerr oscillator

(231) $$H = \hbar\omega \hat{a}^\dagger \hat{a} + \hbar\frac{\eta}{2} \hat{a}^{\dagger 2} \hat{a}^2$$



generic for two-particle scattering

E.g. s-wave interaction of homogeneous Bose gas

$$\hat{H}_{int} = \frac{g}{2}\int d^3r\, \hat{\psi}^\dagger(\vec{r})\hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}(\vec{r}) =$$

(232) $$= \frac{g}{2}\sum_{\vec{k},\vec{k}',\vec{q}} \hat{a}^\dagger_{\vec{k}+\vec{q}}\, \hat{a}^\dagger_{\vec{k}'-\vec{q}}\, \hat{a}_{\vec{k}}\, \hat{a}_{\vec{k}'}$$

is a generalization of eq. (231)

With the transformation rules for the P function (211) we find

(233) $$\boxed{\frac{\partial P}{\partial t} = \left\{ \frac{\partial}{\partial \alpha}\left(i\omega + i\eta|\alpha|^2\right)\alpha - \frac{\partial}{\partial \alpha^*}\left(i\omega + i\eta|\alpha|^2\right)\alpha^* - \frac{i\eta}{2}\left(\frac{\partial^2}{\partial \alpha^2}\alpha^2 - \frac{\partial^2}{\partial \alpha^{*2}}\alpha^{*2}\right)\right\} P}$$

So we recognize

(i) there is a nonlinear term in the drift $\sim \eta|\alpha|^2$

(ii) the diffusion matrix is not positive

$$\dot{P} = \left(-\sum_j \frac{\partial}{\partial x_j} A + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_i \partial x_j} D_{ij}\right)P$$

(234) $$\underline{\underline{D}} = \begin{pmatrix} i\eta\alpha^2 & 0 \\ 0 & -i\eta\alpha^{*2} \end{pmatrix}$$

imaginary eigenvalues!

Interactions lead in general to EOM of the P function which are not Fokker-Planck eqs.!

$\Rightarrow$ Interacting bosons cannot be described by a nonlinear, classical stochastic process in P representation

In a similar way as above we can derive the EOM for the Q function. Using the rules (226) we find

$$\boxed{\frac{\partial Q}{\partial t} = \left[\frac{\partial}{\partial \alpha}\left(i\omega + i\eta(|\alpha|^2-2)\right)\alpha - \frac{\partial}{\partial \alpha^*}\left(i\omega + i\eta(|\alpha|^2-2)\right)\alpha^* + i\frac{\eta}{2}\left(\frac{\partial^2}{\partial \alpha^2}\alpha^2 - \frac{\partial^2}{\partial \alpha^{*2}}\alpha^{*2}\right)\right] Q} \tag{235}$$

I.e. the diffusion matrix has the opposite sign but is still imaginary. Furthermore $|\alpha|^2$ in the nonlinear part of the drift term for P is replaced by $|\alpha|^2-2$ in that for Q

Interactions lead in general to EOM of the Q function which are not Fokker-Planck eqs.!

Since the Wigner function is "in between" P and Q which both had diffusion terms of opposite sign, we can make the guess that the diffusion term may vanish altogether in the W equation?

And indeed we find for W

$$\boxed{\frac{\partial W}{\partial t} = \left[\frac{\partial}{\partial \alpha}\left(i\omega + i\eta(|\alpha|^2-1)\right)\alpha - \frac{\partial}{\partial \alpha^*}\left(i\omega + i\eta(|\alpha|^2-1)\right)\alpha^* - i\frac{\eta}{4}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^*\right)\right]W}$$

(236) there are no 2nd derivatives due to interactions!

If we add decay there will be second derivatives but with positive diffusion.
so apart from higher derivatives this is a FPE!

but: there are now 3rd order derivatives

We will now argue that the 3rd derivatives can often be neglected, which leads to a powerful approximation, the truncated Wigner approximation

If the mode $\hat{a}$ is highly occupied, we can make the scaling ansatz

(237) $$\alpha = \sqrt{N}\,\tilde{\alpha} \qquad \frac{\partial}{\partial\alpha} = \frac{1}{\sqrt{N}}\frac{\partial}{\partial\tilde{\alpha}}$$

then

(238) $$\frac{\partial W}{\partial t} = \left[\frac{\partial}{\partial\tilde{\alpha}}\left(i\omega + 2i\eta(N|\tilde{\alpha}|^2-1)\right)\tilde{\alpha} - c.c. \cancel{- i\frac{\eta}{4}\frac{1}{N}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\left(\frac{\partial}{\partial\tilde{\alpha}}\tilde{\alpha} - \frac{\partial}{\partial\tilde{\alpha}^*}\tilde{\alpha}^*\right)}\right]W$$

≈ 0