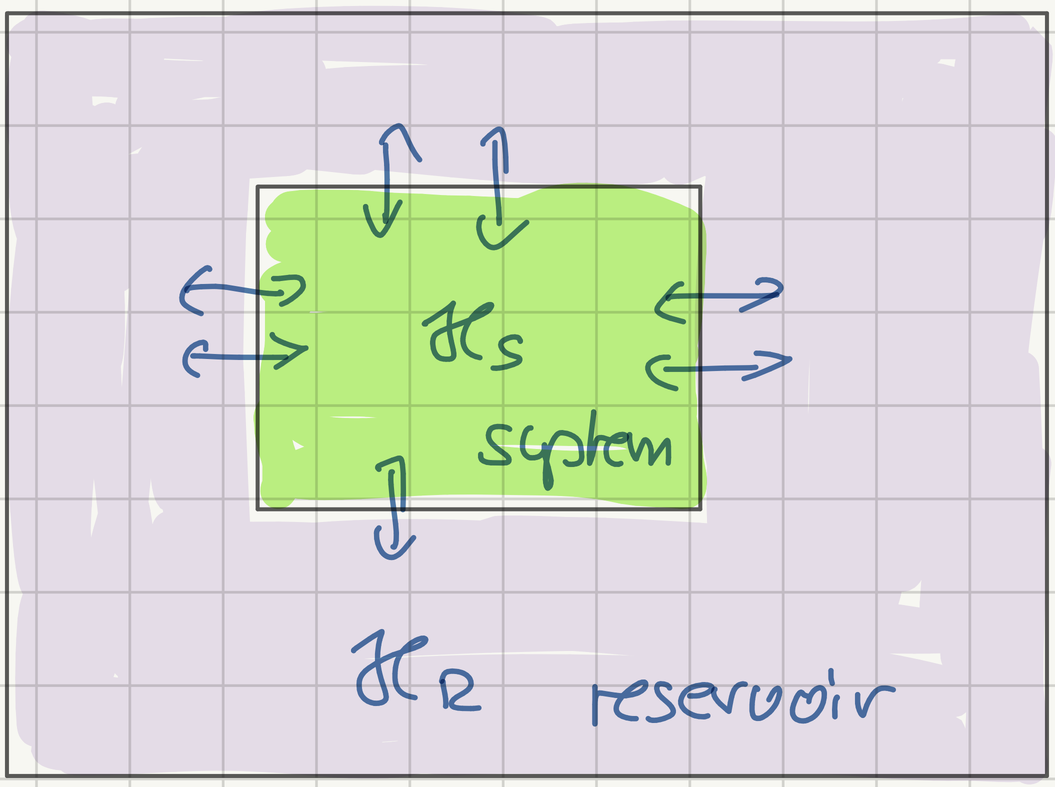
IV. Interlude: Quantum mechanics of Markovian open systems

An advantage of phase-space methods is that they allow for a description of unitary and open quantum systems on the same footing.

Of particular interest are couplings of small systems to Markovian reservoirs



$\dim(\mathcal{H}_S) \ll \dim(\mathcal{H}_R)$

Can we find an effective description of the small system without the uninteresting degrees of freedom of the reservoir?

$$\hat{H} = \underbrace{\hat{H}_S}_{\text{system}} + \underbrace{\hat{H}_R}_{\text{reservoir}} + \underbrace{\hat{H}_{SR}}_{\text{system-reservoir coupling}} \tag{172}$$

Liouville equation of total state

(173) $$\chi \in \mathcal{H}_S \otimes \mathcal{H}_R \qquad \dot{\chi} = -\frac{i}{\hbar}[H,\chi]$$

If we only want to calculate observables $\hat{A}$ that act in the Hilbert space of the system, then

(174) $$\begin{aligned} \langle \hat{A} \rangle &= \mathrm{Tr}_{S+R}\{\chi \hat{A}\} \\ &= \mathrm{Tr}_S\{\mathrm{Tr}_R\{\chi\}\hat{A}\} \end{aligned}$$

it is sufficient to consider the reduced density matrix

(175) $$\rho \equiv \mathrm{Tr}_R\{\chi\}$$

ρ is in general a mixed state even if χ was a pure state. This is the case if and only if system and reservoir are in an entangled state.

(176) $$\chi = |\psi\rangle\langle\psi|$$

Using a Schmidt decomposition $|\psi\rangle$ can be expressed as

(177) $$|\psi\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j}\, |j\rangle_S |j\rangle_R$$

where $\{|j\rangle_S\}$ and $\{|j\rangle_R\}$ are ONB in $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively and $d \leq \min\{\dim(\mathcal{H}_S), \dim(\mathcal{H}_R)\}$

The λ_j are called Schmidt coefficients. If only one coefficient is non-zero the state $|\psi\rangle$ is factorized if more than one value $\lambda_j \neq 0$ the state $|\psi\rangle$ is entangled between S and R. Then

$$\rho = \mathrm{Tr}_R \{ |\psi\rangle\langle\psi| \} = \sum_{j=1}^{d} \lambda_j \, |j\rangle_{S\,S}\langle j|$$

Since

$$\mathrm{Tr}\{\rho\} \overset{!}{=} 1 \Rightarrow \sum_{j=1}^{d} \lambda_j \overset{!}{=} 1$$

but

$$\mathrm{Tr}\{\rho^2\} = \sum_{j=1}^{d} \lambda_j^2 \leq 1$$

equality holds if only one $\lambda_j = 1$ is nonzero

(A) EOM of reduced density matrix ρ

(i) unitary trafo to interaction picture

(178) $$\chi(t) \to \tilde{\chi}(t) = e^{\frac{i}{\hbar}(H_S+H_R)t} \chi(t) \, e^{-\frac{i}{\hbar}(H_S+H_R)t}$$

and

$$A \to \tilde{A}(t) = e^{\frac{i}{\hbar}(H_S+H_R)t} A \, e^{-\frac{i}{\hbar}(H_S+H_R)t}$$

$\Rightarrow$

(179) $$\dot{\tilde{\chi}}(t) = -\frac{i}{\hbar} \left[\tilde{H}_{SR}(t), \tilde{\chi}(t) \right]$$

(ii) Born approximation of weak coupling

formal solution of (179)

$$\tilde{\chi}(t) = \tilde{\chi}(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} d\tau\, \left[\tilde{H}_{SR}(\tau), \tilde{\chi}(\tau)\right] \tag{180}$$

substitute this into r.h.s. of eq. (179)

$$\dot{\tilde{\chi}}(t) = -\frac{i}{\hbar}\cancel{\left[\tilde{H}_{SR}(t), \tilde{\chi}(t_0)\right]} - \frac{1}{\hbar^2}\int_{t_0}^{t} d\tau\, \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(\tau), \tilde{\chi}(\tau)\right]\right] \tag{181}$$

often negligible

already 2nd order in H_{SR} $\Longrightarrow$

$$\tilde{\chi}(\tau) \approx \tilde{\rho}(\tau) \otimes \tilde{\rho}_R(\tau) \tag{182}$$

factorized!

reservoir in stationary state

$$\tilde{\rho}_R(\tau) = \tilde{\rho}_R(0) \equiv \rho_R \tag{183}$$

taking the trace with respect to reservoir degrees

$$\begin{aligned} \dot{\tilde{\rho}}(t) &= \mathrm{Tr}_R\left(\dot{\tilde{\chi}}(t)\right) \\ &= -\frac{1}{\hbar^2}\int_{t_0}^{t} d\tau\, \mathrm{Tr}_R\left\{\left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(\tau), \tilde{\rho}(\tau)\rho_R\right]\right]\right\} \end{aligned} \tag{184}$$

typical form of H_{SR}

(185) $$H_{SR} = \hbar g \sum_j \hat{S}_j \hat{R}_j$$

system reservoir

Then trace $\mathrm{Tr}_R\{\cdots \rho_R\}$ in eq. (184) yields reservoir correlations:

(186) $$\langle \hat{R}_j(t)\hat{R}_e(\tau)\rangle_R = \mathrm{Tr}_R\{\rho_R \hat{R}_j(t)\hat{R}_e(\tau)\}$$

(iii) Markov approximation

In a stationary reservoir state the reservoir correlations are stationary, i.e depend only on time difference

(187) $$\langle \hat{R}_j(t)\hat{R}_e(\tau)\rangle = R_{je}(t-\tau)$$

Often they decay very fast to zero

(188) $$R_{je}(t-\tau) \longrightarrow 0$$

for $(t-\tau)$ large

Markov approximation: in eq. (184)

(189) $$\boxed{S_j(\tau) \simeq S_j(t) \qquad \tilde{\rho}(\tau) \simeq \rho(t)}$$

Due to double commutator in eq. (184) one eventually arrives at a differential equation (drop tilda's)

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_{eff}, \rho\right] + \frac{1}{2}\sum_{jk} \Gamma_{jk}\left\{2\hat{S}_j \rho \hat{S}_k - \hat{S}_k \hat{S}_j \rho - \rho \hat{S}_k \hat{S}_j\right\} \tag{190}$$

where $H_{eff} = H +$ correction and $\Gamma_{jk}^{+} = \Gamma_{jk}$ is hermitian and positive matrix

Diagonalization of Γ_{jk} yields the

Lindblad-Masterequation

$$\boxed{\dot{\rho} = -\frac{i}{\hbar}\left[H_{eff}, \rho\right] + \frac{1}{2}\sum_{\mu} \Gamma_{\mu}\left\{2\hat{L}_\mu \rho \hat{L}_\mu^{+} - \hat{L}_\mu^{+}\hat{L}_\mu \rho - \rho \hat{L}_\mu^{+}\hat{L}_\mu\right\}} \tag{191}$$

Lindblad-Masterequation

$\Gamma_\mu \geq 0$ decay rates

$\hat{L}_\mu$ Lindblad operators or jump operators

$\dot{\rho} = -\frac{i}{\hbar}\left[H_{eff}, \rho\right] + \frac{1}{2}\sum_{\mu}\left\{2\hat{L}_\mu \rho \hat{L}_\mu^{+} - \hat{L}_\mu^{+}\hat{L}_\mu\right.$

(B) Examples for Lindblad operators

- incoherent particle loss

(192) $$\boxed{\hat{L} = \hat{a}}$$ (single-particle loss)

(193) $$\dot{\rho}\Big|_{diss} = \frac{\Gamma_{loss}}{2}\left(2\hat{a}\rho\hat{a}^{+} - \hat{a}^{+}\hat{a}\rho - \rho\hat{a}^{+}\hat{a}\right)$$

probability of n excitations:

(194) $$p(n) = \langle n|\hat{\rho}|n\rangle$$

(195) $$\dot{p}(n) = \Gamma_{loss}\left[(n+1)\,p(n+1) - n\,p(n)\right]$$

- increase of probability $p(n)$ due to decay from $n+1$ to n
- decrease of probability $p(n)$ due to decay from n to $n-1$

- incoherent particle gain (pump)

(196) $$\boxed{\hat{L} = \hat{a}^{+}}$$

(197) $$\dot{\rho}\Big|_{diss} = \frac{\Gamma_{gain}}{2}\left(2\hat{a}^{+}\rho\hat{a} - \hat{a}\hat{a}^{+}\rho - \rho\hat{a}\hat{a}^{+}\right)$$

this gives

$$\dot{p}(n) = \Gamma_{gain}\left(n\,p(n-1) - (n+1)\,p(n)\right) \tag{198}$$

- increase of probability $p(n)$ due to pump from $n-1$ to n
- decrease of probability $p(n)$ due to pump from n to $n+1$. Note pre-factor $n+1$ i.e. even from $n=0$ there is loss of probability due to transition $n=0 \rightarrow n=1$

▶ dephasing

$$\boxed{\hat{L} = \hat{a}^\dagger \hat{a}} \tag{199}$$

$$\dot{\rho}\big|_{diss} = \frac{\Gamma_{deph}}{2}\left[2\hat{n}\rho\hat{n} - \hat{n}^2\rho - \rho\hat{n}^2\right] \tag{200}$$

$p(n)$ is unaffected but coherent superpositions are destroyed

▶ two-particle loss

$$\boxed{\hat{L} = \hat{a}^2} \tag{201}$$

$p(n)$ jumps in units of 2