III. Coherent-state representation of states and operators

III.1 General remarks

We have seen that classical fields describing the condensed atoms in a Bose-Einstein condensate or the e.m. field of a laser are well approximated by (multi-mode) coherent states. So can we make use of them to also take into account (quantum) fluctuations on top of the condensate or nonclassical quantum state of light such as squeezed light?

Let us discuss now a single mode/oscillator. The generalization is straightforward

$$\frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha| = \mathbb{1}$$

i.e. for arbitrary state

(113) $$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha|\psi\rangle$$

there is a one-to-one correspondence

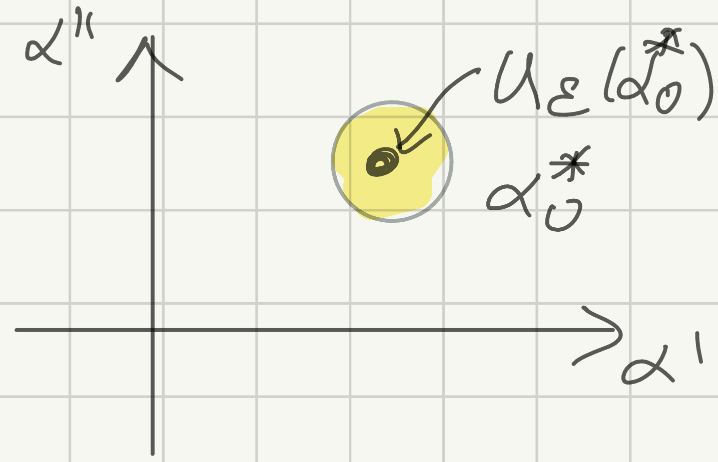
(114) $$|\psi\rangle \leftrightarrow \langle\alpha|\psi\rangle$$

$$\langle\alpha|\psi\rangle = e^{-|\alpha|^2/2} \underbrace{\sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \langle n|\psi\rangle}_{f_\psi(\alpha^*)} \tag{115}$$

$f_\psi(\alpha^*) = \langle\alpha||\psi\rangle$ is an analytic function in α^*! Now we can make use of the power of complex analyticity

$f_\psi(\alpha^*)$ is <u>completely</u> determined by its behaviour in an ε-neighbourhood of one value α_0^*

$\alpha = \alpha' + i\alpha''$



Similarly we can argue for an <u>operator</u>

$$\hat{A} = \frac{1}{\pi^2}\int d^2\alpha \int d^2\alpha' \, |\alpha\rangle\langle\alpha|\hat{A}|\alpha'\rangle\langle\alpha'| \tag{116}$$

$$\langle\alpha|\hat{A}|\alpha'\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\alpha'|^2}{2}} \sum_{n,m} \frac{\alpha^{*n}\alpha'^m}{\sqrt{n!m!}} \langle n|\hat{A}|m\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\alpha'|^2}{2}} f_{\hat{A}}(\alpha^*, \alpha')$$

$f_{\hat{A}}(\alpha^*, \alpha') = \langle\alpha||\hat{A}||\alpha'\rangle$ is analytic in both α^* and α'.

Consequence of analyticity

$$\boxed{\text{An operator } \hat{A} \text{ is completely characterized by its diagonal representation} \quad \hat{A} \longleftrightarrow f_{\hat{A}}(\alpha^*, \alpha) \equiv \langle\alpha\|\hat{A}\|\alpha\rangle}$$

proof: assume $\langle\alpha\|\hat{A}\|\alpha\rangle = \langle\alpha\|\hat{B}\|\alpha\rangle$

$$\langle\alpha\|\hat{A}-\hat{B}\|\alpha\rangle = f_{\hat{A}-\hat{B}}(\alpha^*, \alpha) = 0 \quad \forall \alpha$$

since $f(\alpha^*, \alpha)$ is analytic in (e.g.) second argument it also holds that

$$f_{\hat{A}-\hat{B}}(\alpha^*, \alpha') \equiv 0$$

$$\Rightarrow \hat{A} = \hat{B} \qquad \square$$

This means every operator (here for single oscillator) can be represented as superposition of coherent-state projectors

$$\boxed{\hat{A} = \int d^2\alpha \; \phi_{\hat{A}}(\alpha) \, |\alpha\rangle\langle\alpha|} \tag{117}$$

III.2 Glauber-Sudarshan representation of the density operator

Using eq. (117) and following Roy Glauber (1963) and George Sudarshan (1963) we can represent the density operator (of a single mode) in terms of coherent-state projectors

(118)
$$\boxed{\hat{\rho} = \int d^2\alpha \; P(\alpha) \, |\alpha\rangle\langle\alpha|}$$

Glauber-Sudarshan P-representation

Some obvious properties

- $\hat{\rho}^\dagger = \hat{\rho}$ (119) $$\boxed{P(\alpha) = P^*(\alpha)}$$
- $\mathrm{Tr}\{\hat{\rho}\} = 1$

$$\mathrm{Tr}\{\hat{\rho}\} = 1 = \mathrm{Tr}\left\{\int d^2\alpha \; P(\alpha) \, |\alpha\rangle\langle\alpha|\right\}$$

$$= \sum_n \int d^2\alpha \; P(\alpha) \langle n|\alpha\rangle \langle\alpha|n\rangle$$

$$= \int d^2\alpha \; P(\alpha) \underbrace{\sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle}_{=\mathbb{I}} = \int d^2\alpha \; P(\alpha)$$

(120)
$$\boxed{\int d^2\alpha \; P(\alpha) = 1}$$

(118) – (120) make the impression that every quantum state can be written as probability distribution of coherent states. However $P(\alpha)$ is in general not positive!

Def: a state ρ is called classical state iff $P_\rho(\alpha) \geq 0 \quad \forall \alpha$

If $P_\rho(\alpha_0) < 0$ for some α_0, $\hat{\rho}$ is called non-classical

- How to obtain $P(\alpha)$ from ρ? $|\beta\rangle$ coherent state

$$\langle -\beta | \hat{\rho} | \beta \rangle = \int d^2\alpha \, P(\alpha) \langle -\beta | \alpha \rangle \langle \alpha | \beta \rangle$$

(121) $$= \int d^2\alpha \, P(\alpha) e^{-|\beta|^2} e^{-|\alpha|^2} e^{\alpha^*\beta - \alpha\beta^*}$$

so

(122) $$\langle -\beta | \hat{\rho} | \beta \rangle e^{|\beta|^2} = \underbrace{\int d^2\alpha \, P(\alpha) e^{-|\alpha|^2} e^{\alpha^*\beta - \alpha\beta^*}}_{\text{2dim. Fourier-trafo}}$$

since $\alpha = u + iv \quad \beta = x + iy$

$$\int d^2\alpha = \int du \int dv \qquad \alpha^*\beta - \alpha\beta^* = 2iuy - 2ivx$$

so

$$\int d^2\alpha \, e^{\alpha^*\beta - \alpha\beta^*} = \underbrace{\int du \, e^{2iuy}}_{\text{F-Trafo } y \leftrightarrow u} \underbrace{\int dv \, e^{-2ivx}}_{\text{F-Trafo } v \mapsto x}$$

The inverse 2-dim. Fouriertrafo of (122) yields

$$\boxed{P(\alpha)e^{-|\alpha|^2} = \frac{1}{\pi^2}\int d^2\beta\, e^{|\beta|^2}\langle-\beta|\hat\rho|\beta\rangle e^{\alpha\beta^* - \alpha^*\beta}} \tag{123}$$

examples

(i) coherent state $\hat\rho = |\beta\rangle\langle\beta|$

$$\boxed{P_\beta(\alpha) = \delta^{(2)}(\alpha-\beta) \geq 0} \tag{124}$$ classical state

(ii) thermal state

$$\hat\rho = \frac{1}{Z}\, e^{-\frac{\hbar\omega a^\dagger a}{k_B T}} \qquad Z = \mathrm{Tr}\left\{e^{-\frac{\hbar\omega a^\dagger a}{k_B T}}\right\} \tag{125}$$

with $x = \frac{\hbar\omega}{k_B T}$

$$\hat\rho = e^{-x\hat a^\dagger \hat a}(1-e^{-x})$$ is diagonal in $|n\rangle$

$$\rho_{nn} = \langle n|\hat\rho|n\rangle = e^{-nx}(1-e^{-x}) \tag{126}$$

i.e. $$\bar n = \sum_{n=0}^{\infty} \rho_{nn}\, n = \sum_{n=0}^{\infty} n e^{-nx}(1-e^{-x})$$

$$\frac{\bar n}{(1-e^{-x})} = -\frac{\partial}{\partial x}\sum_{n=0}^{\infty} e^{-nx} = -\frac{\partial}{\partial x}\frac{1}{1-e^{-x}}$$

$$= \frac{e^{-x}}{(1-e^{-x})^2}$$

so

(127) $$\bar{n} = \frac{e^{-x}}{(1-e^{-x})} = \frac{1}{e^{x}-1}$$

and

(128) $$\varrho_{nn} = \frac{1}{\bar{n}+1} \frac{1}{\left(1+\frac{1}{\bar{n}}\right)^{n}} \qquad \varrho = \sum_n \varrho_{nn} |n\rangle\langle n|$$

$$\langle -\beta|\varrho|\beta\rangle = \sum_n \varrho_{nn} \underbrace{\langle -\beta|n\rangle\langle n|\beta\rangle}$$

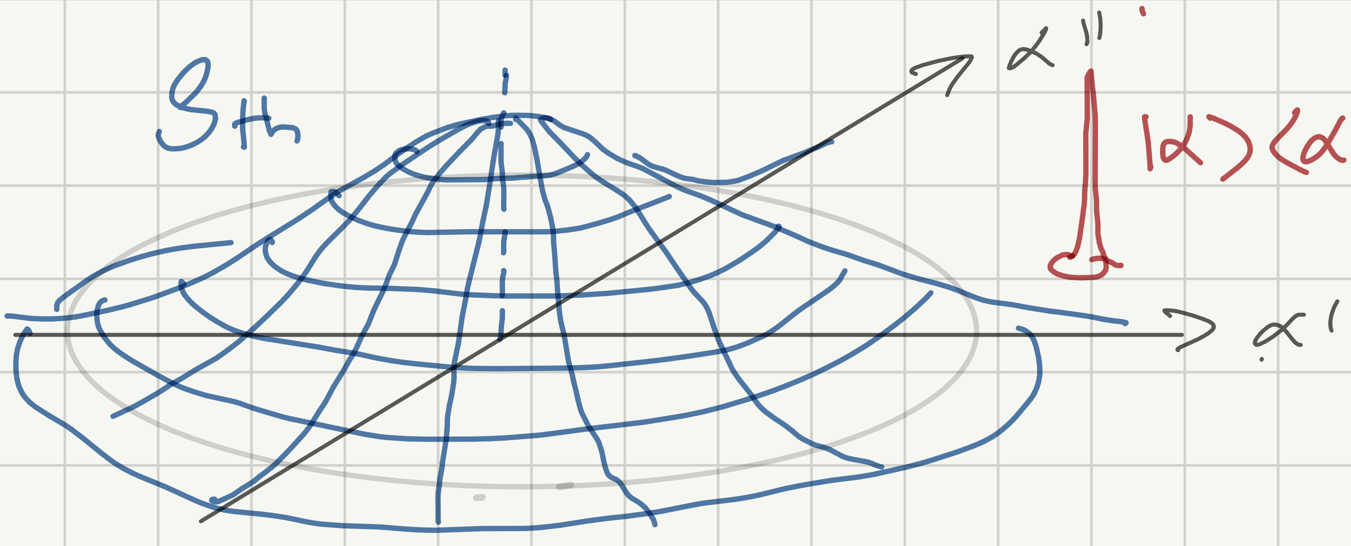
(129) $$= \sum_n \frac{1}{\bar{n}+1} \frac{1}{\left(1+\frac{1}{\bar{n}}\right)^{n}} \left(\frac{\left(-|\beta|^2\right)^{n}}{n!} e^{-|\beta|^2}\right)$$

$$= \frac{1}{\bar{n}+1} e^{-|\beta|^2} \exp\left\{-\frac{|\beta|^2}{1+1/\bar{n}}\right\}$$

so

$$P_{th}(\alpha) = \frac{e^{|\alpha|^2}}{1+\bar{n}} \underbrace{\int \frac{d^2\beta}{\pi^2} e^{\alpha\beta^* - \alpha^*\beta} \exp\left\{-\frac{|\beta|^2}{1+\frac{1}{\bar{n}}}\right\}}_{\text{2D Gauß integral}}$$

(130) $$\boxed{P_{th}(\alpha) = \frac{1}{\pi\bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}}} \geq 0 \quad \text{Gaussian classical}$$



(iii) Fock state

(131) $$\rho = |n\rangle\langle n|$$

$$\langle -\beta|\rho|\beta\rangle = \langle -\beta|n\rangle\langle n|\beta\rangle = \frac{e^{-|\beta|^2}(-|\beta|^2)^n}{n!}$$

So

$$P_n(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int d^2\beta \underbrace{\frac{(-|\beta|^2)^n}{n!} e^{\alpha\beta^* - \beta\alpha^*}}_{\text{highly singular!}}$$

The powers of β can be represented as derivatives of the exponential

$$P_n(\alpha) = \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} \underbrace{\frac{1}{\pi^2}\int d^2\beta\, e^{\alpha\beta^* - \beta\alpha^*}}_{\delta^{(2)}(\alpha)}$$

so

(132) $$\boxed{P_n(\alpha) = \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n} \delta^{(2)}(\alpha)} \not\geq 0$$

non-classical

highly singular!

(iv) Squeezed state

even worse!

expectation values

To evaluate observables from the P-distribution of the density matrix we have to calculate

$$\langle \hat{A} \rangle = \mathrm{Tr}\{\varrho \hat{A}\} = \sum_n \langle n| \int d^2\alpha\, P(\alpha) |\alpha\rangle\langle\alpha| \hat{A} |n\rangle$$

$$= \int d^2\alpha\, P(\alpha) \sum_n \langle\alpha| \hat{A} |n\rangle \langle n|\alpha\rangle$$

(133) $$\boxed{\langle \hat{A} \rangle = \int d^2\alpha\, P(\alpha)\, \langle\alpha| \hat{A} |\alpha\rangle}$$

The evaluation of $\langle\alpha|\hat{A}|\alpha\rangle$ is particularly simple if $\hat{A}$ is normal ordered

$$\hat{A}[\hat{a}^\dagger, \hat{a}] = \sum_{n,m} A_{nm}\, \hat{a}^{\dagger n} \hat{a}^m \equiv A_N[\hat{a}^\dagger, \hat{a}]$$

$$\langle\alpha| \hat{A} |\alpha\rangle = \sum_{n,m} A_{nm}\, \alpha^{*n} \alpha^m = A_N[\alpha^*, \alpha]$$

i.e.

(134) $$\boxed{\langle \hat{A} \rangle = \int d^2\alpha\, P(\alpha)\, A_N[\alpha^*, \alpha]}$$

$P(\alpha) \longleftrightarrow$ normal order of operators

Every operator can be written in normal order. The normal ordering of an operator function may however be rather involved.

e.g. consider the special case

(135) $$f(a^+, a) = f(a^+ a) = e^{\lambda a^+ a}$$

$$\langle \alpha | e^{\lambda a^+ a} | \alpha \rangle = \sum_n |\langle \alpha | n \rangle|^2 e^{\lambda n}$$

$$= \sum_n \frac{1}{n!} |\alpha|^{2n} e^{\lambda n} e^{-|\alpha|^2}$$

(136) $$= \sum_n \frac{1}{n!} (|\alpha|^2 e^{\lambda})^n e^{-|\alpha|^2}$$

$$= \exp\{ |\alpha|^2 (e^{\lambda} - 1) \}$$

Eq. (134) allows for the following interpretation: If $P(\alpha) \geq 0$ (classical) than quantum averages can be interpreted as classical averages.

Normal-ordered correlation functions can be obtained from the (normal-ordered) quantum characteristic function

(137) $$\boxed{\begin{aligned} \chi_N[\xi, \xi^*] &\equiv \langle e^{\xi \hat{a}^+} e^{-\xi^* \hat{a}} \rangle \\ &= \mathrm{Tr}\{ \varrho\, e^{\xi \hat{a}^+} e^{-\xi^* \hat{a}} \} \end{aligned}}$$

(138) $$\boxed{\langle \hat{a}^{+n} \hat{a}^m \rangle = \frac{\partial^{n+m}}{\partial^n \xi\, \partial^m (-\xi^*)} \chi_N[\xi, \xi^*] \Big|_{\xi = \xi^* = 0}}$$

It is near at hand that $\chi_N[\xi,\xi^*]$ and $P(\alpha)$ must be related. In fact it is easy to see from eq.(137)

$$\boxed{\begin{aligned}\chi_N[\xi,\xi^*] &= \int d^2\alpha\; P(\alpha)\, e^{\xi\alpha^*} e^{-\xi^*\alpha} \\ &= \int d^2\alpha\; P(\alpha)\, e^{\xi\alpha^* - \xi^*\alpha}\end{aligned}} \tag{139}$$

this is a 2-dim. Fourier transformation.

Eq. (139) allows for an interesting inversion

$$\boxed{P(\alpha) = \frac{1}{\pi^2}\int d^2\xi\; \chi_N[\xi,\xi^*]\, e^{-\xi\alpha^* + \xi^*\alpha}} \tag{140}$$

To associate operator expectation values to classical (quasi-) expectation values ["quasi" because classical distributions such as $P(\alpha)$ may not be positive] requires the fixing of an operator ordering. Above we have used normal ordering but this is only one of many possible orderings.

Can we find an analogue to the P-function for other orderings?

$\Rightarrow$ look at different characteristic functions!

III.3 The Q-representation (Husimi-representation)

Similarly to (137) we can define an anti-normal ordered quantum characteristic function

$$\boxed{\begin{aligned} \chi_A[\xi,\xi^*] &= \langle e^{-\xi^*\hat{a}} e^{\xi\hat{a}^\dagger}\rangle \\ &= \mathrm{Tr}\{\rho\, e^{-\xi^*\hat{a}} e^{\xi\hat{a}^\dagger}\} \end{aligned}} \tag{141}$$

which is the generator of anti-normal ordered correlations

$$\boxed{\langle \hat{a}^n \hat{a}^{\dagger m}\rangle = \frac{\partial^{n+m}}{\partial^n(-\xi^*)\,\partial^m\xi} \chi_A[\xi,\xi^*]\Big|_{\xi=\xi^*=0}} \tag{142}$$

χ_A and χ_N can easily be related by Baker-Hausdorff

$$e^{-\xi^*\hat{a}} e^{\xi\hat{a}^\dagger} = e^{\xi\hat{a}^\dagger} e^{-\xi^*\hat{a}} e^{-|\xi|^2}$$

thus

$$\boxed{\chi_A[\xi,\xi^*] = \chi_N[\xi,\xi^*]\, e^{-|\xi|^2}} \tag{143}$$

We have seen that the P-function was the

Fourier transform of χ_N. In a similar way we introduce the Husimi or Q-function

(144) $$\boxed{\chi_A[\xi,\xi^*] = \int d^2\alpha\, Q(\alpha)\, e^{\xi\alpha^* - \xi^*\alpha}}$$

2d-Fourier trafo

Now

$$\langle \hat{a}^n \hat{a}^{+m}\rangle = \frac{\partial^{n+m}}{\partial^n(-\xi^*)\partial^m\xi}\chi_A\Big|_{\xi=0}$$

(145) $$= \int d^2\alpha\, Q(\alpha) \frac{\partial^{n+m}}{\partial^n(-\xi^*)\partial^m\xi} e^{\xi\alpha^* - \xi^*\alpha}\Big|_{\xi=\xi^*=0}$$

$$= \int d^2\alpha\, Q(\alpha)\, \alpha^n \alpha^{*m}$$

i.e. any operator expressed in terms of anti-normal ordered products of $\hat{a}$ and $\hat{a}^+$

(146) $$\hat{A}[\hat{a}^+,\hat{a}] = \sum_{n,m} \mathcal{A}^A_{nm}\, \hat{a}^n \hat{a}^{+m} \equiv \mathcal{A}_A(\hat{a},\hat{a}^+)$$

(147) $$\boxed{\langle \hat{A}[\hat{a}^+,\hat{a}]\rangle = \int d^2\alpha\, Q(\alpha)\, \mathcal{A}_A(\alpha,\alpha^*)}$$

$Q(\alpha,\alpha^*) \longleftrightarrow$ anti-normal ordered operators

So if $Q(\alpha) > 0 \Rightarrow$ quantum average of anti-normal ordered operator expressions = classical averages

How to obtain the Q function from a quantum state ϱ?

<u>a short interlude:</u>

$$\mathrm{Tr}\{\hat{O}\} = \sum_n \langle n|\hat{O}|n\rangle = \sum_n \int \frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi} \langle n|\alpha\rangle\langle\alpha|\hat{O}|\beta\rangle\langle\beta|n\rangle$$

$$= \int\frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi} \underbrace{\sum_n \langle\beta|n\rangle\langle n|\alpha\rangle}_{\mathbb{1}} \langle\alpha|\hat{O}|\beta\rangle$$

$$= \underbrace{\int\frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi} \langle\beta|\alpha\rangle\langle\alpha|}_{\mathbb{1}}\hat{O}|\beta\rangle$$

$$= \int\frac{d^2\beta}{\pi} \langle\beta|\hat{O}|\beta\rangle$$

$$\boxed{\mathrm{Tr}\{\hat{O}\} = \int\frac{d^2\beta}{\pi}\langle\beta|\hat{O}|\beta\rangle} \quad (148)$$

- Inverse Fourier-transform of (144)

$$Q(\alpha) = \frac{1}{\pi^2}\int d^2\xi \, \chi_A[\xi,\xi^*]\, e^{\alpha\xi^*-\alpha^*\xi}$$

$$= \frac{1}{\pi^2}\int d^2\xi \, \mathrm{Tr}\{\varrho e^{-\xi^*\hat{a}} e^{\xi\hat{a}^\dagger}\} e^{\alpha\xi^*-\alpha^*\xi}$$

$$= \frac{1}{\pi^2}\int d^2\xi \int\frac{d^2\beta}{\pi} \underbrace{\langle\beta| e^{\xi\hat{a}^\dagger}\varrho e^{-\xi^*\hat{a}}|\beta\rangle}_{= e^{\xi\beta^*}\langle\beta|\varrho|\beta\rangle e^{-\xi^*\beta}} e^{\alpha\xi^*-\alpha^*\xi}$$

$$Q(\alpha) = \frac{1}{\pi^3} \int d^2\beta \, \underbrace{\int d^2\xi \; e^{\xi(\beta^* - \alpha^*) - \xi^*(\beta - \alpha)}}_{= \pi^2 \delta^{(2)}(\alpha - \beta)} \langle \beta | \hat{\varrho} | \beta \rangle$$

therefore

$$(149) \quad \boxed{Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\varrho} | \alpha \rangle}$$

I.e. the Q-function is just the expectation value of $\hat{\varrho}$ in coherent states!

properties of Husimi- or Q distribution

$$(150) \quad \boxed{Q(\alpha) = Q^*(\alpha)} \qquad \boxed{0 \le Q(\alpha) \le \frac{1}{\pi}}$$

from $Tr\{\hat{\varrho}\} = 1$ follows

$$(151) \quad \boxed{Tr\{\hat{\varrho}\} = \int \frac{d^2\alpha}{\pi} \langle \alpha | \hat{\varrho} | \alpha \rangle = \int d^2\alpha \, Q(\alpha) = 1}$$

Eqs. (150) + (151) seem to suggest that $Q(\alpha)$ is a true probability distribution in phase space spanned by

$$Re(\alpha) = \alpha' \sim x$$
$$Im(\alpha) = \alpha'' \sim p \qquad ?$$

One finds however that the marginal distributions

$$(152) \quad F(x) = \int dp \, Q(x,p) \sim \int d\alpha'' Q(\alpha)$$

(153) $$F(p) = \int dx\, Q(x,p) \sim \int d\alpha'\, Q(\alpha)$$

do not fulfill properties of a probability distribution

examples

(i) coherent state $\qquad \hat{\rho} = |\beta\rangle\langle\beta|$

$$Q_\beta(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2$$

(154) $$\boxed{Q_\beta(\alpha) = \frac{1}{\pi}\exp\left(-|\alpha-\beta|^2\right)}$$

note $P_\beta(\alpha) = \delta^{(2)}(\alpha-\beta)$

$\hat{=}$ 2-dimensional Gaussian centered at β

(ii) thermal state

$$\hat{\rho} = e^{-x\hat{n}}(1-e^{-x}) \qquad x = \frac{\hbar\omega}{k_B T}$$

$$Q_{th}(\alpha) = \frac{1}{\pi}(1-e^{-x})\langle\alpha|e^{-x\hat{a}^+\hat{a}}|\alpha\rangle$$

Making use of eq. (136) (normal ordering of $e^{\lambda a^+ a}$) we find

(155) $$Q_{th}(\alpha) = \frac{1}{\pi}(1-e^{-x})\, e^{|\alpha|^2(e^{-x}-1)}$$

using furthermore that for a thermal state

$$\bar{n} = \frac{1}{e^{x}-1} = \frac{e^{-x}}{1-e^{-x}} \quad \text{or} \quad e^{x}-1 = \frac{1}{\bar{n}}$$

$$e^{x} = \frac{1}{\bar{n}}+1 \qquad e^{-x} = \frac{1}{1+\frac{1}{\bar{n}}} \qquad 1-e^{-x} = \frac{1}{\bar{n}+1}$$

gives

(156) $$\boxed{Q_{th}(\alpha) = \frac{1}{\pi(\bar{n}+1)}\, e^{-\frac{|\alpha|^2}{\bar{n}+1}}}$$

compare to P distribution (eq. (130))

$$P_{th}(\alpha) = \frac{1}{\pi\bar{n}}\, e^{-\frac{|\alpha|^2}{\bar{n}}} \qquad Q \text{ is broader!}$$

(iii) Fock state

$$\rho = |n\rangle\langle n|$$

then

$$Q_n(\alpha) = \frac{1}{\pi}\langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{1}{\pi}|\langle\alpha|n\rangle|^2$$

(157) $$\boxed{Q_n(\alpha) = \frac{1}{\pi}\frac{|\alpha|^{2n}}{n!}\, e^{-|\alpha|^2}}$$

ring in phase space

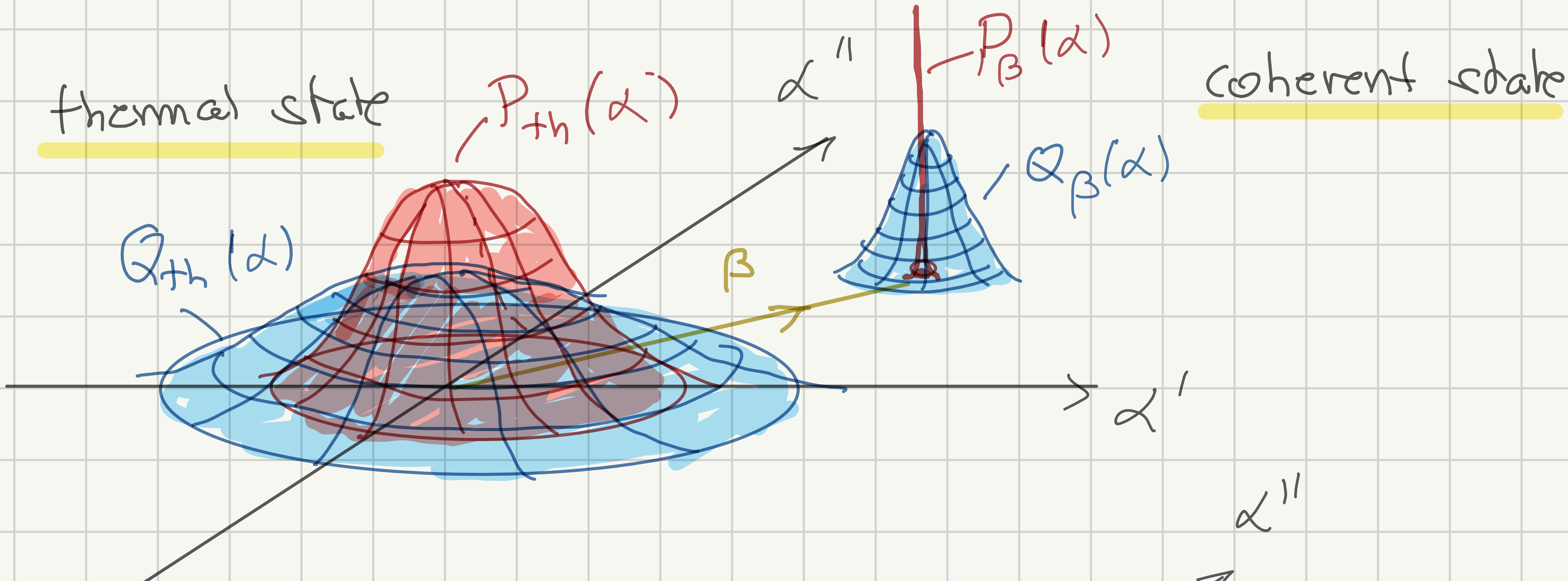
(iv) Squeezed vacuum

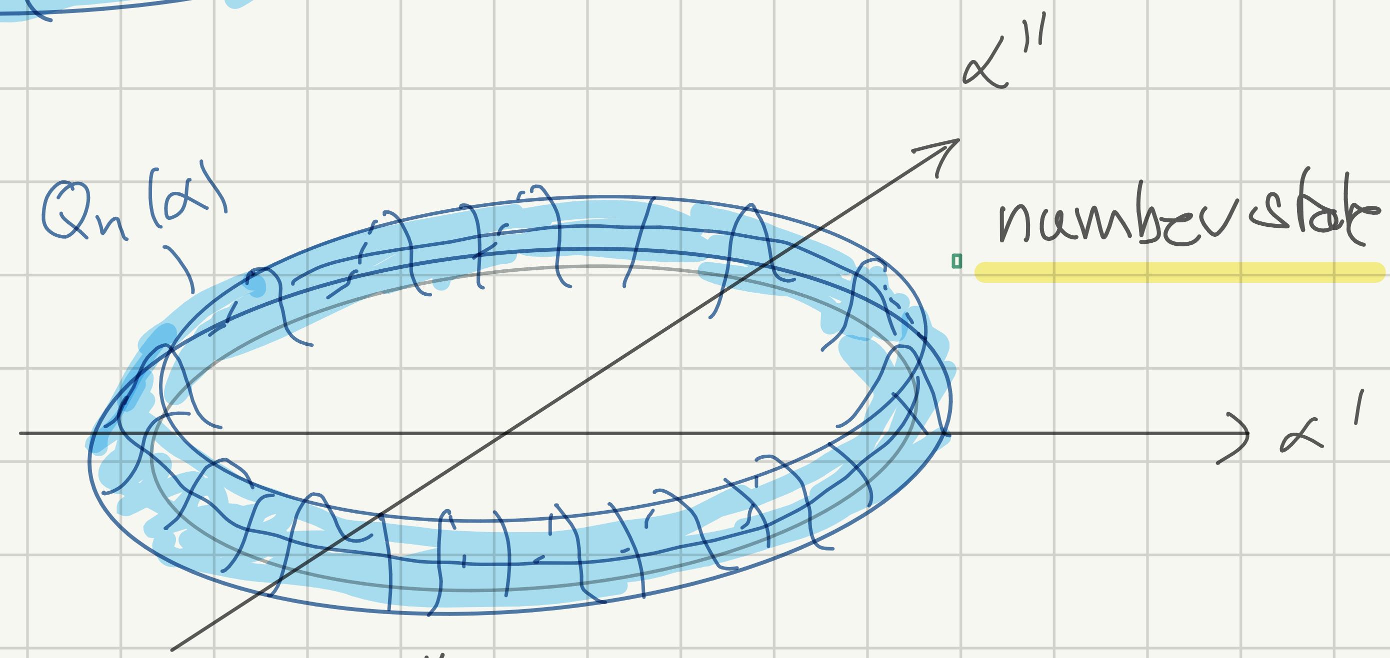
$$\rho = |0,\xi\rangle\langle 0,\xi| \qquad \xi = re^{i\theta}$$

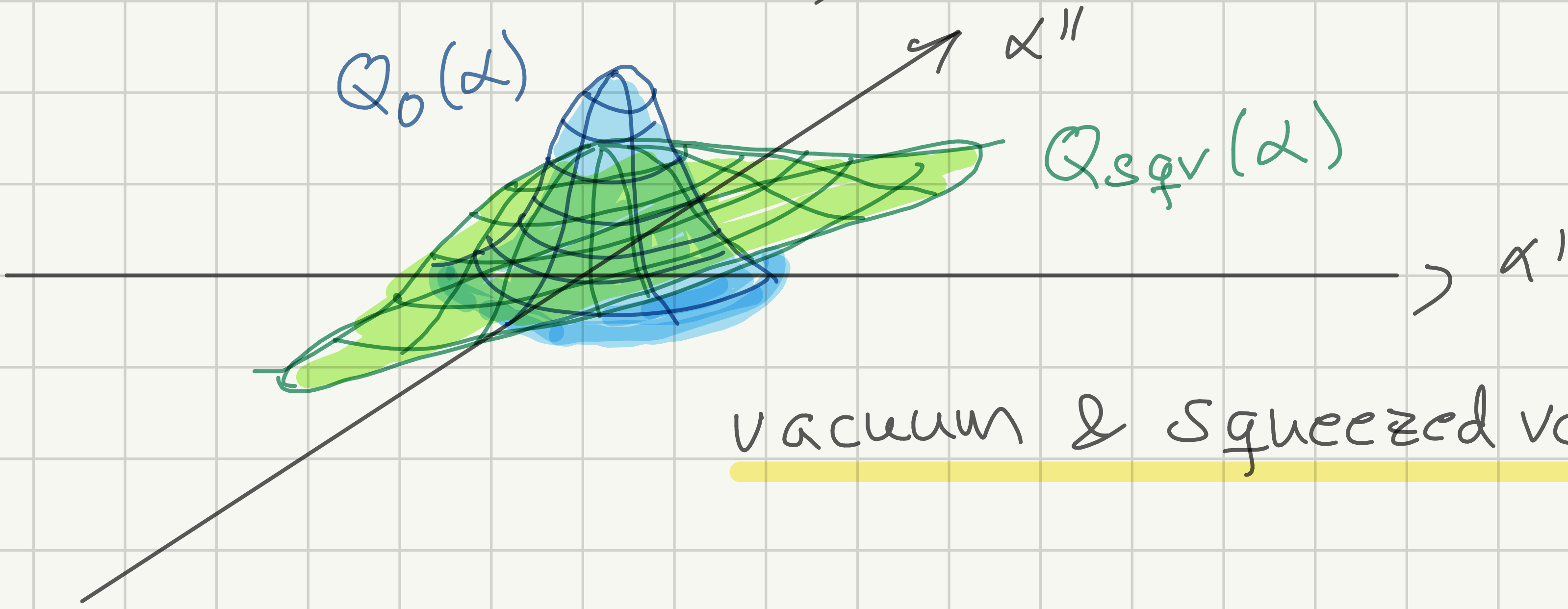
$$Q_{sq.v}(\alpha) = \frac{1}{\pi} \langle \alpha | 0, \xi \rangle \langle 0, \xi | \alpha \rangle$$

$$= \frac{1}{\pi} \left| \langle \alpha | \hat{S}(\xi) | 0 \rangle \right|^2$$

$$\boxed{Q_{sq.v}(\alpha) = \frac{e^{-|\alpha|^2} \operatorname{sech} r}{\pi} \exp\left\{ -\frac{1}{2} \left(e^{i\theta} \alpha^{*2} + e^{-i\theta} \alpha^2 \right) \tanh r \right\}} \tag{158}$$







The Husimi (Q) distribution is well suited for visualization of quantum states in phase space. We recognize that it is a "smoothened" distribution

$$P(\alpha) \overset{FT}{\longleftrightarrow} \chi_N \longleftrightarrow \chi_A = \chi_N e^{-|\xi|^2} \overset{FT}{\longleftrightarrow} Q(\alpha)$$

It is easy to see that

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle = \frac{1}{\pi} \int d^2\beta \, P(\beta) \underbrace{\langle \alpha | \beta \rangle \langle \beta | \alpha \rangle}_{= |\langle \alpha | \beta \rangle|^2}$$

(159) $$\boxed{Q(\alpha) = \frac{1}{\pi} \int d^2\beta \, \underbrace{e^{-|\alpha-\beta|^2}}_{\text{Smoothening kern}} P(\beta)}$$

For practical calculations (see also later) the Q function is not very useful, but there is a third representation which is sort of "inbetween" P and Q

III.4 Wigner-Weyl representation

We have seen

$$P(\alpha) \longleftrightarrow \chi_N = \langle e^{\xi a^\dagger} e^{-\xi^* a} \rangle$$

$$Q(\alpha) \longleftrightarrow \chi_A = \langle e^{-\xi^* \hat{a}} e^{\xi \hat{a}^\dagger} \rangle$$

how about symmetric ordering?

(160) $$\boxed{\chi_W[\xi,\xi^*] \equiv \mathrm{Tr}\{\varrho\, e^{-\xi^*\hat{a}+\xi\hat{a}^\dagger}\}}$$

Wigner characteristic function

The Wigner characteristic function is the generating function of

symmetrically ordered operator products

(161) $$\langle(\hat{a}^{\dagger m}\hat{a}^{n})_{symm}\rangle = \frac{\partial^{n+m}}{\partial\xi^{m}\partial(-\xi^*)^{n}}\chi_W\Big|_{\xi=\xi^*}$$

e.g. $$(\hat{a}^{\dagger 2}\hat{a})_{symm} = \frac{1}{3}\left(\hat{a}^{\dagger 2}\hat{a}+\hat{a}^\dagger\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^{\dagger 2}\right)$$

Using the Baker Hausdorff relation one easily sees that the Wigner characteristic function is in between normal- and anti-normal ordered ones:

(162) $$\chi_W = \chi_N[\xi,\xi^*]\,e^{-\frac{1}{2}|\xi|^2} = \chi_A[\xi,\xi^*]\,e^{\frac{1}{2}|\xi|^2}$$

The 2-dimensional Fourier transform of $\chi_W[\xi,\xi^*]$ defines the Wigner-Weyl distribution

(163) $$\boxed{\chi_W[\xi,\xi^*] = \int d^2\alpha\, W(\alpha)\, e^{\xi\alpha^*-\xi^*\alpha}}$$

Since every operator can be expanded into symmetrically ordered products

$$\hat{A}[\hat{a}^\dagger \hat{a}] = \sum_{n,m} A^s_{nm} \left(\hat{a}^{\dagger n} \hat{a}^m\right)_{\text{symm}} \tag{164}$$

$$= A_s[\hat{a}^\dagger, \hat{a}]$$

one finds for expectation values

$$\boxed{\langle \hat{A}[\hat{a}^\dagger, \hat{a}] \rangle = \int d^2\alpha \, W(\alpha) \, A_s(\alpha^*, \alpha)} \tag{165}$$

$W(\alpha) \Longleftrightarrow$ symmetrically-ordered operator expressions

From the relation between the characteristic functions (162) follows a relation between $P(\alpha)$, $Q(\alpha)$ and $W(\alpha)$

$$W(\alpha) = \frac{2}{\pi} \int d^2\beta \, e^{-2|\alpha-\beta|^2} P(\beta) \tag{166}$$

Wigner is a smoothened P-distribution

$$Q(\alpha) = \frac{2}{\pi} \int d^2\beta \, e^{-2|\alpha-\beta|^2} W(\beta) \tag{167}$$

$$= \frac{1}{\pi} \int d^2\beta \, e^{-|\alpha-\beta|^2} P(\beta)$$

Q-function is a smoothened Wigner distribution

normal order	symmetric order	anti-normal order
$P(\alpha)$	$W(\alpha)$	$Q(\alpha)$
very singular	weakly singular	regular
vacuum = trivial	vacuum = "$\frac{1}{2}$ photon"	vacuum = "1 photon"
$P_0(\alpha) = \delta^{(2)}(\alpha)$	$W_0(\alpha) = \frac{2}{\pi} e^{-2\|\alpha\|^2}$	$Q_0(\alpha) = \frac{1}{\pi} e^{-\|\alpha\|^2}$

example of Wigner

(i) coherent state $\qquad \rho = |\beta\rangle\langle\beta|$

(168) $$\boxed{W_\beta(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\beta|^2}}$$

Gaussian but narrower that $Q_\beta(\alpha)$

(ii) Fock state $\qquad \rho = |n\rangle\langle n|$

$$W_n(\alpha) = \frac{2}{\pi}\int d^2\beta\; e^{-2|\alpha-\beta|^2}\, \frac{e^{|\beta|^2}}{n!}\, \frac{\partial^{2n}}{\partial(\beta)^n\partial(-\beta^*)^n}\,\delta^{(2)}(\beta)$$

partial integration ...

(169) $$\boxed{W_n(\alpha) = \frac{2}{\pi}(-1)^n L_n\left[4|\alpha|^2\right] e^{-2|\alpha|^2}}$$

$L_n(x)$ Laguerre polynomials

We see that $W_n(\alpha)$ is in general not positive

In general one finds

$$\boxed{-\frac{2}{\pi} \leq W(\alpha) \leq \frac{2}{\pi}} \tag{170}$$

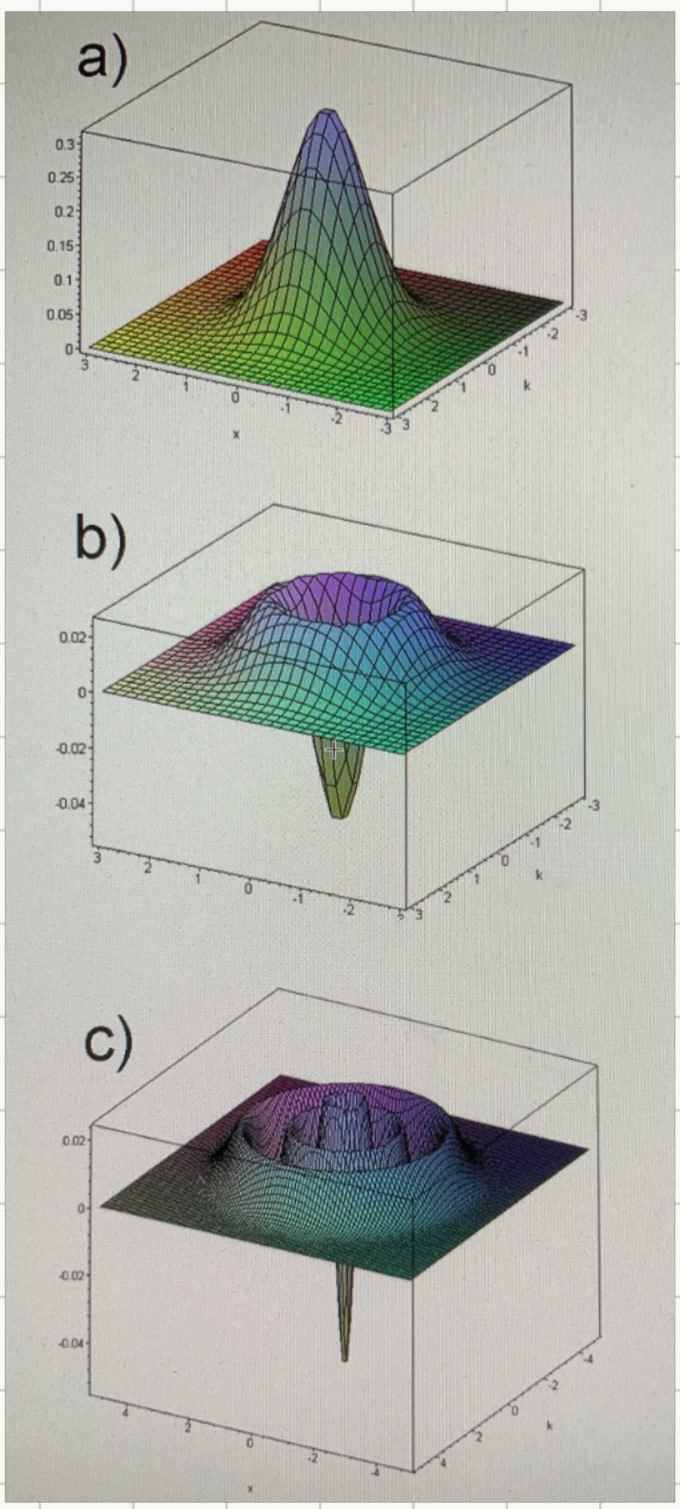
(iii) squeezed vacuum and $\theta = 0$

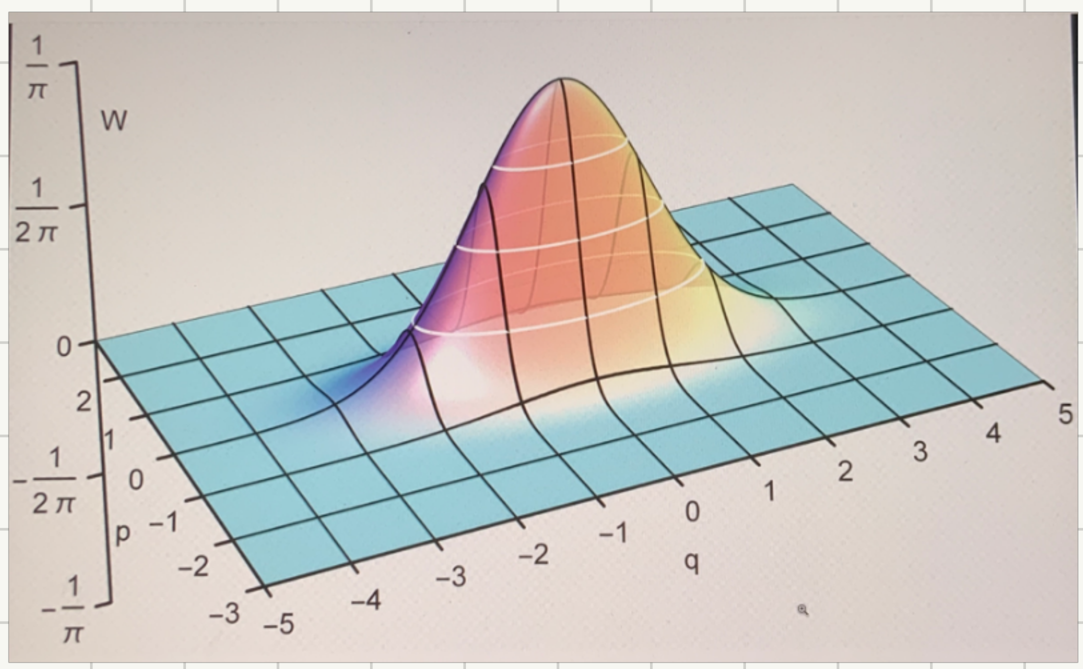
$$\boxed{W_{sq.v}(\alpha) = \frac{2}{\pi} \exp\left\{-2\left(\frac{\alpha'^2}{e^{-2\xi}} + \frac{\alpha''^2}{e^{2\xi}}\right)\right\}} \tag{171}$$

$|0\rangle$

$|n=1\rangle$

$|n=5\rangle$





squeezed vacuum

© wikipedia