II. Quantum states of bosons

We have seen that bosonic quantum fields can be decomposed into suitable complete and orthonormal sets of eigenmodes where the pre-coefficients are bosonic annihilation and creation operators which are formally identical to those which we know from the harmonic oscillator in elementary quantum mechanics:

Schrödinger

(63)
$$\hat{\Psi}(\vec{r},t) \sim \sum_n \hat{a}_n(t)\,\phi_n(\vec{r})$$
$$\hat{\Psi}^+(\vec{r},t) \sim \sum_n \hat{a}_n^+(t)\,\phi_n^*(\vec{r})$$

EDM

(64)
$$\hat{E}(\vec{r},t) = \hat{E}^+(\vec{r},t)$$
$$\sim \sum_n \underbrace{\hat{a}_n(t) f_n(\vec{r})}_{\equiv \hat{E}^{(+)}(\vec{r},t)} + \underbrace{\hat{a}_n^+(t) f_n^*(\vec{r})}_{\equiv \hat{E}^{(-)}(\vec{r},t)}$$

positive/negative frequency components

(65)
$$[\hat{a}_n, \hat{a}_m] = 0 \qquad [\hat{a}_n, \hat{a}_m^+] = \delta_{nm}$$

It is useful to (re)consider particular sets of basis states for a given mode "m"

II.1 Fock states, coherent states, squeezed states

(A) Fock states = number states

$$\hat{n} = \hat{a}^{+}\hat{a} \qquad \hat{n}|n\rangle = n|n\rangle \qquad n = 0,1,2,\dots$$

(66)
$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{annihilation}$$
$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \text{creation}$$

(67)
$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}\,\hat{a}^{+n}|0\rangle}$$

since $|n\rangle$ is eigenstate of $\hat{n}$

$$\langle n|\hat{n}|n\rangle = n \qquad \langle n|\Delta\hat{n}^{2}|n\rangle = 0$$

However

(68)
$$\langle n|\hat{\varphi}|n\rangle = 0 \qquad \langle n|\hat{E}^{(+)}|n\rangle = 0$$

since

$$\langle n|\hat{a}|n\rangle = \langle n|\hat{a}^{+}|n\rangle = 0$$

In an eigenstate of the number of particles / photons the expectation value of the (frequency component of the) field vanishes!

(B) Coherent states

Let us define an important class of states, but in a different way as you may have done this in quantum mechanics.

coherent displacement operator

(69) $$\boxed{\hat{D}(\alpha) \equiv \exp\{\alpha \hat{a}^{+} - \alpha^{*} \hat{a}\}} \qquad \alpha \in \mathbb{C}$$

properties:

(70) $$\hat{D}(\alpha=0) = \mathbb{1}$$

$$\hat{D}^{+}(\alpha) = \exp\{\alpha^{*} \hat{a} - \alpha \hat{a}^{+}\} = \hat{D}(-\alpha)$$

Baker-Hausdorff

(71) $$\boxed{\begin{aligned} &\text{if } [A,[A,B]] = [B,[A,B]] = 0 \\ &e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]} = e^{B} e^{A} e^{+\frac{1}{2}[A,B]} \end{aligned}}$$

i.e. $$\hat{D}(\alpha) = e^{\alpha \hat{a}^{+} - \alpha^{*} \hat{a}} = e^{|\alpha|^2/2} e^{-\alpha^{*} \hat{a}} e^{\alpha \hat{a}^{+}}$$

$$= e^{-|\alpha|^2/2} e^{\alpha \hat{a}^{+}} e^{-\alpha^{*} \hat{a}}$$

$$\Rightarrow \quad \hat{D}(\alpha)\hat{D}(\beta) = e^{\alpha \hat{a}^{+} - \alpha^{*} \hat{a}} e^{\beta \hat{a}^{+} - \beta^{*} \hat{a}}$$

$$= \underbrace{e^{(\alpha+\beta)\hat{a}^{+} - (\alpha^{*}+\beta^{*})\hat{a}}}_{D(\alpha+\beta)} e^{\frac{1}{2}[\alpha \hat{a}^{+} - \alpha^{*} \hat{a}, \beta \hat{a}^{+} - \beta^{*} \hat{a}]}$$

(72) $$\boxed{\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\, e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}} \neq \hat{D}(\beta)\hat{D}(\alpha)$$

$$\Rightarrow \quad \hat{D}(\alpha)\hat{D}(-\alpha) = \mathbb{1} \quad \Rightarrow \hat{D}(-\alpha) = \hat{D}^{-1}(\alpha)$$

(73) $$\boxed{\hat{D}^{+}(\alpha) = \hat{D}^{-1}(\alpha)}$$ unitary

Now we define the unitary shift of the vacuum state (ground state of harmonic oscillator)

(74) $$\boxed{|\alpha\rangle \equiv \hat{D}(\alpha)\,|0\rangle}$$ coherent state

From Baker-Hausdorff (77) one finds ($\underbrace{[A,[A,B]] = [B,[A,B]]}_{=0}$)

(75) $$e^{-\alpha A} B e^{\alpha B} = B - \alpha[A,B] + \underbrace{\frac{\alpha^2}{2!}[A,[A,B]] + \dots}_{=0}$$

with this one finds for the unitary transformation of annihilation and creation operators

(76) $$\boxed{\begin{aligned} \hat{D}^{+}(\alpha)\, \hat{a}\, \hat{D}(\alpha) &= \hat{D}(-\alpha)\, \hat{a}\, \hat{D}(\alpha) = \hat{a} + \alpha \\ \hat{D}^{+}(\alpha)\, \hat{a}^{+} \hat{D}(\alpha) &= \hat{D}(-\alpha)\, \hat{a}^{+} \hat{D}(\alpha) = \hat{a}^{+} + \alpha^* \end{aligned}}$$

shift by classical amplitude

From these relations we can now see that $|\alpha\rangle$ is a right eigenstate of $\hat{a}$ (and left eigenstate of $\hat{a}^\dagger$)

(77)
$$\boxed{\begin{aligned} \hat{a}\,|\alpha\rangle &= \alpha\,|\alpha\rangle \\ \langle\alpha|\,\hat{a}^\dagger &= \alpha^*\,\langle\alpha| \end{aligned}}$$

<u>proof:</u> $\hat{a}\,|\alpha\rangle = \hat{a}\,\hat{D}(\alpha)|0\rangle = \hat{D}(\alpha)(\hat{a}+\alpha)|0\rangle$
$$= \hat{D}(\alpha)\,\alpha\,|0\rangle = \alpha\,|\alpha\rangle \qquad \square$$

properties of coherent states

- while for Fock states we had $\langle n|\hat{\Psi}|n\rangle = 0$ $\langle n|\hat{\vec{E}}^{(+)}|n\rangle = 0$ we now have in a multi-mode coherent state

(78)
$$\langle\{\alpha\}|\,\hat{\Psi}(\vec{r})\,|\{\alpha\}\rangle = \sum_n \varphi_n(\vec{r})\,\langle\{\alpha\}|\,\hat{a}_n\,|\{\alpha\}\rangle$$
$$= \sum_n \varphi_n(\vec{r})\,\alpha_n \neq 0$$

We will see that a quantum field in a coherent state behaves essentially as the classical field

- furthermore

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle = e^{-|\alpha|^2/2}\,e^{\alpha\hat{a}^\dagger}\,\underbrace{e^{-\alpha^*\hat{a}}\,|0\rangle}_{=|0\rangle}$$
$$= e^{-|\alpha|^2/2}\,e^{\alpha\hat{a}^\dagger}\,|0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n \underbrace{\hat{a}^{\dagger n} |0\rangle}_{= \sqrt{n!}\,|n\rangle}$$

78) $$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

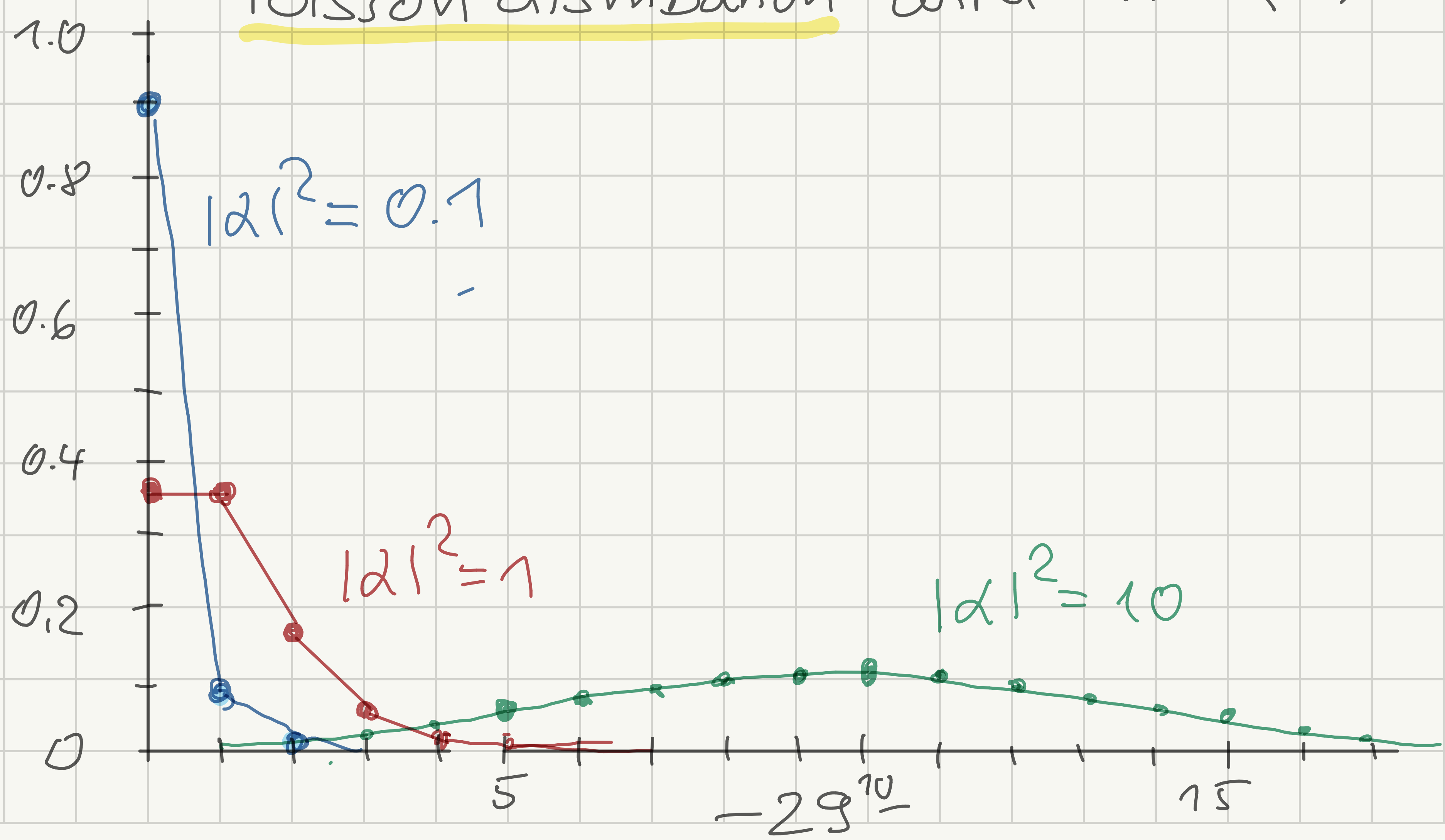
$\Rightarrow$ coherent state does <u>not</u> have well defined number of particles / photons, but

(79) $$\boxed{\langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2}$$

number distribution

(80) $$\boxed{P_\alpha(n) = |\langle \alpha | n \rangle|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!}}$$

Poisson distribution with $\bar{n} = \langle \hat{n} \rangle = |\alpha|^2$



one finds

$$\langle\alpha|\Delta\hat{n}^2|\alpha\rangle = \langle\alpha|(\hat{n} - \langle\hat{n}\rangle)^2|\alpha\rangle = \langle\alpha|\hat{n}^2|\alpha\rangle - |\alpha|^4$$

$$= \langle\alpha|\underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}}_{=\hat{n}^2}|\alpha\rangle - |\alpha|^4$$

$$= |\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2 = \langle\alpha|\hat{n}|\alpha\rangle$$

(81) $$\langle\Delta\hat{n}^2\rangle_\alpha = \langle\hat{n}\rangle_\alpha$$ Poisson

- coherent states are only "almost" orthogonal

$$\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*n}\beta^m}{\sqrt{n!\,m!}} \underbrace{\langle n|m\rangle}_{=\delta_{nm}} e^{-|\beta|^2/2}$$

$$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n!}$$

$$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} e^{\alpha^*\beta}$$

(82) $$\boxed{\langle\alpha|\beta\rangle = \exp\left\{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2-2\alpha^*\beta\right)\right\} \neq 0}$$

however

$$|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right) \xrightarrow[\text{for } |\alpha-\beta|\to\infty]{} 0$$

- coherent states are over-complete

$$\boxed{\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}} \tag{83}$$

α complex!

$\alpha = re^{i\varphi}$

proof:

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d^2\alpha \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^{*n}\alpha^m}{\sqrt{n!m!}} |n\rangle\langle m|$$

$$= \frac{1}{\pi}\sum_{nm} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int_0^\infty dr\, e^{-r^2} r^{n+m+1} \underbrace{\int_0^{2\pi} d\varphi\, e^{-i(n-m)\varphi}}_{2\pi\delta_{nm}}$$

$$= \frac{2\pi}{\pi}\sum_n \frac{|n\rangle\langle n|}{n!} \underbrace{\int_0^\infty dr\, e^{-r^2} r^{2n+1}}_{=\frac{1}{2}n!} = \sum_n |n\rangle\langle n| = \mathbb{1}$$

$\square$

- coherent states are states of minimum uncertainty

in analogy to the harmonic oscillator we introduce two hermitian operators

$$\hat{q} \leftrightarrow \hat{a}_1 = \frac{\hat{a}+\hat{a}^\dagger}{2} \qquad \hat{p} \leftrightarrow \hat{a}_2 = \frac{\hat{a}-\hat{a}^\dagger}{2i} \tag{84}$$

$$[\hat{a}_1, \hat{a}_2] = \frac{i}{2} \tag{85}$$

Heisenberg uncertainty

(86) $$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle \geq \frac{1}{16}$$

now for coherent state

$$\langle \alpha | \hat{a}_1 | \alpha \rangle = \frac{1}{2} \langle \alpha | \hat{a} + \hat{a}^\dagger | \alpha \rangle = \frac{\alpha + \alpha^*}{2} = \mathrm{Re}[\alpha]$$

$$\langle \alpha | \hat{a}_2 | \alpha \rangle = \frac{1}{2i} \langle \alpha | \hat{a} - \hat{a}^\dagger | \alpha \rangle = \frac{\alpha - \alpha^*}{2i} = \mathrm{Im}[\alpha]$$

one finds after simple calculation

(87) $$\boxed{\langle \alpha | \Delta \hat{a}_1^2 | \alpha \rangle = \langle \alpha | \Delta \hat{a}_2^2 | \alpha \rangle = \frac{1}{4}}$$

$$\langle \alpha | \Delta \hat{a}_1^2 | \alpha \rangle \langle \alpha | \Delta \hat{a}_2^2 | \alpha \rangle = \frac{1}{16}$$ minimum uncertainty

- What is the action of

$$\hat{a}^\dagger | \alpha \rangle = ? \qquad \langle \alpha | \hat{a} = ?$$

for this it is convenient to introduce the non normalized Bargmann states

(88) $$\| \alpha \rangle \equiv | \alpha \rangle e^{|\alpha|^2/2}$$

$$\hat{a}^\dagger \| \alpha \rangle = \hat{a}^\dagger \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \alpha^n | n \rangle = \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{n!}} \alpha^n | n+1 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{n+1}{\sqrt{(n+1)!}} \alpha^n | n+1 \rangle = \sum_{m=0}^{\infty} \frac{m}{\sqrt{m!}} \alpha^{m-1} | m \rangle$$

$$= \frac{\partial}{\partial\alpha} \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} \alpha^m |m\rangle = \frac{\partial}{\partial\alpha} \| \alpha\rangle$$

$\Rightarrow$

$$\hat{a}^\dagger \|\alpha\rangle = \frac{\partial}{\partial\alpha} \|\alpha\rangle = \frac{\partial}{\partial\alpha}\left(e^{-|\alpha|^2/2} |\alpha\rangle\right) \tag{89}$$

$$= e^{-|\alpha|^2/2} \left(-\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle$$

this then yields

$$\boxed{\begin{aligned} \hat{a}^\dagger |\alpha\rangle &= \left(-\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle \\ \langle\alpha| \hat{a} &= \left(-\alpha + \frac{\partial}{\partial\alpha^*}\right)\langle\alpha| \end{aligned}} \tag{90}$$

which will become important later on.

(c) Squeezed coherent states*

Although they will be of less importance for the purpose of this lecture let us introduce another interesting class of states

squeezing operator

$$\boxed{\hat{S}(\xi) \equiv \exp\left\{\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)\right\}} \tag{91}$$

$\hat{S}$ is again unitary

(92) $$\hat{S}^{-1}(\xi) = \hat{S}^{+}(\xi) = \hat{S}(-\xi)$$

a unitary transformation of annihilation and creation operators is a Bogoliubov transformation

(93)
$$\boxed{\begin{aligned} \hat{S}^{+}(\xi)\,\hat{a}\,\hat{S}(\xi) &= \hat{a}\cosh r - \hat{a}^{+} e^{i\theta} \sinh r \\ \hat{S}^{+}(\xi)\,\hat{a}^{+}\hat{S}(\xi) &= \hat{a}^{+}\cosh r - \hat{a}\, e^{-i\theta} \sinh r \\ \xi &= r e^{i\theta} \end{aligned}}$$

squeezed coherent states

(94) $$\boxed{|\alpha, \xi\rangle \equiv \hat{S}(\xi)\hat{D}(\alpha)\,|0\rangle}$$

squeezed vacuum $\quad |0, \xi\rangle = \hat{S}(\xi)\,|0\rangle$

properties

(95) $$\langle \alpha, \xi|\,\hat{a}\,|\alpha, \xi\rangle = \alpha\cosh r - \alpha^{*} \sinh r\, e^{i\theta}$$

squeezed vacuum $\quad \alpha \equiv 0$

$$\langle \hat{a} \rangle_{sqv} = 0 \qquad \langle a^{+} a\rangle_{sqv} = \sinh^{2} r \qquad \langle \hat{a}^{2}\rangle_{sqv} = -e^{i\theta}\cosh r \sinh r$$

rotated quadrature components

$$\hat{a}_{1\theta} \equiv \frac{1}{2}\left(\hat{a} e^{-\frac{i\theta}{2}} + \hat{a}^{+} e^{\frac{i\theta}{2}}\right) \tag{96}$$

$$\hat{a}_{2\theta} \equiv \frac{1}{2i}\left(\hat{a} e^{-\frac{i\theta}{2}} - \hat{a}^{+} e^{\frac{i\theta}{2}}\right)$$

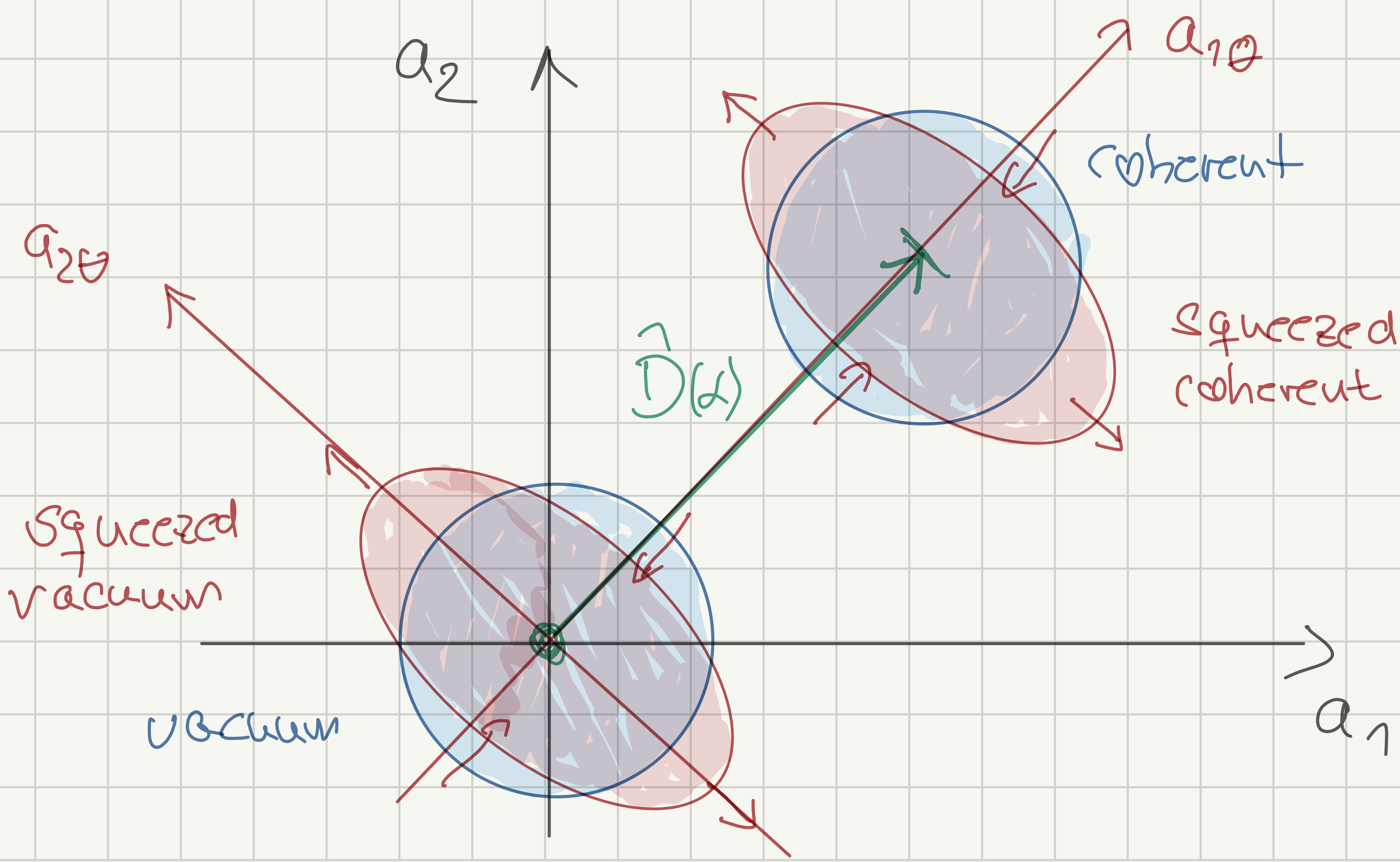
$$\boxed{\begin{aligned} \langle \alpha, \xi | \Delta\hat{a}_{1\theta}^{2} | \alpha, \xi \rangle &= \frac{1}{4} e^{-2r} \\ \langle \alpha, \xi | \Delta\hat{a}_{2\theta}^{2} | \alpha, \xi \rangle &= \frac{1}{4} e^{+2r} \end{aligned}} \tag{97}$$

smaller than vacuum

larger than vacuum

"squeezed" fluctuations

$$\langle \Delta\hat{a}_{1\theta}^{2} \rangle \langle \Delta\hat{a}_{2\theta}^{2} \rangle = \frac{1}{16} \tag{98}$$ minimum uncertainty



II.2 Classical approximation to quantum fields

We will now argue that the classical equations of motion for the Schrödinger field of e.g. an ultra-cold Bose gas or the elm. field are obtained assuming that the many-body state is close to a multi-mode coherent state.

Let us assume that the many-body state of e.g. an ultra-cold quantum gas is approximately a multi-mode coherent state

$$|\psi(t)\rangle \approx \prod_n \hat{D}_n(\alpha_n(t))\,|0\rangle \tag{99}$$

with time dependent amplitudes $\alpha_n(t)$

(A) Schrödinger field of ultra-cold, neutral atoms

Let us first consider ultra-cold atoms with s-wave interaction, eq. (46). The Hamiltonian, eq. (46) conserves the total particle number

$$\hat{N} = \int d^3r\; \hat{\psi}^\dagger(\bar{r})\hat{\psi}(\bar{r}) \qquad [\hat{N},\hat{H}] = 0 \tag{100}$$

$\hat{N}$ is a Noether charge related to the global U(1) symmetry of $\hat{H}$, i.e the invariance under

$$\hat{\psi}(\bar{r},t) \rightarrow \hat{\psi}(\bar{r},t)\, e^{i\phi} \tag{101}$$

This symmetry is explicitely broken with the ansatz for $|\psi(t)\rangle$ in eq. (95). Still number fluctuations in a coherent state are Poissonian i.e.

$$\frac{\sqrt{\langle \Delta \hat{N}^2 \rangle_{coh}}}{\langle \hat{N} \rangle_{coh}} \sim \frac{1}{\sqrt{\langle \hat{N} \rangle_{coh}}} \rightarrow 0 . \tag{102}$$

ground-state variational approach

If particle number is not conserved we should not minimize the expectation value of $\hat{H}$ but of the grand-canonical Hamiltonian

$$\boxed{\hat{H} - \mu \hat{N}} \tag{103}$$

where μ is the chemical potential that fixes $\langle \hat{N} \rangle$. This is equivalent to minimization with a constraint $\langle \hat{N} \rangle =$ fixed where μ is a Lagrange parameter. Then the minimization

$$\langle \psi | \hat{H} - \mu \hat{N} | \psi \rangle = \min \tag{104}$$

yields

$$\boxed{\left(\frac{\hbar^2}{2m}\Delta + V(\vec{r}) + g\,|\psi(\vec{r})|^2\right)\psi(\vec{r}) = \mu\,\psi(\vec{r})}$$

(105) time-independent Gross-Pitaevski equation

time-dependent variational approach (Jakin)

(106) $$\mathcal{L} \hat{=} \langle \psi(t)|\, i\hbar\frac{\partial}{\partial t} - \hat{H}\, |\psi(t)\rangle$$

Euler-Lagrange equations

(107) $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\alpha}_k} - \frac{\partial \mathcal{L}}{\partial \alpha_k} = 0$$

compact form: c-number field $\hat{=} \{\alpha_k(t)\}$

(108) $$\psi(\vec{r},t) = \sum_k \alpha_k(\vec{r},t)\,\phi_k(\vec{r})$$

Eigenmodes

Euler Lagrange $\Rightarrow$

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{r}) + g\,|\psi(\vec{r},t)|^2\right)\psi(\vec{r},t)}$$

(109) Gross-Pitaevski equation

(B) Nonlinear Quantum optics

In a very similar way one can treat problems from nonlinear quantum optics. For example second harmonic generation of monochromatic light can be described by a quantum Hamiltonian of two modes

$$\hat{a}_1 \quad \text{frequency } \omega \qquad \hat{a}_2 \quad \text{frequency } 2\omega$$

(110) $$\hat{H} = \hbar\omega \hat{a}_1^+ \hat{a}_1 + 2\hbar\omega \hat{a}_2^+ \hat{a}_2 + \hbar \frac{g}{2} \hat{a}_2^+ \hat{a}_1^2 + \hbar \frac{g}{2} \hat{a}_1^{+2} \hat{a}_2$$

The variational ansatz

(111) $$|\psi(t)\rangle = \hat{D}_{a_1}(\alpha_1) \hat{D}_{a_2}(\alpha_2) |0\rangle$$

then yields the classical equations

(112) $$\begin{aligned} \dot{\alpha}_1 &= -i\omega \alpha_1 - i g \alpha_1^* \alpha_2 \\ \dot{\alpha}_2 &= -2i\omega \alpha_2 - i \frac{g}{2} \alpha_1^2 \end{aligned}$$