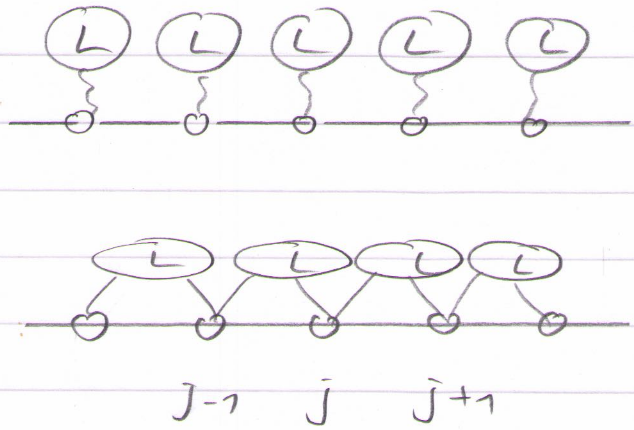
Translationsinvarianz um einen Gitterplatz



periodische RB mit Länge N

$\Rightarrow$ alle Matrizen sind translationsinvariant

$$\Gamma^{\alpha\beta}_{jk} = \Gamma^{\alpha\beta}_{j-k} \qquad \text{Kovarianzmatrix}$$

$$M^{\alpha\beta}_{jk} = M^{\alpha\beta}_{j-k} \qquad \text{Badmatrix}$$

$$h^{\alpha\beta}_{jk} = h^{\alpha\beta}_{j-k} \qquad \text{Hamiltonmatrix}$$

$\Rightarrow X^{\alpha\beta}_{jk}, Y^{\alpha\beta}_{jk}$ translationsinvariant

Fouriertransformation

$$A^{\alpha\beta}_{j-k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \; \underbrace{a^{\alpha\beta}(\varphi)}_{2\times 2 \text{ Matrix}} \, e^{i\varphi(j-k)}$$

F-transformierte Sylvester Lyapunov Gleichung

$$\boxed{x^T(-\varphi)\,\Gamma(\varphi) + \Gamma(\varphi)\,x(\varphi) = y(\varphi)}$$

alles 2×2 Matrizen

<u>Korrelationsfunktionen</u>

$$\langle a_j^+ a_{j+d} \rangle = \frac{1}{4}\langle (u_j^1 - i u_j^2)(u_{j+d}^1 + i u_{j+d}^2) \rangle$$

$$= \frac{1}{4}\Big\{ \langle u_j^1 u_{j+d}^1 \rangle - i\langle u_j^2 u_{j+d}^1 \rangle + i\langle u_j^1 u_{j+d}^2 \rangle + \langle u_j^2 u_{j+d}^2 \rangle \Big\}$$

es gilt

$$[u_j^1, u_k^1] = [u_j^2, u_k^2] = 0$$

$$[u_j^1, u_k^2] = -2i\delta_{jk}$$

$\Rightarrow$

$$\langle a_j^+ a_{j+d} \rangle \longleftrightarrow \Gamma_d^{\alpha\beta}$$

$$\Gamma_d^{\alpha\beta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\, \Gamma^{\alpha\beta}(\varphi)\, e^{i\varphi d}$$

$$\varphi \to z = e^{i\varphi} \qquad dz = i z \, d\varphi$$

$$\boxed{\Gamma_d^{\alpha\beta} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \, g^{\alpha\beta}(z) \, z^d = \sum_{z_i} g^{\alpha\beta}(z_i) \, z_i^{d-1}}$$

z_i – Residuen von $g^{\alpha\beta}(z)$ in $|z| \le 1$

Mit Hilfe dieses Ausdrucks können z.B. <u>Korrelationslängen</u> einfach berechnet werden

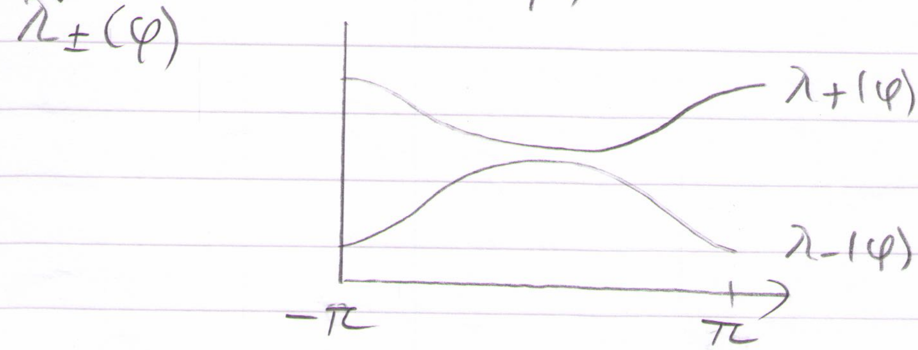
$$\xi^{-1} \equiv \lim_{d\to\infty} \frac{\ln |\langle a_j^+ a_{j+d} \rangle|}{d}$$

ξ divergiert (langreichweitige Korrelationen) wenn ein Pol z_0 von $g^{\alpha\beta}(z)$ gegen den Einheitskreis läuft, d.h. $|z_0| \to 1-0$

<u>kritisches Verhalten</u> $\quad \xi \to \infty$

<u>Dämpfungsspektrum</u>

Eigenwerte von $X(\varphi)$ (2×2 Matrix)



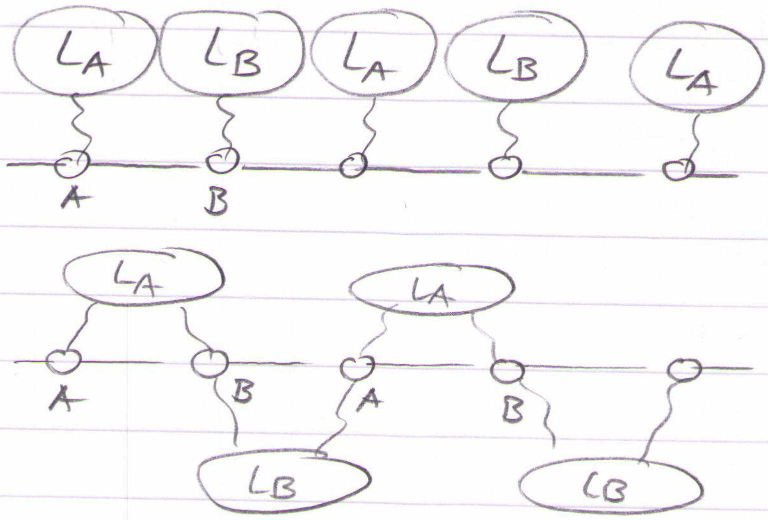
$$\frac{d}{dt}\,\delta\Gamma = X^T\delta\Gamma + \delta\Gamma X$$

$$\hookrightarrow$$

$$\frac{d}{dt}\,\delta\gamma(\varphi) = X(-\varphi)\,\delta\gamma(\varphi) + \delta\gamma(\varphi)X(\varphi)$$

Eigenwerte dieser Gleichung liefern Dämpfungs-spektrum

<u>Translationsinvarianz um zwei Gitterplätze</u>



periodische RB mit Länge 2N

Einheitszelle mit 2 Gitterplätzen

$\Rightarrow$ 4 Sorten u's und w's

$$u_j^1 = a_{Aj} + a_{Aj}^+ \qquad u_j^2 = i(a_{Aj} - a_{Aj}^+)$$

$$u_j^3 = a_{Bj} + a_{Bj}^+ \qquad u_j^4 = i(a_{Bj} - a_{Bj}^+)$$

$\gamma(\varphi),\ m(\varphi),\ x(\varphi),\ y(\varphi)$ – 4x4 Matrizen!

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Quantum phase transitions & criticality

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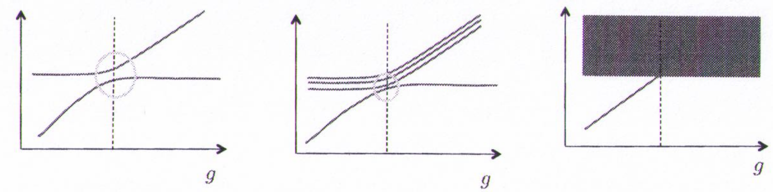
quantum phase transition

$\boxed{H = H_1 + gH_2}$ $[H_1, H_2] \neq 0$



- avoided crossing → smooth change of ground-state properties
- transition gets less smooth the more states involved (large number of lattice sites)
- non-analytic behaviour for continuum (infinite lattice)

→ **quantum phase transition**

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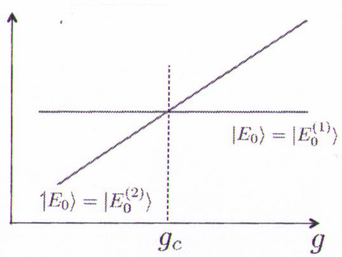
quantum phase transition

- consider ground state of a Hamiltonian

$\boxed{H = H_1 + gH_2}$

$[H_1, H_2] = 0$



- non-analytic change of ground-state properties at $g = g_c$
- however not generic, since $[H_1, H_2] = 0$

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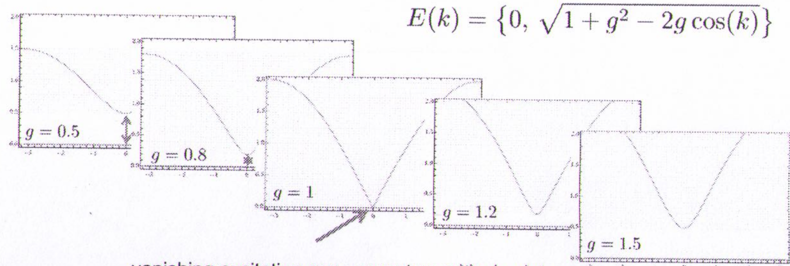
quantum phase transition

- 1D Ising model $H_I = -\sum_i \left(g\,\hat{\sigma}_i^z + \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \right)$

$g \to 0 \quad |+-+\cdots+-\rangle$

$g \to \infty \quad |\uparrow\uparrow\uparrow\cdots\uparrow\uparrow\rangle$

$$E(k) = \left\{0,\ \sqrt{1 + g^2 - 2g\cos(k)}\right\}$$



vanishing excitation gap = quantum critical point

criticality

- non-analytic behaviour at critical point

  vanishing of excitation gap:

$$\boxed{\Delta \sim |g-g_c|^{\lambda z}}$$

diverging correlation length:

$$\boxed{\xi^{-1} \sim |g-g_c|^{\lambda}} \qquad \xi^{-1} = -\lim_{|i-j|\to\infty} \frac{\log|\langle \hat{o}_i \hat{o}_j\rangle|}{|i-j|}$$

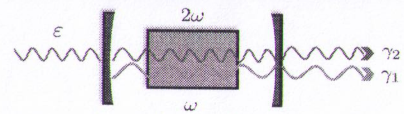
λ critical exponent

z dynamical critical exponent

Open systems & flux-equilibrium phase transitions

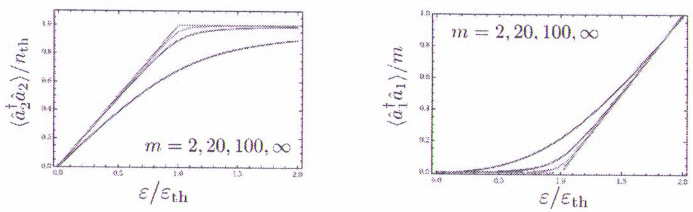
phase transitions in open systems ?

- optical parametric oscillator



$$H_{\mathrm{OPO}} = \frac{K}{2}\left(\hat{a}_2 \hat{a}_1^{\dagger 2} + h.a.\right) + \varepsilon\left(\hat{a}_2 + \hat{a}_2^\dagger\right)$$

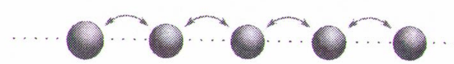
system size scales $\quad n_{\mathrm{th}} = \frac{\gamma_1^2}{K} \quad m = \frac{2\gamma_1\gamma_2}{K}$



non-analytic behaviour at threshold for $m \to \infty$

criticality ?

- lattice models



criticality = diverging correlation length

$$\xi^{-1} = -\lim_{|i-j|\to\infty} \frac{\log|\langle \hat{o}_i \hat{o}_j\rangle|}{|i-j|}$$

$$\xi^{-1} \sim |g-g_c|^{\lambda}$$

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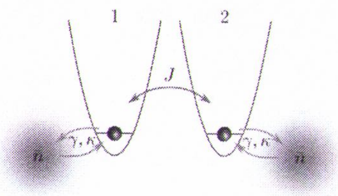
two-site problem

- model with single-site reservoirs $\quad \mathrm{H} = -J(a_1^\dagger a_2 + a_2^\dagger a_1)$

$$\mathcal{L}\rho = \gamma(\bar{n}+1)(2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a\rho) + \gamma\bar{n}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger)$$
$$+\frac{\kappa}{2}(2a\rho a - a^2\rho - \rho a^2) + \frac{\kappa^*}{2}(a^\dagger\rho a^\dagger - a^{\dagger 2}\rho - \rho a^{\dagger 2})$$

- equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^2\rangle = 2iJ\langle a_1 a_2\rangle - \gamma\langle a_1^2\rangle - \kappa^*$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1^\dagger a_2\rangle = iJ\left(\langle\hat{n}_1\rangle - \langle\hat{n}_2\rangle\right) - \gamma\langle a_1^\dagger a_2\rangle$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a_1 a_2\rangle = iJ\left(\langle a_1^2\rangle + \langle a_2^2\rangle\right) - \gamma\langle a_1 a_2\rangle$$



translational invariance → no normal correlations

however: anormal correlations can build up (bosons only)

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translational invariant linear models

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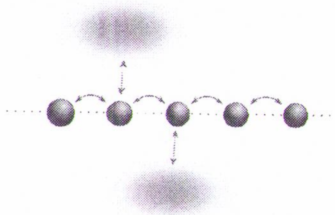
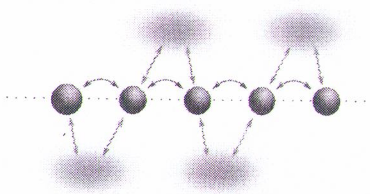
- **bosons:**



single-site reservoirs
but phase-sensitive

$$L_j = \varepsilon(a_j + a_j^\dagger) + i\varepsilon(a_j - a_j^\dagger)e^{ig}$$

- **fermions:**



N-site reservoirs

$$L_j = \sum_{m=0}^{N-1}\frac{\nu_m}{2}e^{ig_m}(c_{j+m} + c_{j+m}^\dagger)$$

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linear models

- stationary states = Gaussian states

covariance matrix:

$$(\gamma_b)_{jk} = \frac{1}{2}\mathrm{Tr}\Big(\rho(u_j u_k + u_k u_j)\Big) \qquad (\gamma_f)_{jk} = \frac{i}{2}\mathrm{Tr}\Big(\rho(w_j w_k - w_k w_j)\Big)$$

- equation of motion of covariance matrix

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\gamma_\alpha = X_\alpha^{\mathrm{T}}\gamma_\alpha + \gamma_\alpha X_\alpha - Y_\alpha} \qquad \alpha = b, f$$

$$X_b = \sigma\,(2H_b + 2\mathrm{Im}[M]) \qquad X_f = -2iH_f + 2\mathrm{Re}[M]$$
$$Y_b = 4\sigma^{\mathrm{T}}\mathrm{Re}[M]\sigma \qquad Y_f = 4\mathrm{Im}[M]$$

translational invariance

- stationary state: Ljapunov-Sylvester equation $X_\alpha^{\mathrm{T}}\gamma_\alpha + \gamma_\alpha X_\alpha = Y_\alpha$
- matrices are circulant → $A_{m,n} = A_{m-n} \sim \int \mathrm{d}\phi\, \widetilde{a}(\phi)\, \mathrm{e}^{i(m-n)\phi}$

  symbol functions (2x2 matrices for two types of quadrature / Majoranas)
  $\widetilde{x}(\phi),\ \widetilde{y}(\phi),\ \widetilde{\gamma}(\phi)$

$$\boxed{\widetilde{x}_\alpha^{\mathrm{T}}(-\phi)\widetilde{\gamma}_\alpha(\phi) + \widetilde{\gamma}_\alpha\widetilde{x}_\alpha(\phi) = \widetilde{y}_\alpha(\phi)}$$ simple 2 x 2 matrix equation

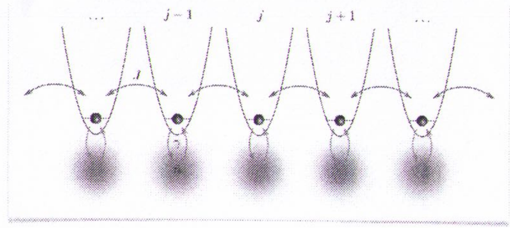
- correlation functions

$$\boxed{\langle a_j^\dagger a_{j+d}\rangle \leftrightarrow \gamma_d} \qquad \boxed{\gamma_d = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi\, \widetilde{\gamma}(\phi)\, e^{id\phi} = \oint \frac{\mathrm{d}z}{z}\,\widetilde{\gamma}(z)\, z^d}$$

lattice bosons

linear bosonic model

$$L_j = \varepsilon(a_j + a_j^\dagger) + i\varepsilon(a_j - a_j^\dagger)e^{ig}$$

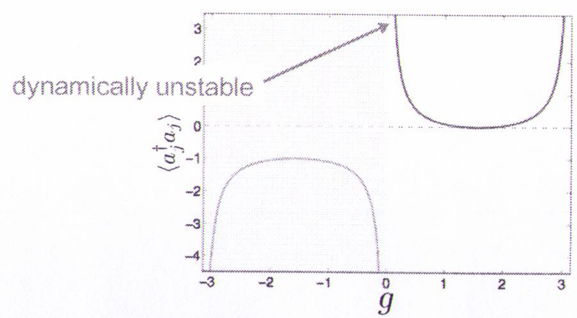


- criticality of anomal correlations

$$\boxed{\xi^{-1} \approx \frac{\epsilon}{2J}|\sin g|} \qquad \boxed{g_c = 0, \pm\pi}$$

linear bosonic model

- local particle number $\langle n\rangle = \dfrac{1-\sin g}{2\sin g}$



➡ **generic property of linear bosonic models!!**

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nonlinear boson model

M. Moos, M. Höning, M.F. in preparation

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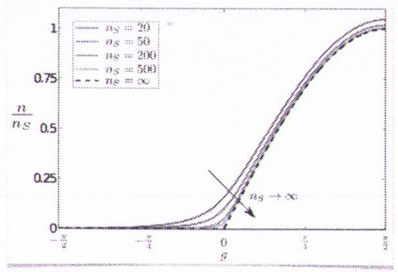
continuous vs. lattice models

- non-analytic behaviour at infinite system size $n_S \to \infty$

$$\boxed{\frac{\bar{n}}{n_s} = \begin{cases} 0, & \text{for } -\pi \le g \le 0 \\ \sin g, & \text{for } 0 \le g \le \pi \end{cases}}$$

- correlation length:

$$\boxed{\left(\frac{\xi}{l_s}\right)^{-1} = \begin{cases} \sin g, & \text{for } -\pi \le g \le 0 \\ 0, & \text{for } 0 \le g \le \pi \end{cases}}$$



- critical exponent:

$$\boxed{\lambda = 1}$$

- amplitude of critical component

$$|\langle a_j a_{j+d} \rangle| \approx \frac{\cos g}{2 l_S}, \text{ for } 0 \le g \le \pi$$

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particle-number dependent loss

- add nonlinear damping: $\mathcal{L} \to \mathcal{L} + \mathcal{L}_{nl}$

$$\boxed{\mathcal{L}_{\rm nl}\rho = 2\kappa^2 \left(2\sqrt{\hat{n}_j}\hat{a}_j \rho \hat{a}_j^\dagger \sqrt{\hat{n}_j} - \hat{a}_j^{\dagger 2}\hat{a}_j^2 \rho - \rho \hat{a}_j^{\dagger 2}\hat{a}_j^2 \right)}$$

- mean-field approximation: $\hat{n}_j \to \bar{n}_j$
- self-consistent solution:

$$\frac{\bar{n}}{n_S} = \frac{1}{2}\sin g + \frac{1}{2}\sqrt{\sin^2 g + 2\frac{\sin g + 1}{n_S}}$$

$$\left(\frac{\xi}{l_S}\right)^{-1} \approx \frac{1}{2}\left|\sin g - \sqrt{\sin^2 g + 2\frac{\sin g + 1}{n_S}}\right|$$

- system size parameter:

$$n_S = \frac{\epsilon^2}{\kappa^2}$$

$$l_S = \frac{J}{\epsilon^2}$$

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lattice fermions & critical exponents

M. Höning, M. Moos, M.F. PRA 2012

two-site reservoir coupling

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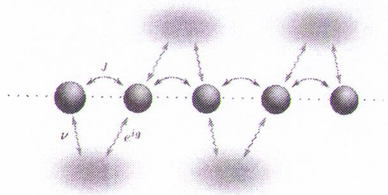

$$L_j = \nu w_{2j-1} + w_{2j+1} e^{ig}$$

- symbol function (F-Trafo) of correlations:

$$\widetilde{\gamma}(\phi) = \frac{\nu \sin g \sin \phi}{\frac{1+\nu^2}{2} + \nu \cos g \cos \phi} \mathbf{1}_{2\times 2}$$

correlations

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two-site reservoirs

$$\langle c_j^\dagger c_{j+n} \rangle = -\frac{i}{2} z^n \tan g$$

$\boxed{\nu = 1}$

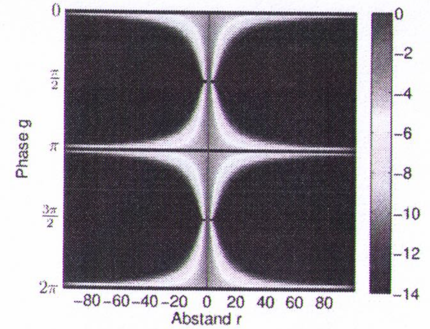
$$z = \frac{1 - |\sin g|}{\cos g}$$

$$|z| = \sqrt{\frac{1 - |\sin g|}{1 + |\sin g|}} \le 1$$

phase transition with critical points

critical exponent

$$\xi^{-1} \sim |g - g_c|^1$$



critical points

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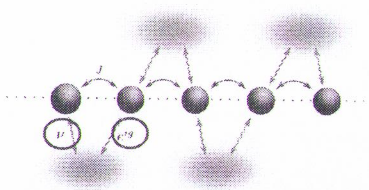
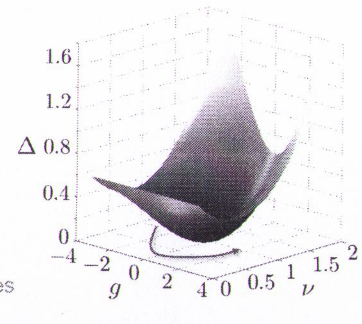
two-site reservoirs

$$\frac{d\rho}{dt} = -i\left[\hat{H}, \rho\right] + \frac{1}{2}\sum_\mu \left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\}\right) = 0$$

$$L_j = (c_j + c_j^\dagger) + \nu e^{-ig}(c_{j+1} + c_{j+1}^\dagger)$$





only isolated critical points / hyperspheres

three-site model

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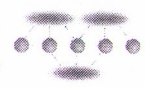
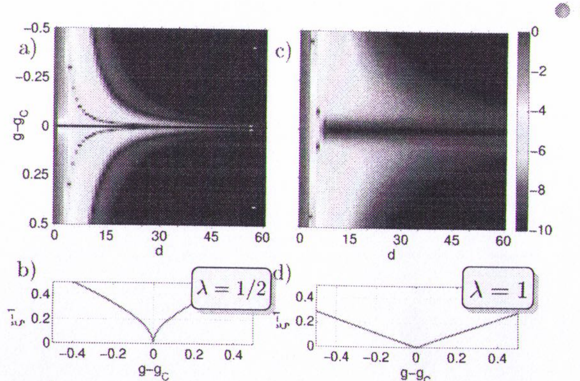
$\langle c_j^\dagger c_{j+d} \rangle$



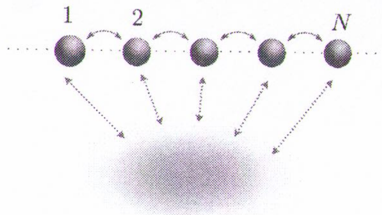


$\nu_0 = \frac{1}{2}\nu_1 = \nu_2 \quad g_0 = g_1 = g_2 = 0$

$\nu_0 = \nu_1 = \nu_2 \quad g_0 = -g_2 = \frac{2\pi}{3}, \ g_1 = 0$

critical exponents

- reservoirs coupling to N adjescent sites:



$$\xi^{-1} \sim |g - g_c|^{\lambda}$$

$$\Delta \sim |g - g_c|^{\lambda z}$$

$$\boxed{\lambda = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{N-1}\right\}}$$

$$\boxed{z\lambda = 2}$$