Numerical implementation of the TWA

Overview:

1) Tools: Julia language, Jupyter-Notebooks
   DifferentialEquations package
2) Workflow: From initial Wigner function to numerical results:
   Anharmonic oscillator
3) Representation of states, Monte Carlo sampling
4) Numerical integrators
5) Dynamics of coupled BECs

1) Tools

- Instead of solving the master equation in $\hat{S}$ we can solve $2N$-dimensional PDE in phase-space
- Approximate PDE with a FPE $\rightarrow$ set of SDE
- Stochastic trajectories can be solved independently $\Longrightarrow$ can be done in parallel

2) Workflow

How to go from $\hat{S}(t=0)$ to $W(\alpha^*, \alpha, t)$?

Master equation in $\hat{S}(t)$

$\downarrow$ operator-differential mappings

PDE in $W(\alpha^*, \alpha, t)$

$\downarrow$ TWA

FPE in $W(\alpha^*, \alpha, t)$

$\downarrow$

SDE in $\alpha^*(t)$, $\alpha(t)$

$\downarrow$

Sample and evolve $\alpha(t)$

Consider the anharmonic oscillator

$$\hat{H} = \hbar\omega a^\dagger a + \frac{\hbar\eta}{2} a^{\dagger 2} a^2 \Longrightarrow \frac{d}{dt}\hat{S} = -\frac{i}{\hbar}[\hat{H}, \hat{S}]$$

$$\frac{\partial}{\partial t} W \approx \{ + \partial_\alpha \, i[\omega + \eta(|\alpha|^2 - 1)]\alpha + c.c. \} W$$

$$d\alpha = - i[\omega + \eta(|\alpha|^2 - 1)]\alpha \, dt$$

$$d\alpha^* = + i[\omega + \eta(|\alpha|^2 - 1)]\alpha^* \, dt \implies \text{complex conjugate of } d\alpha$$

3) Representation of states, Monte Carlo sampling

Evaluate $W(\alpha^*, \alpha, t = 0)$ corresponding to $\hat{S}(0)$.

Wigner function represents non-deterministic initial conditions of the SDE.

Draw coherent amplitudes α^*, α randomly according to $W(\alpha^*, \alpha, 0)$.

But $W(\alpha^*, \alpha, 0) > 0 \quad \forall \alpha^*, \alpha \in \mathbb{C}$.

Common initial states:

Coherent state $\hat{S} = |\beta\rangle\langle\beta|$, $W(\alpha^*, \alpha) = \frac{2}{\pi} \exp[-2|\alpha - \beta|^2]$

Fock state $\hat{S} = |n\rangle\langle n|$, $W(\alpha^*, \alpha) = \frac{2}{\pi}(-1)^n \exp(-2|\alpha|^2) \underbrace{L_n(4|\alpha|^2)}_{\text{n'th Laguerre polynomial}}$

Monte Carlo sampling

Coherent state: $\alpha = \beta + \frac{1}{\sqrt{2}} \eta_C$ $\quad \eta_C$: complex normally distributed

Fock state: $\alpha \overset{n \gg 1}{\approx} (p + q\, \eta_R) e^{2\pi i \zeta}$ $\quad \eta_R$: real " "

$0 \le \zeta \le 1$: uniformly distributed

Statistics in Hilbert space:

$\langle \beta | \hat{a} | \beta \rangle = \beta$ $\qquad \langle n | a | n \rangle = 0$

$\langle \beta | a^\dagger a | \beta \rangle = |\beta|^2$ $\qquad \langle n | a^\dagger a | n \rangle = n$

$\langle \beta | (a^\dagger a)^2 | \beta \rangle = |\beta|^4 + |\beta|^2$ $\qquad \langle n | (a^\dagger a)^2 | n \rangle = n^2$

Statistics in Wigner phase space using Bopp operators:

$\hat{a} = \alpha + \frac{1}{2} \partial_{\alpha^*}$, $\quad \hat{a}^\dagger = \alpha^* - \frac{1}{2} \partial_\alpha$,

therefore

average over many samples

$a = (\alpha + \frac{1}{2} \partial_{\alpha^*}) 1 = \alpha$ $\iff$ $\langle a \rangle = \overline{\alpha}$

$a^\dagger a = (\alpha^* - \frac{1}{2} \partial_\alpha)(\alpha + \frac{1}{2} \partial_{\alpha^*}) 1$

$= (\alpha^* - \frac{1}{2} \partial_\alpha) \alpha$

$= |\alpha|^2 - \frac{1}{2}$ $\iff$ $\langle a^\dagger a \rangle = \overline{|\alpha|^2} - \frac{1}{2}$

$(a^\dagger a)^2 = |\alpha|^4 - |\alpha|^2$ $\iff$ $\langle (a^\dagger a)^2 \rangle = \overline{|\alpha|^4} - \overline{|\alpha|^2}$

$\Rightarrow$ Approximation of infinite matrix $\hat{S}$ with a finite amount of c-numbers.

4) Numerical integrators, Introduction to DifferentialEquations (Julia)

Consider a univariate SDE

$$dx(t) = A(x,t)\,dt + B(x,t)\,dW_t$$

with the differential Wiener process $\overline{dW_t} = 0$ and $\overline{dW_t\, dW_{t'}} = \delta(t-t')\,dt$ $(dW^2 = dt)$

Find some numerical approximation to $x(t+\Delta t)$:

$$x(t+\Delta t) = \int_t^{t+\Delta t} dt'\, A(x,t') + \int_t^{t+\Delta t} dt'\, B(x,t')\, dW_{t'},$$

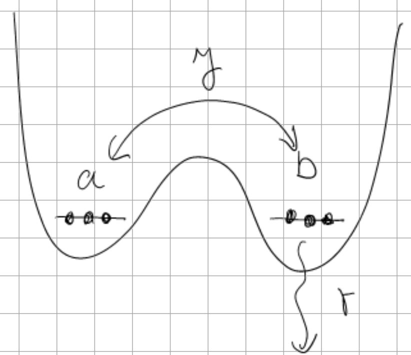
for very small time steps Δt we can use the lowes-order approximation:

$$x(t+\Delta t) = A(x(t),t)\,\Delta t + B(x(t),t)\,\Delta W_t + \mathcal{O}(\Delta W_t^2 = \Delta t),$$

where $\Delta W_t = \sqrt{\Delta t}\,\eta$, η is normally distributed.

This is called the Euler-Maruyama integrator.

Example: Driven-dissipative Bose-Hubbard dimer



$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \frac{1}{2}(2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L)$$

$$H = -\hbar J(a^\dagger b + b^\dagger a) + \frac{\hbar \eta}{2}(a^{\dagger 2} a^2 + b^{\dagger 2} b^2)$$

$$L = \sqrt{\gamma}\, b$$

In phase-space this yields (TWA):

$$\partial_t W(\alpha^*,\alpha,\beta^*,\beta,t) = \{+\partial_\alpha[iJ\beta + i\eta|\alpha|^2\alpha] + c.c.$$
$$+ \partial_\beta[iJ\alpha + i\eta|\beta|^2\beta + \gamma\beta] + c.c.$$
$$+ \partial_{\beta^*}\partial_\beta \frac{\gamma}{2} + c.c.\}W$$

$$d\alpha = [-iJ\beta - i\eta|\alpha|^2\alpha]\,dt$$

$$d\beta = [-iJ\alpha - i\eta|\beta|^2\beta - \gamma\beta]\,dt + \sqrt{\gamma}\,dW$$

where dW is complex normally distributed.