II. Vektoranalysis Teil 1

II.1 Gradient

- $f(x)$ $\quad \frac{df}{dx} = f'(x)$ Maß für die Änderung der Funktion f bei $x \to x + dx$

$$\boxed{df = f'(x)\, dx}$$

- $f(x,y,z)$ $\quad \frac{\partial f}{\partial x}$ Änderg bei $x \to x + dx$

  $\frac{\partial f}{\partial y}$ — " — $y \to y + dy$ etc.

$d\vec{r} = (dx, dy, dz)$ Vektor

$$\boxed{df = \vec{v} \cdot d\vec{r}} \qquad \vec{v} = ?$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (dx, dy, dz)$$

$$\vec{v} = \left(\frac{\partial}{\partial x} \vec{e}_x + \frac{\partial}{\partial y} \vec{e}_y + \frac{\partial}{\partial z} \vec{e}_z\right) f$$

$$\boxed{\vec{v} = \operatorname{grad} f = \vec{\nabla} f} \qquad \text{Gradient von } f$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \text{Nabla Operator}$$

Rechenregeln

$$\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$$

$$\vec{\nabla} fg = f \vec{\nabla} g + g \vec{\nabla} f$$

Bspl:

(i) $$f(\vec{r}) = \vec{a}\cdot\vec{r} = a_x x + a_y y + a_z z$$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (a_x, a_y, a_z)$$

(ii) $$f(\vec{r}) = f(r) \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r}$$ analog y, z

$$\vec{\nabla} f(r) = f'(r)\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

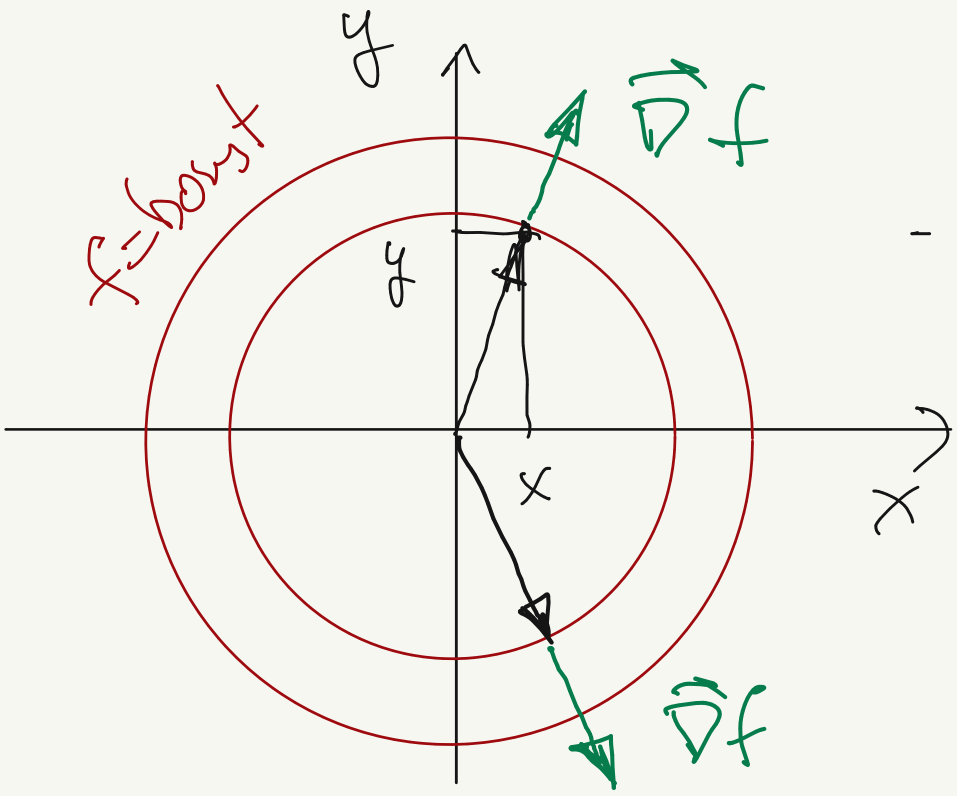
$$= f'(r)\,\frac{\vec{r}}{r} = f'(r)\,\vec{e}_r$$

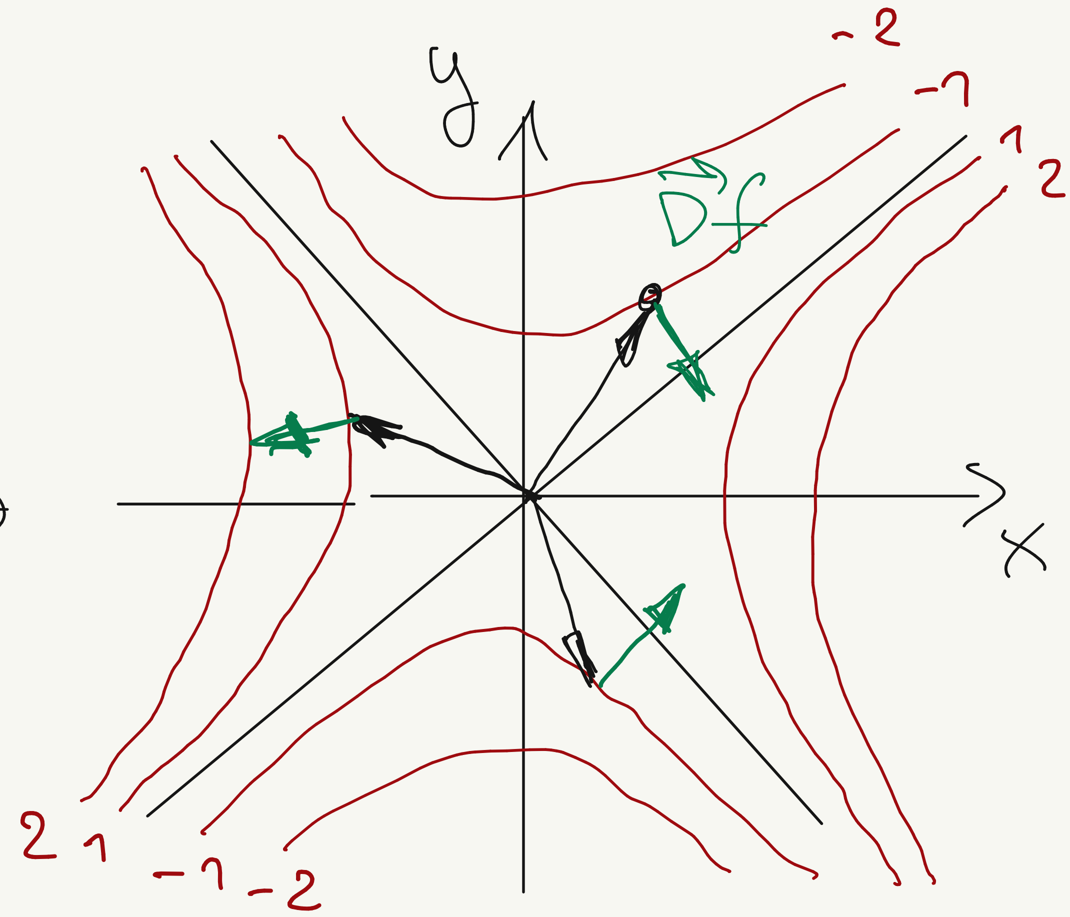
$$\boxed{\text{Der Gradient } \operatorname{grad} f = \vec{\nabla} f \text{ steht senkrecht auf den Flächen } f(x,y,z) = \text{konst}}$$

Bew.: sei $d\vec{r}$ in Fläche mit $f = \text{konst}$

$$0 \overset{!}{=} df = \vec{\nabla} f \cdot d\vec{r}$$

$$\Rightarrow \vec{\nabla} f \perp d\vec{r}$$





$$f = \frac{1}{4}(x^2+y^2) \qquad\qquad f = \frac{1}{4}(x^2-y^2)$$

$$\frac{\partial f}{\partial x} = \frac{x}{2} \quad \frac{\partial f}{\partial y} = \frac{y}{2} \qquad\qquad \frac{\partial f}{\partial x} = \frac{x}{2} \quad \frac{\partial f}{\partial y} = -\frac{y}{2}$$

$$\vec\nabla f = \frac{1}{2}(x,y) = \frac{1}{2}\vec r \qquad\qquad \vec\nabla f = \frac{1}{2}(x,-y)$$

- Taylorentwicklung von Funktionen mehrerer Veränderlicher

$$f(\vec r) \cong f(\vec r_0) + \vec\nabla f\Big|_{\vec r_0} \cdot (\vec r - \vec r_0)$$

Bspl.

$\vec r \approx 0$

$$\frac{1}{|\vec r - \vec a|} \approx \frac{1}{|\vec a|} + \vec\nabla \frac{1}{|\vec r - \vec a|}\Bigg|_{\vec r = 0} \cdot \vec r$$

$$= \frac{1}{a} + \frac{1}{a^3}\,\vec a \cdot \vec r$$

II.2 Die Divergenz

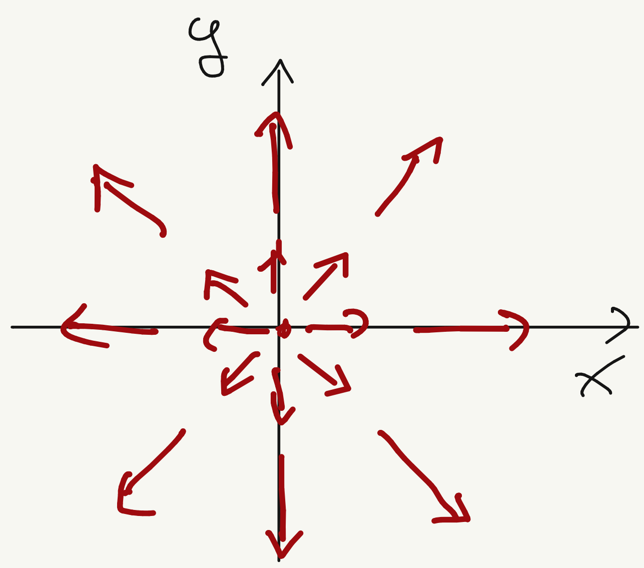
$$\operatorname{grad} f = \vec{\nabla} f$$

$$\boxed{\begin{aligned} \operatorname{div} \vec{F} &\equiv \vec{\nabla} \cdot \vec{F} \\ &= \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z \end{aligned}}$$

Quellstärke oder Divergenz von $\vec{F}$

Bspl:

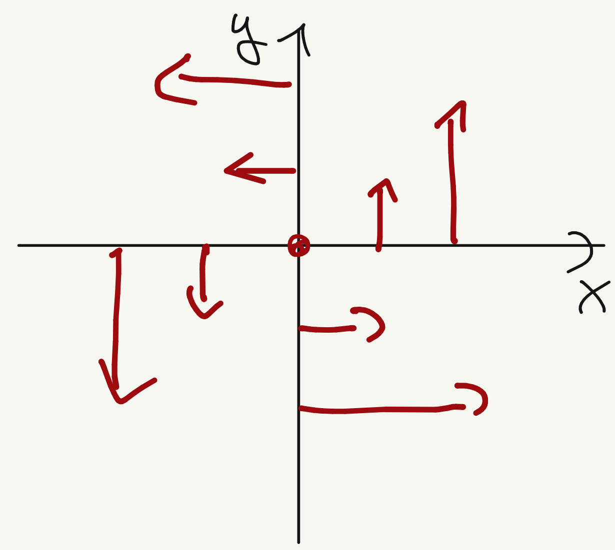
$\vec{F} = \vec{r}$



$$\operatorname{div} \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$\vec{F} = \vec{\omega} \times \vec{r}$

(hier $\vec{\omega} = \vec{e}_z$)



$$\vec{F} = -y\vec{e}_x + x\vec{e}_y$$

$$\operatorname{div} \vec{F} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0$$

Rechenregeln

$$\operatorname{div}(\vec{A} + \vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}$$

$$\operatorname{div}(g\vec{F}) = g \operatorname{div}\vec{F} + (\operatorname{grad} g)\,\vec{F}$$
$$= g\,\vec{\nabla}\cdot\vec{F} + (\vec{\nabla} g)\cdot\vec{F}$$

Bew: $\operatorname{div}(g\vec{F}) = \frac{\partial}{\partial x}(gF_x) + \frac{\partial}{\partial y}(gF_y) + \frac{\partial}{\partial z}(gF_z)$

$$= \frac{\partial g}{\partial x}F_x + g\frac{\partial F_x}{\partial x} + \frac{\partial g}{\partial y}F_y + g\frac{\partial F_y}{\partial y} + \frac{\partial g}{\partial z}F_z + g\frac{\partial F_z}{\partial z}$$

$$= \vec{\nabla} g\cdot\vec{F} + g\,\vec{\nabla}\cdot\vec{F} \qquad \square$$

Bspl: Radialfeld $\quad \vec{F}(\vec{r}) = f(r)\,\vec{r}$

$$\vec{\nabla}\cdot\vec{F}(\vec{r}) = \vec{\nabla}\cdot\left(f(r)\,\vec{r}\right) = \vec{\nabla} f(r)\cdot\vec{r} + f(r)\underbrace{\vec{\nabla}\cdot\vec{r}}_{=3}$$
$$= 3f(r) + \vec{r}\cdot\vec{\nabla} f(r)$$

- wichtige Kombination aus div und grad

$$\operatorname{div}\operatorname{grad} f(\vec{r}) = \vec{\nabla}\cdot\vec{\nabla} f(\vec{r}) =$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$= \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \frac{\partial^2}{\partial z^2}f \equiv \Delta f$$

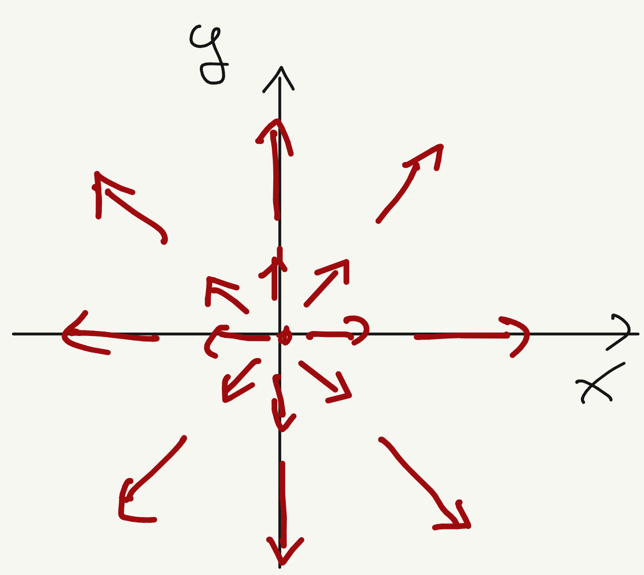
Laplace Operator $\quad \boxed{\Delta = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$

II.3 Rotation

$$\boxed{\begin{aligned}\operatorname{rot}\vec F &= \vec\nabla\times\vec F = \sum_{ijk}\varepsilon_{ijk}\frac{\partial}{\partial x_i}F_j\,\hat e_k\\ &= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\end{aligned}}$$

Wirbelstärke oder Rotation

Bspl: $\vec F=\vec r$

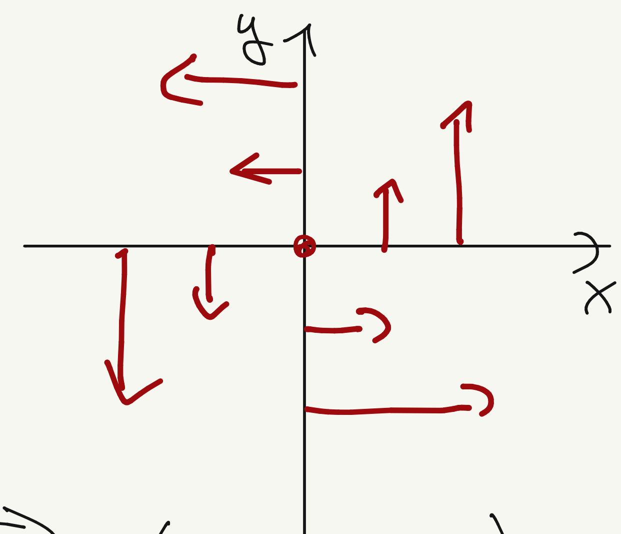


$\vec F=(x,y,z)$

$\operatorname{rot}\vec F=0$

$\vec F=\vec\omega\times\vec r$

(hier $\vec\omega=\hat e_z$)



$\vec F=(-y,x,0)$

$\operatorname{rot}\vec F=(0,0,1-(-1))$

$=(0,0,2)$

Rechenregeln

$$\operatorname{rot}(\vec F+\vec G)=\operatorname{rot}\vec F+\operatorname{rot}\vec G$$

$$\operatorname{rot}(g\vec F)=g\operatorname{rot}\vec F+(\vec\nabla g)\times\vec F$$

← 3.11.2020

Es gelten folgende Regeln:

$$\boxed{\operatorname{div}(\vec{F}\times\vec{G})} = \vec{\nabla}\cdot(\vec{F}\times\vec{G})$$

Produktregel der Differenz.
$\vec{\nabla} = \vec{\nabla}_F + \vec{\nabla}_G$

$$= \vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$$

Spatprod.

$$\boxed{= \vec{G}\cdot \operatorname{rot}\vec{F} - \vec{F}\cdot\operatorname{rot}\vec{G}}$$

außerdem (nicht angeben ist Übungsaufgabe 8)

$$\boxed{\operatorname{rot}(\vec{F}\times\vec{G})} = \Big[\vec{\nabla}\times(\vec{F}\times\vec{G})$$

Produktregel
$\vec{\nabla} = \vec{\nabla}_F + \vec{\nabla}_G$

$$\underset{\text{bac-cab}}{=} \vec{F}(\vec{\nabla}_G\cdot\vec{G}) - (\vec{F}\cdot\vec{\nabla}_G)\vec{G}$$

$$+(\vec{G}\cdot\vec{\nabla}_F)\vec{F} - (\vec{G}\cdot\vec{\nabla}_F)\vec{F}\Big]$$

$$\boxed{= \vec{F}(\vec{\nabla}\cdot\vec{G}) - (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{G}\cdot\vec{\nabla})\vec{F}}$$

Ein Gradientenfeld ist wirbelfrei

$$\boxed{\operatorname{rot}\operatorname{grad} f = \vec{\nabla}\times(\vec{\nabla}f) = \vec{0}}$$

Ein Wirbelfeld ist quellenfrei

$$\boxed{\operatorname{div}\operatorname{rot}\vec{F} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0}$$

Beweis

(a) $\vec{\nabla}\times(\vec{\nabla}f) = \vec{\nabla}\times(\partial_x f, \partial_y f, \partial_z f)$

$$= (\partial_y\partial_z f - \partial_z\partial_y f, \partial_z\partial_x f - \partial_x\partial_z f, \partial_x\partial_y f - \partial_y\partial_x f)$$

$$= (0,0,0)$$

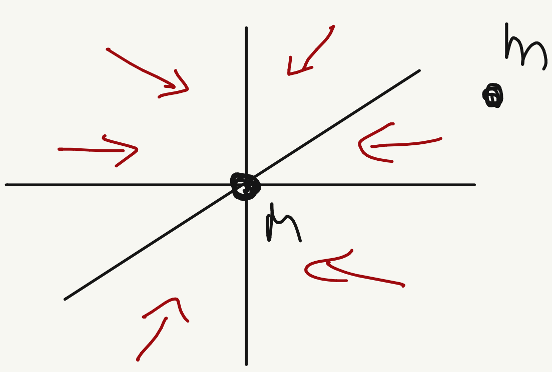
(b) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \vec{\nabla}\cdot\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$

$$= \cancel{\partial_x\partial_y F_z} - \cancel{\partial_x\partial_z F_y} + \cancel{\partial_y\partial_z F_x} - \cancel{\partial_y\partial_x F_z}$$
$$+ \cancel{\partial_z\partial_x F_y} - \cancel{\partial_z\partial_y F_x} = 0$$

$\square$

Beispiel

- Gravitationskraft einer Punktmasse M



$$\vec{F}(\vec{r}) = -\frac{mMG\,\vec{r}}{r^3}$$

$$= -k\frac{\hat{e}_r}{r^2} = k\vec{\nabla}\frac{1}{r}$$

$$\vec{F}(\vec{r}) = -k\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$$

$$\vec{\nabla}\times\vec{F} = -k\left(\partial_y\frac{z}{r^3}-\partial_z\frac{y}{r^3}, \partial_z\frac{x}{r^3}-\partial_x\frac{z}{r^3}, \partial_x\frac{y}{r^3}-\partial_y\frac{x}{r^3}\right)$$

$$\partial_y \frac{1}{r^3} = -3\frac{1}{r^4}\frac{\partial r}{\partial y} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$= -\frac{3}{r^4}\frac{1}{2r}2y = -\frac{3y}{r^5} \qquad \text{etc.}$$

$$\partial_y \frac{z}{r^3} - \partial_z \frac{y}{r^3} = -z\frac{3y}{r^5} + y\frac{3z}{r^5} = 0 \qquad \text{etc.}$$

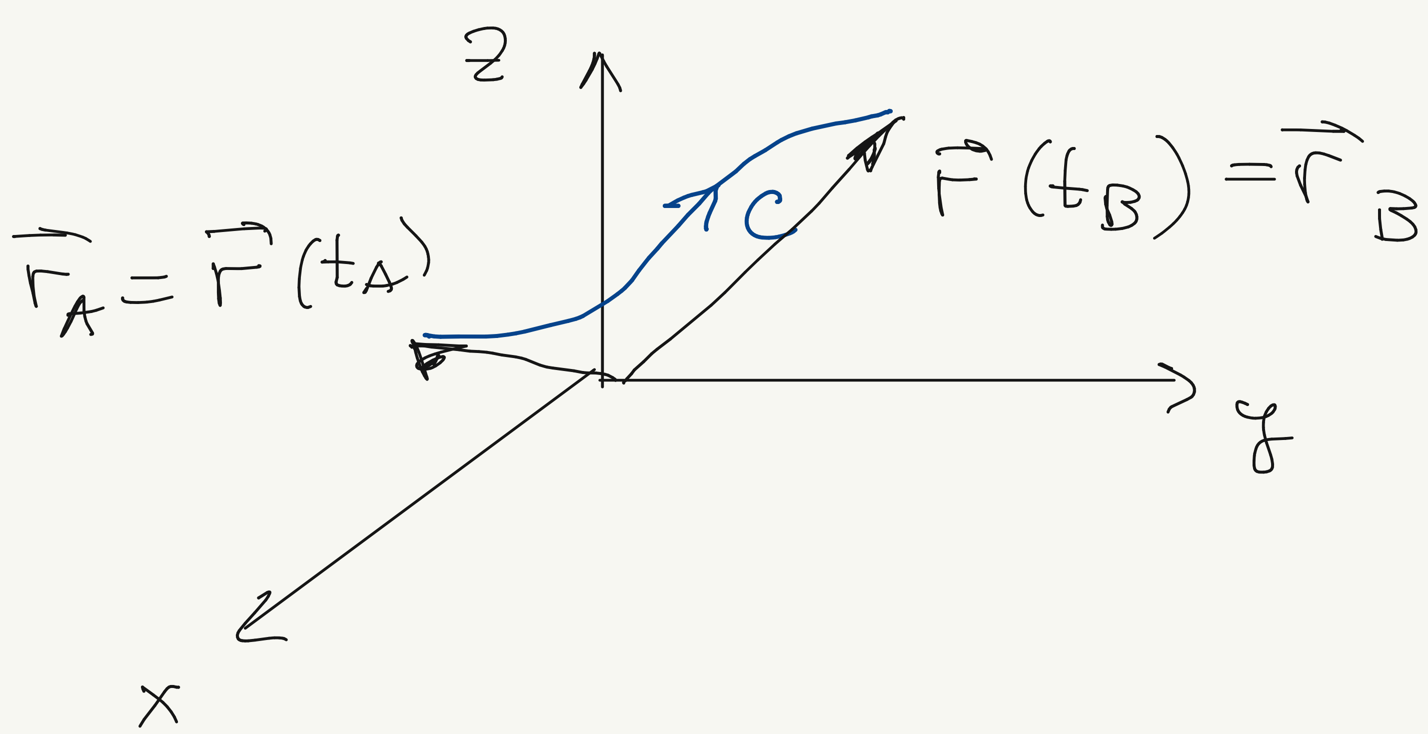
$$\vec{\nabla} \times \vec{F} = 0$$

II.4 Integrale über Vektorfeldern

(A) Kurvenintegrale von Vektorfeldern

- $\boxed{\text{Kurve:} \qquad C = \{ \vec{r}(t) \mid t_A \le t \le t_B \}}$

Parameter t der die Kurve parametrisiert



- Vektorfeld $\quad \vec{F}(\vec{r})$

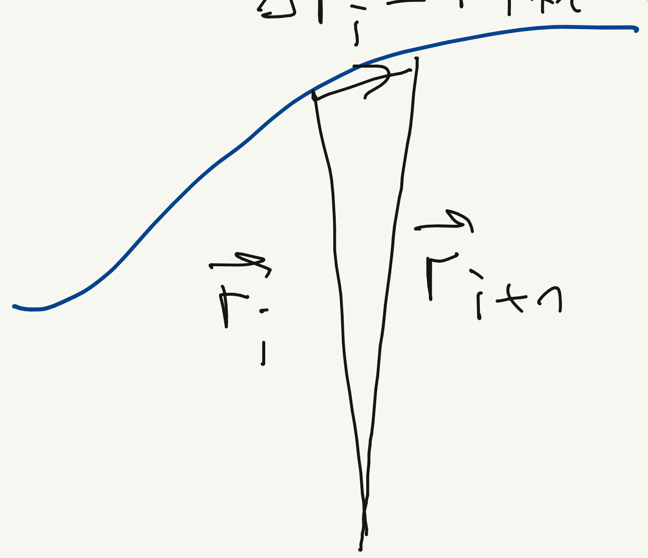
Kurvenintegral

Zahl wg. Skalarprod

$$\boxed{\int_C d\vec r \cdot \vec F(\vec r) \equiv \lim_{N\to\infty} \sum_{i=1}^{N} \vec F(\vec r_i)\cdot \Delta \vec r_i}$$

wobei

$$\Delta \vec r_i = \vec r_{i+1} - \vec r_i$$



mit Parametrisierung $\vec r = \vec r(t)$

$$\int_C d\vec r \cdot \vec F(t) = \lim_{N\to\infty} \sum_{i=1}^{N} \vec F(\vec r_i)\cdot \frac{d\vec r}{dt}\Big|_{t_i} \Delta t_i$$

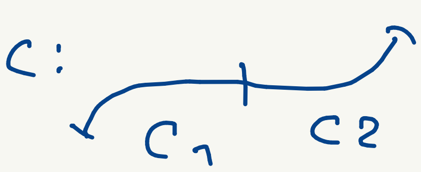
$$= \int_{t_A}^{t_B} dt\, \vec F(\vec r(t))\cdot \frac{d\vec r}{dt}$$

Eigenschaften

(i) $C \to -C$ $\quad \int_C d\vec r\cdot \vec F = -\int_{-C} d\vec r \cdot \vec F$

$(d\vec r \to -d\vec r)$

(ii) $C = C_1 + C_2$



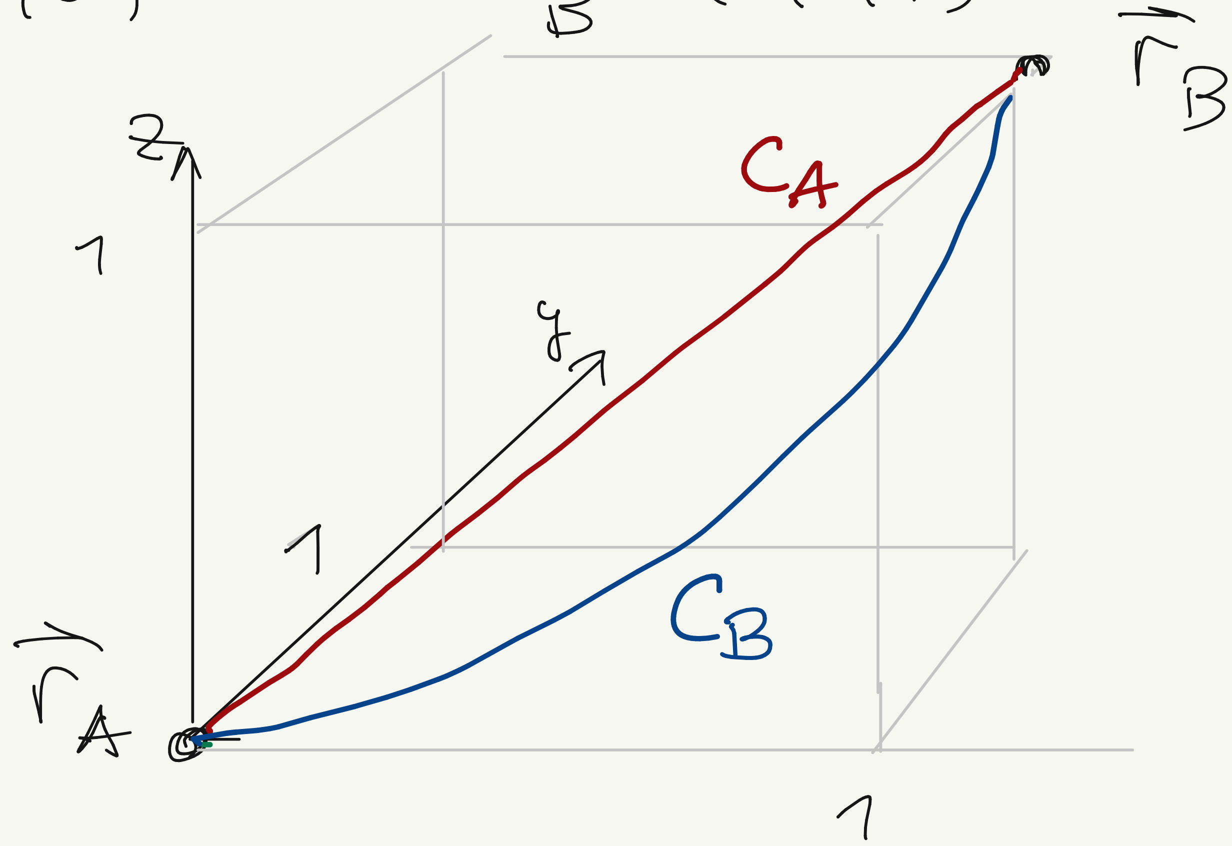
$$\int_C d\vec r\cdot \vec F = \int_{C_A} d\vec r\cdot \vec F + \int_{C_B} d\vec r\cdot \vec F$$

Beispiel:

(i) $\vec{F}(\vec{r}) = \left(3x^2+6y, -14yz, 20xz^2\right)$

$\vec{r}_A = (0,0,0) \qquad \vec{r}_B = (1,1,1)$

betrachten 3 Wege



C_A: $\quad \vec{r}(t) = (t,t,t)$

$d\vec{r}(t) = (1,1,1)\,dt \qquad t_A = 0, \; t_B = 1$

$$I_A = \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r}\cdot\vec{F} = \int_0^1 dt\left(3x^2+6y, -14yz, 20xz^2\right)\cdot(1,1,1)$$

$$= \int_0^1 dt\left(3t^2+6t-14t^2+20t^3\right)$$

$$= t^3\Big|_0^1 + 3t^2\Big|_0^1 - \frac{14}{3}t^3\Big|_0^1 + 5t^4\Big|_0^1$$

$$= 1+3-\frac{14}{3}+5 = \underline{\underline{\frac{13}{3}}}$$

C_B | $\vec{r}(t) = (t, t^2, t^3) \qquad d\vec{r} = (1, 2t, 3t^2)\,dt$

$$I_B = \int_{C_B} d\vec{r}\cdot\vec{F} = \int_0^1 dt\, (1, 2t, 3t^2)\cdot\left(3t^2+6t^2, -14t^5, 20t^7\right)$$

$$= \int_0^1 dt\, \left(9t^2 - 28t^6 + 60t^9\right) = \frac{9}{3}t^3\Big|_0^1 - \frac{28}{7}t^7\Big|_0^1 + \frac{60}{10}t^{10}\Big|_0^1$$

$$= 1 - \frac{28}{7} + 6 = \underline{\underline{5}}$$

$\boxed{\Rightarrow \text{Integral hängt i.A. vom Weg ab}}$

• jetzt

(ii) $\vec{F}(\vec{r}) = \left(2x^2 + 2xy + 2xz^2, x^2, 2x^2 z\right)$

C_A | $\vec{r} = (t, t, t)$

$$I_A = \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r}\cdot\vec{F} = \int_0^1 dt\, (1,1,1)\cdot\left(2t^2+2t^2+2t^3, t^2, 2t^3\right)$$

$$= \int_0^1 dt\; 4t^2 + 2t^3 + t^2 + 2t^3 = \int_0^1 dt\; 5t^2 + 4t^3 = \underline{\underline{\frac{8}{3}}}$$

C_B | $\vec{r} = (t, t^2, t^3) \qquad d\vec{r} = (1, 2t, 3t^2)\,dt$

$$I_B = \int_0^1 dt\, (1, 2t, 3t^2)\cdot\left(\underline{2t^2+2t^3+2t^7}, \underline{t^2}, \underline{2t^5}\right)$$

$$= \int_0^1 dt\; \underline{2t^2+2t^3+2t^7} + \underline{2t^3} + \underline{6t^7}$$

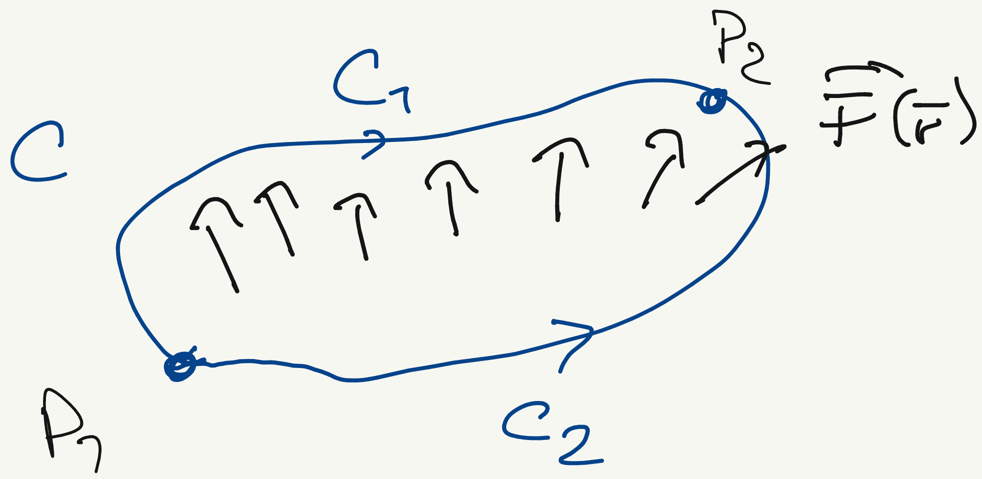
! $=$

$$= \int_0^1 dt\; 2t^2 + 4t^3 + 8t^7 = \underline{\underline{\frac{8}{3}}}$$

Integrale sind jetzt gleich! da steckt was dahinter!

Wegunabhängigkeit von Kurvenintegralen

Def: Zirkulation: Kurvenintegral über geschlossene Kurve C



$C = C_1 - C_2$

$$\oint_C d\vec r \cdot \vec F = \int_{C_1} d\vec r \cdot \vec F - \int_{C_2} d\vec r \cdot \vec F$$

$$\boxed{\text{Ein Kurvenintegral einer Vektorfunktion ist wegunabhängig (in einem Gebiet } \subset \mathbb{R}^3\text{) falls die Zirkulation über alle geschlossenen Pfade innerhalb des Gebietes verschwindet.}}$$

Komponentendarstellung $\vec F = (F_x, F_y, F_z)$ $d\vec r = (dx, dy, dz)$

$$\vec F(\vec r) \cdot d\vec r = F_x\,dx + F_y\,dy + F_z\,dz$$

Def: Ein Ausdruck

$$u(x,y,z)\,dx + v(x,y,z)\,dy + w(x,y,z)\,dz$$

heißt <u>vollständiges Differential</u> einer Funktion $f(x,y,z)$, falls gilt

$$\boxed{u = \frac{\partial f}{\partial x} \qquad v = \frac{\partial f}{\partial y} \qquad w = \frac{\partial f}{\partial z}}$$

Dann gilt

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

- notwendige Bedingung (Satz von Schwarz)

$$(*) \qquad \frac{\partial u}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial v}{\partial x} \quad \text{etc.}$$

Man sieht

(1) $$df = \vec{\nabla} f \cdot d\vec{r} = \vec{F} \cdot d\vec{r}$$

(2) $$\int_C d\vec{r} \cdot \vec{F} = \int_{f(\vec{r}_A)}^{f(\vec{r}_B)} df = f(\vec{r}_B) - f(\vec{r}_A)$$

$$\Rightarrow \oint d\vec{r} \cdot \vec{F} = \oint d\vec{r} \cdot \vec{\nabla} f = \oint df = 0$$

Für ein Gradientenfeld $\vec{F} = \vec{\nabla} f$ ist das Kurvenintegral wegunabhängig. Die Zirkulation verschwindet

Die notwendige Bedingung (*) kann auch wie folgt umgeschrieben werden $\quad u = F_x \,,\; v = F_y \,,\; w = F_z$

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \qquad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$$

$$\Rightarrow \qquad \vec{\nabla} \times \vec{F} = 0$$

$$\boxed{\oint d\vec{r} \cdot \vec{F} = 0 \iff \underbrace{\vec{F} \cdot d\vec{r} = df}_{\text{totales Diff.}} \iff \vec{F} = \vec{\nabla} f \iff \vec{\nabla} \times \vec{F} = 0}$$

man erinnere sich
wir hatten gesehen $\quad \vec{\nabla} \times (\vec{\nabla} f) = 0$

Beispiel von oben (ii) $\vec{F} = (\underbrace{2x^2 + 2xy + 2xz^2}_{u}, \underbrace{x^2}_{v}, \underbrace{2x^2 z}_{w})$

$$\frac{\partial u}{\partial y} = 2x = \frac{\partial u}{\partial x} \qquad \frac{\partial u}{\partial z} = 4xz = \frac{\partial w}{\partial x}$$

$$\frac{\partial v}{\partial z} = 0 = \frac{\partial w}{\partial y} \qquad \Rightarrow \quad \vec{\nabla} \times \vec{F} = 0$$

$$\vec{F} = \vec{\nabla} f \qquad f = \frac{2}{3} x^3 + x^2 y + x^2 z^2 + c$$

(i) $\vec{F} = (\underbrace{3x^2 + 6y}_{u}, \underbrace{-14yz}_{v}, \underbrace{20xz^2}_{w})$

$$\frac{\partial u}{\partial y} = 6 \neq \frac{\partial u}{\partial x} = 0 \;\lightning \qquad \vec{\nabla} \times \vec{F} \neq 0$$