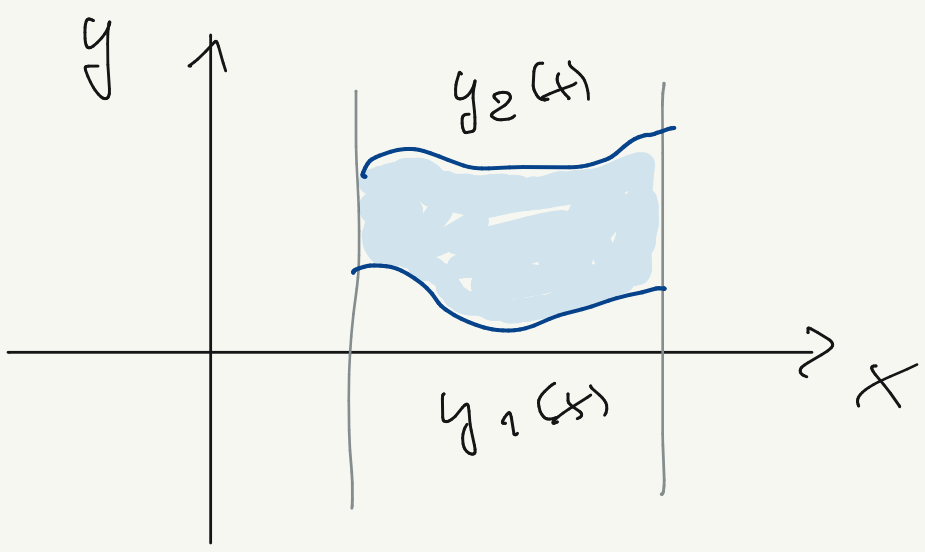
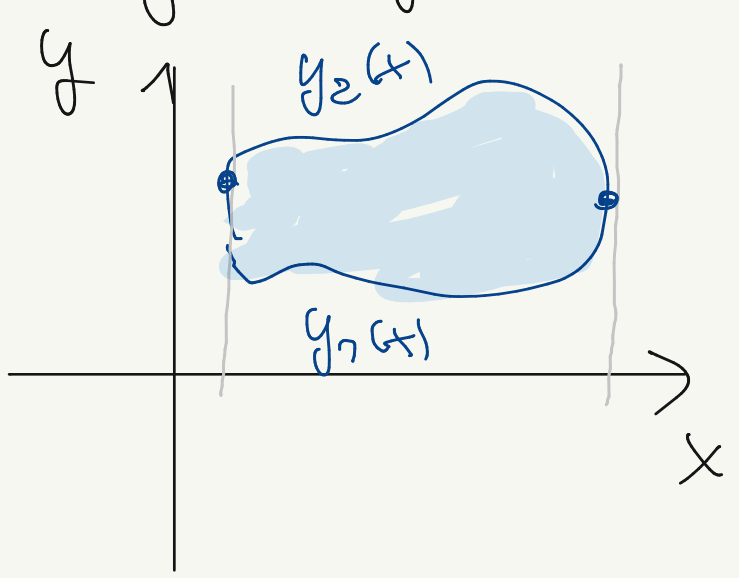
XI Vektoranalysis II

XI.1 Integrale über Vektorfelder II

(A) Flächen- und Volumenintegrale skalarer Fkt.

- Fläche durch Randkurven $y_1(x)$, $y_2(x)$ begrenzt





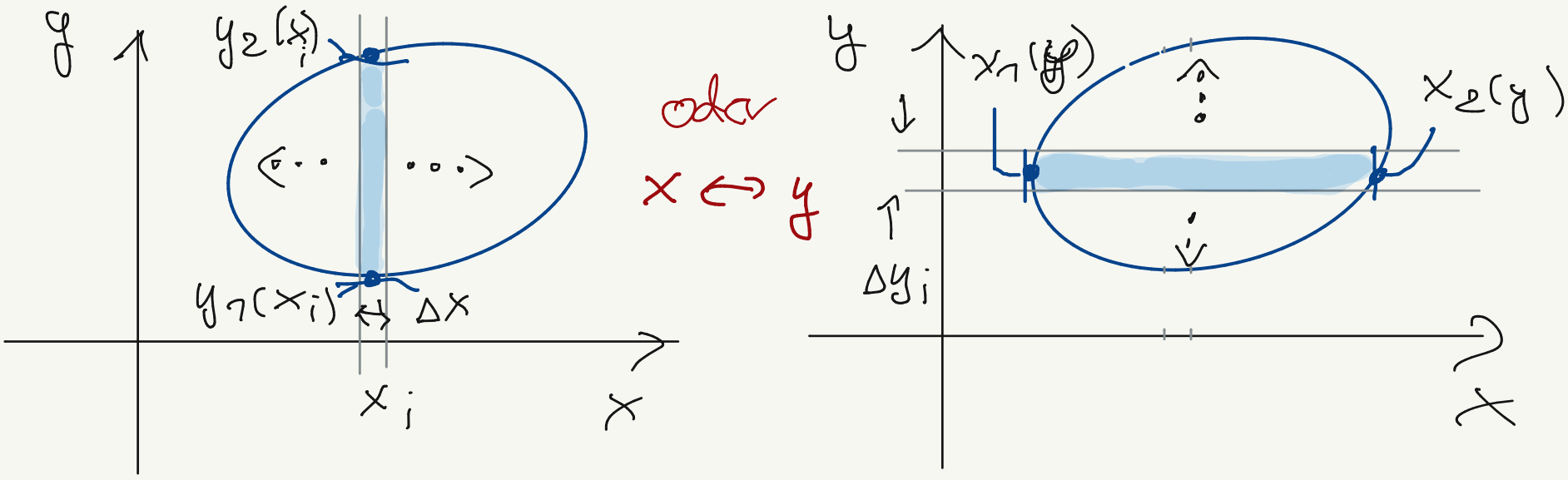
$$\mathcal{A} = \{ (x,y) \mid x_1 \le x \le x_2 \quad y_1(x) \le y \le y_2(x) \}$$

Flächenintegral einer skalaren Funktion $f(x,y)$

$$I = \iint_{\mathcal{A}} dA\, f(x,y) = \iint_{\mathcal{A}} dx\,dy\, f(x,y)$$

$$\equiv \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy\, f(x,y)$$

Flächenelement $dA = dx\,dy = d^2r$



analog Volumenintegral eines skalaren Feldes $f(x,y,z)$

$$V = \{(x,y,z) \mid x_1 \le x \le x_2;\ y_1(x) \le y \le y_2(x);\ z_1(x,y) \le z \le z_2(x,y)\}$$

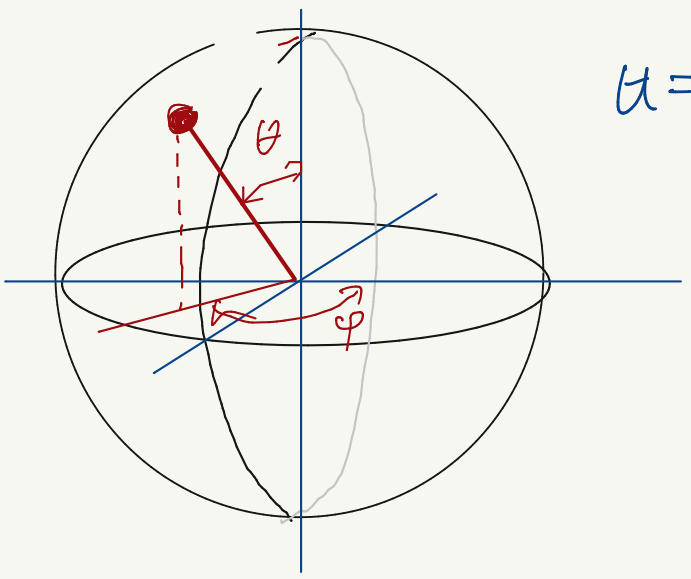
$$I = \iiint_V dV\, f(x,y,z) = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} dz\ f(x,y,z)$$

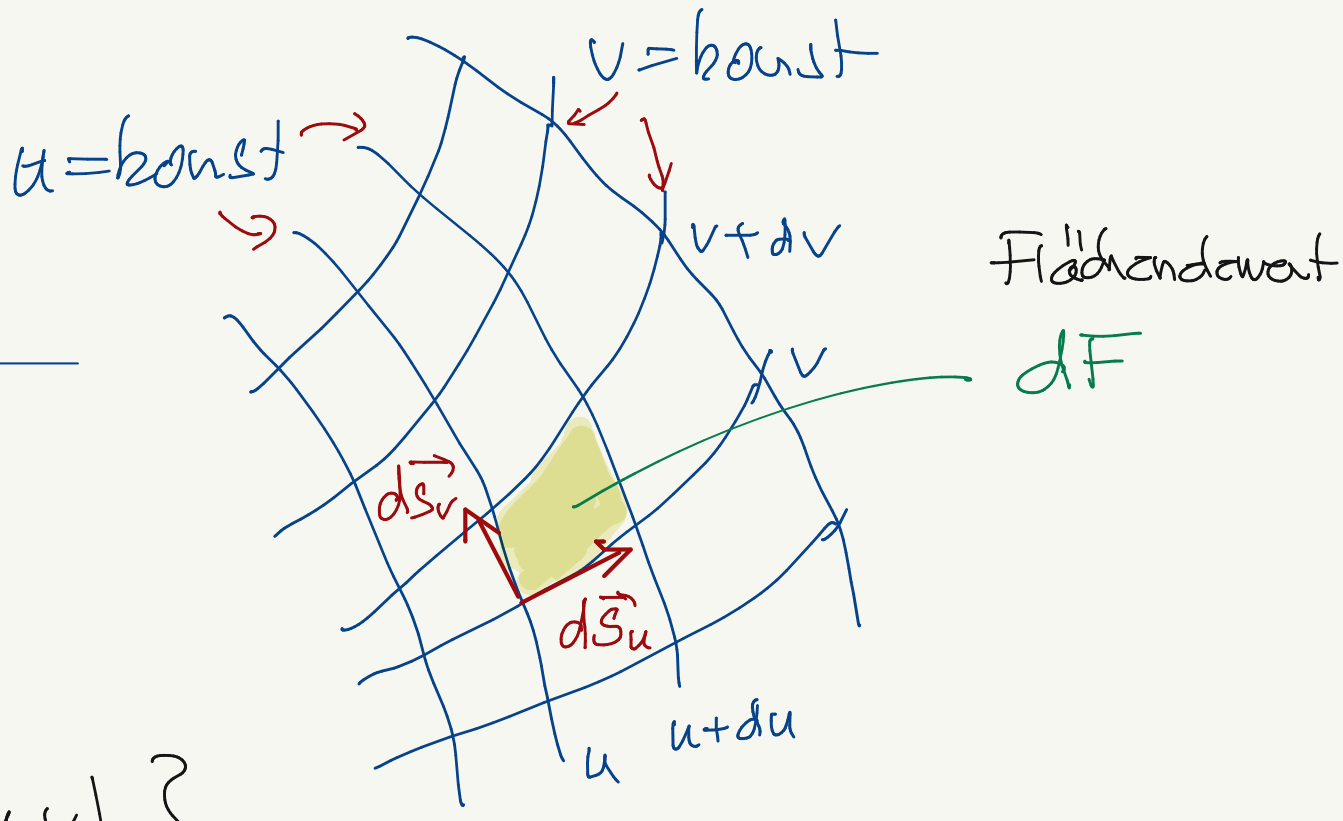
Volumenelement $dV = dx\,dy\,dz = d^3r$

(B) Oberflächenintegrale vektorieller Funktionen

jetzt vektorielle Funktion $\vec{F}(\vec{r})$ und Oberfläche F

Oberfläche parametrisiert durch zwei Parameter u, v





$$\mathcal{A} = \{ \vec{r} = \vec{r}(u,v) \}$$

ebene Fläche

dx dy

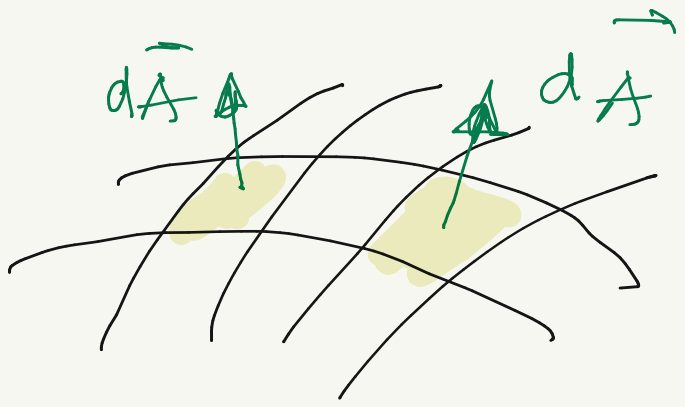
$$d\vec{x} = \hat{e}_x dx$$
$$d\vec{y} = \hat{e}_y dy$$

Oberfläche

$$d\vec{S}_u = \vec{r}(u+du, v) - \vec{r}(u,v) = \frac{\partial \vec{r}}{\partial u} du$$
$$d\vec{S}_v = \vec{r}(u, v+dv) - \vec{r}(u,v) = \frac{\partial \vec{r}}{\partial v} dv$$

Flächenelement: jetzt auch eine Richtung!

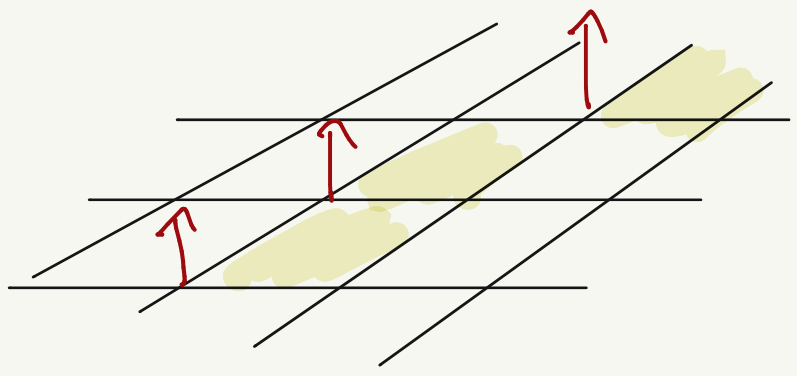
$$d\vec{A} = d\vec{S}_u \times d\vec{S}_v$$
$$= \vec{r}_u \times \vec{r}_v \, du\,dv$$



$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} \qquad \vec{r}_v = \frac{\partial \vec{r}}{\partial v}$$

[in der Ebene ist das nicht wichtig:

$$d\vec{A} = d\vec{s}_x \times d\vec{s}_y$$
$$= dx\,dy\ \vec{e}_x \times \vec{e}_y$$
$$= dx\,dy\ \vec{e}_z$$

]

Einheitsvektor in Richtung der Flächennormalen

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

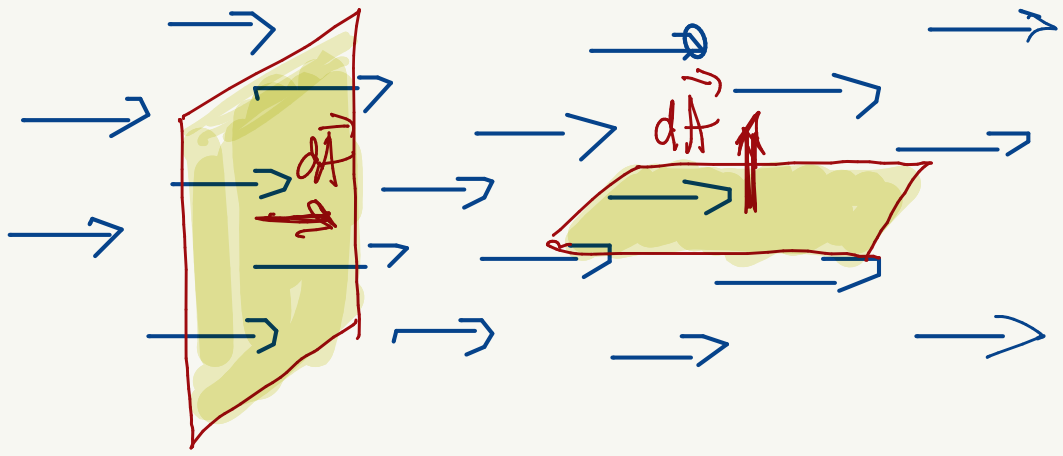
$$d\vec{A} = \hat{n}\,dA \qquad dA = |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

Oberflächenintegral eines vektoriellen Feldes

$$I = \int_A d\vec{A} \cdot \vec{F}(\vec{r}) = \int_A dA\ \underbrace{\hat{n} \cdot \vec{F}(\vec{r})}_{\text{skalares Feld.}}$$

Fluß des Feldes $\vec{F}$ durch infinitesimale Fläche $d\vec{A}$

$$d\Phi = d\vec{A} \cdot \vec{F}$$



$d\Phi = \max$ $\qquad$ $d\Phi = 0$

Fluß durch Fläche A

$$\Phi = \int_A d\vec{A} \cdot \vec{F}$$

Eine besondere Bedeutung hat der Fluß durch eine geschlossene Oberfläche

$$\Phi = \oint_A d\vec{A} \cdot \vec{F} \quad \text{oder} \quad = \oiint_A d\vec{A} \cdot \vec{F}$$

Flächeninhalt der Oberfläche

setze $\vec{F}(\vec{r}) = \hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

$$\int_A d\vec{A} \cdot \hat{n} = \iint_A du\,dv \; \frac{(\vec{r}_u \times \vec{r}_v)^2}{|\vec{r}_u \times \vec{r}_v|}$$

$$= \iint_A du\,dv \; |\vec{r}_u \times \vec{r}_v| = A$$

(C) Koordinatentransformation in Integralen: Jacobi Determinante

Wie transformiert man ein (Ober)fläche oder Volumenintegral zwischen verschiedenen krummlinigen Koordinaten?

- betrachten Flächenintegral in x,y Ebene

$$dA = d\vec{A}\cdot\hat{e}_z = \underbrace{\left(\vec{r}_u \times \vec{r}_v\right)\cdot\hat{e}_z}_{=J}\, du\,dv = dx\,dy$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} =: \frac{\partial(x,y)}{\partial(u,v)}$$

Jacobi Determinante

Transformation zwischen Flächenelemente

$$dx\,dy = |J|\,du\,dv = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Verallgemeinerung auf Transformation zwischen beliebigen (krummlinigen) ebenen Koordinaten

$$(U,V) \longleftrightarrow (u,v)$$

$$dU\,dV = \left|\frac{\partial(U,V)}{\partial(u,v)}\right| du\,dv$$

- Volumenintegrale

$$(U,V,W) \longleftrightarrow (u,v,w)$$

$$dU\,dV\,dW = \left| \frac{\partial(U,V,W)}{\partial(u,v,w)} \right| du\,dv\,dw$$

wobei die Jacobi-Determinante in 3D sich ergibt zu

$$\frac{\partial(U,V,W)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial U}{\partial u} & \frac{\partial V}{\partial u} & \frac{\partial W}{\partial u} \\ \frac{\partial U}{\partial v} & \frac{\partial V}{\partial v} & \frac{\partial W}{\partial v} \\ \frac{\partial U}{\partial w} & \frac{\partial V}{\partial w} & \frac{\partial W}{\partial w} \end{vmatrix}$$

Bspl: (i) $(x,y) \longleftrightarrow (r,\varphi)$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$$\int dx \int dy = \int dr \int d\varphi\, r$$

(ii) $(x,y,z) \longleftrightarrow (r,\vartheta,\varphi)$

$$x = r\sin\vartheta\cos\varphi$$
$$y = r\sin\vartheta\sin\varphi$$
$$z = r\cos\vartheta$$

$$\frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \vartheta} & \frac{\partial y}{\partial \vartheta} & \frac{\partial z}{\partial \vartheta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \\ r\cos\vartheta\cos\varphi & r\cos\vartheta\sin\varphi & -r\sin\vartheta \\ -r\sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi & 0 \end{vmatrix}$$

$$= \dots = r^2 \sin\vartheta$$

$$\int dx \int dy \int dz = \int dr \int d\vartheta \int d\varphi \; r^2 \sin\vartheta$$

XI.2 Integraldarstellung von Divergenz und Rotation

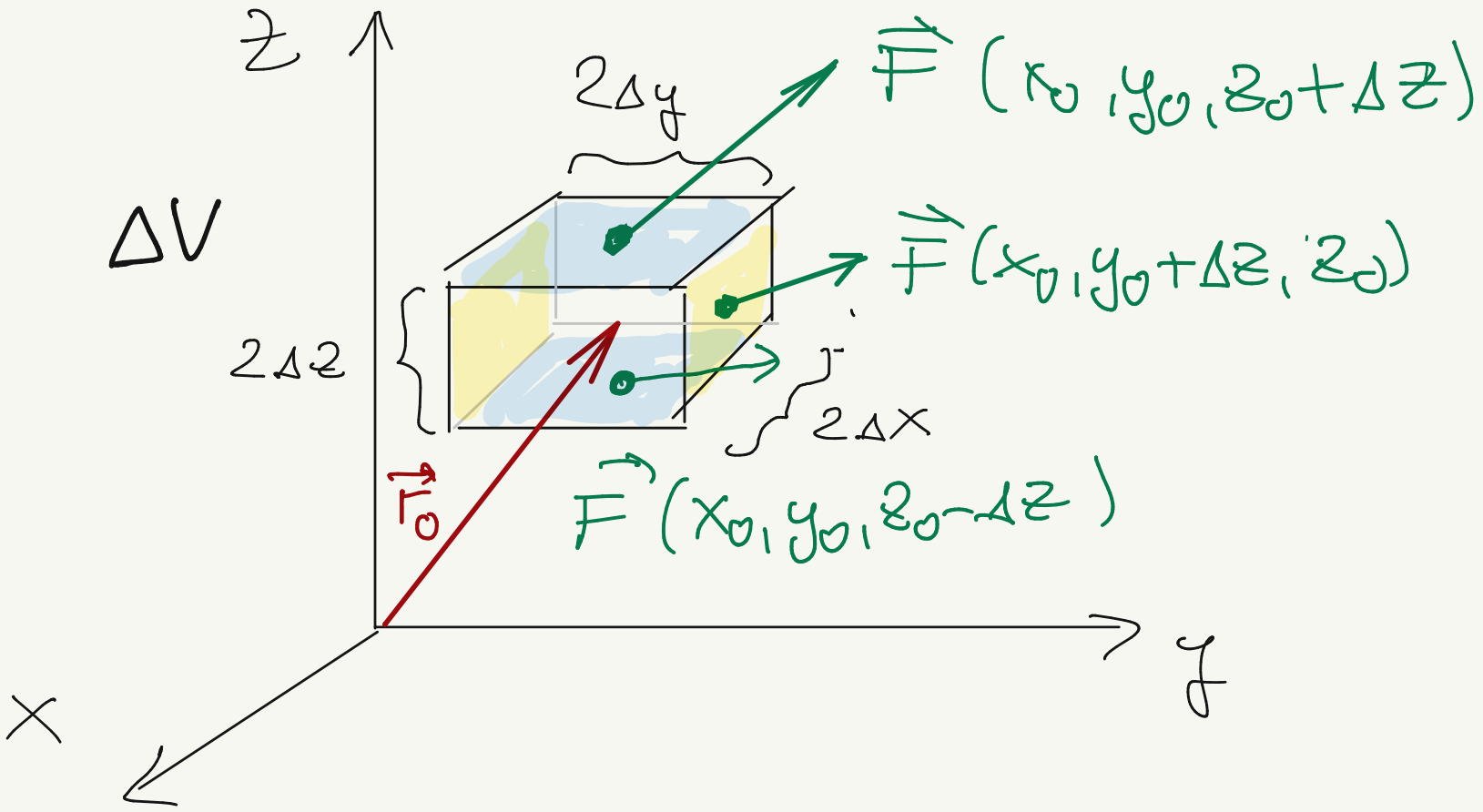
Betrachten $\quad \operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

$$\operatorname{rot} \vec{F} = \vec{\nabla} \times \vec{F}$$

(A) Divergenz als Quellstärke

Wir zeigen

$$\operatorname{div} \vec{F}(\vec{r}_0) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{O(\Delta A)} d\vec{A} \cdot \vec{F}$$



$$\oiint_{\partial(\Delta V)} d\vec{A}\cdot\vec{F} = \iint dy\,dz \left[A_x(x_0+\Delta x, y, z) - A_x(x_0-\Delta x, y, z)\right]$$

$$+ \iint dx\,dz \left[A_y(x, y_0+\Delta y, z) - A_y(x, y_0-\Delta y, z)\right]$$

$$+ \iint dx\,dy \left[A_z(x, y, z_0+\Delta z) - A_z(x, y, z_0-\Delta z)\right]$$

$\Delta z \to 0 \qquad 2\Delta z \dfrac{\partial A_z}{\partial z}(x, y, z_0)$

etc.

Mittelwertsatz der Integralrechnung

$$\frac{1}{\Delta V}\iint dx\,dy\,[\dots] = \frac{\partial A_z}{\partial z}(x_0, y_0, z_0)\,\frac{2\Delta z}{8\Delta x\Delta y\Delta z}\underbrace{\iint dx\,dy}_{4\Delta x\Delta y}$$

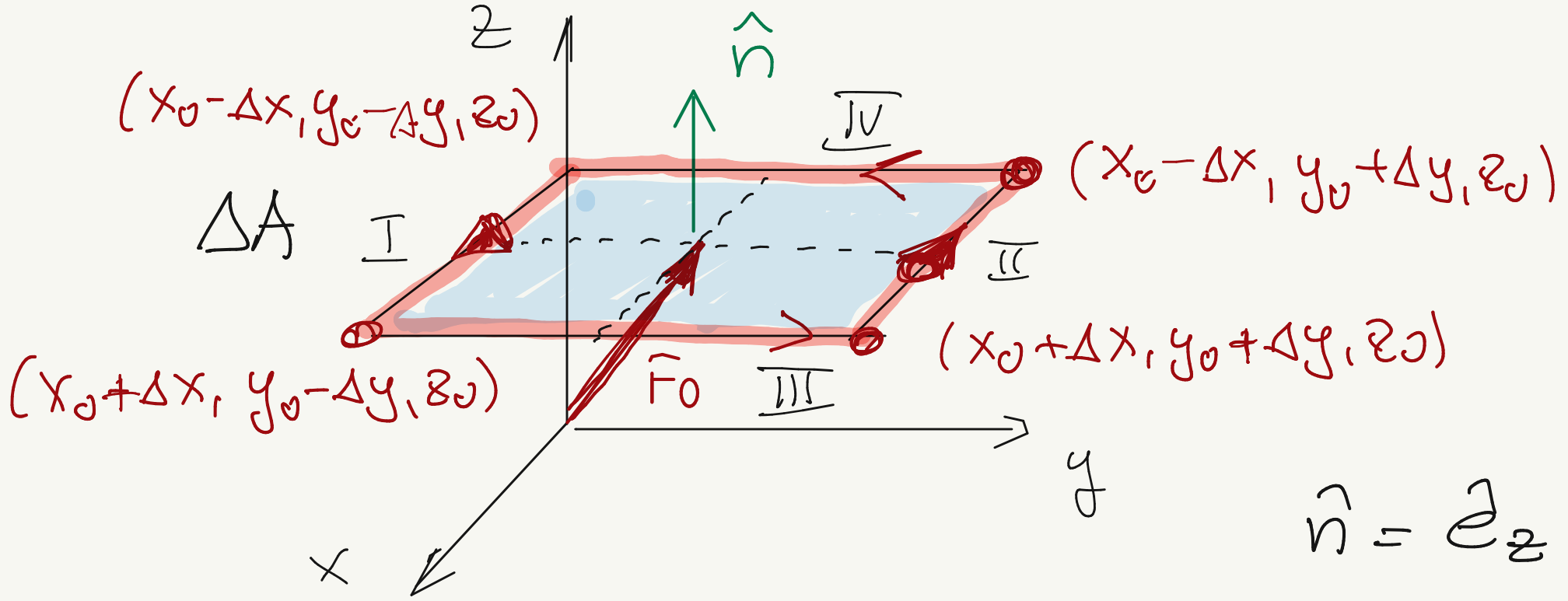
$$= \frac{\partial A_z}{\partial z}(x_0, y_0, z_0) \qquad \square$$

+ andere Seiten

(B) Rotation als Wirbelstärke

Wir zeigen

$$\hat{n} \cdot \operatorname{rot} \vec{F} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{O(\Delta A)} d\vec{r} \cdot \vec{F}$$



$\hat{n} = e_z$

$$\oint d\vec{r} \cdot \vec{F} = \int_{x_0-\Delta x}^{x_0+\Delta x} dx \underbrace{\left[F_x(x, y_0-\Delta y, z_0) \right]}_{I} - \underbrace{F_x(x, y_0+\Delta y, z_0)}_{II}$$

$$+ \int_{y_0-\Delta y}^{y_0+\Delta y} dy \underbrace{\left[F_y(x_0+\Delta x, y, z_0) \right]}_{III} - \underbrace{F_y(x_0-\Delta x, y, z_0)}_{IV}$$

$\Delta x, \Delta y \to 0$

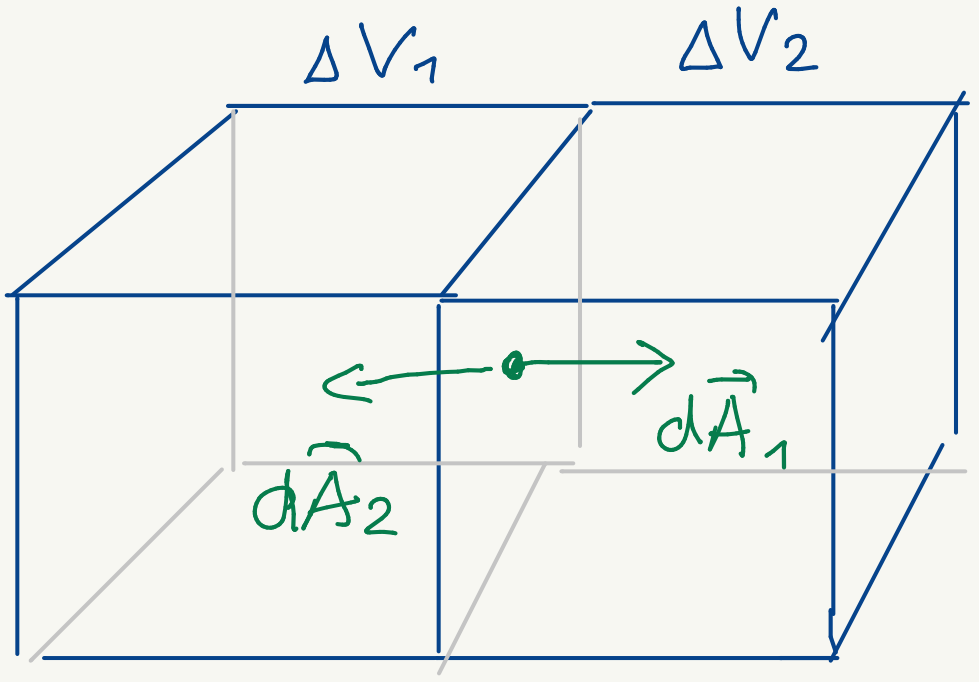
$$= - \frac{\partial F_x}{\partial y}(x_0, y_0, z_0)\, 2\Delta y\, 2\Delta x + \frac{\partial F_y}{\partial x}(x_0, y_0, z_0)\, 2\Delta x\, 2\Delta y$$

$$\Rightarrow \lim_{\Delta x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \oint_{\partial(\Delta x, \Delta y)} d\vec{A} \cdot \vec{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$= \left(\mathrm{rot}\, \vec{F} \right)_z \qquad \square$$

XI. 3 Integralsätze von Gauß und Stokes

- betrachten zwei zusammengesetzte Volumenelemente



$$\Delta V = \Delta V_1 + \Delta V_2$$

$$\oiint d\vec{A} \cdot \vec{F} = \oiint d\vec{A} \cdot \vec{F} + \oiint d\vec{A} \cdot \vec{F}$$

$$(\Delta V) \qquad \Delta V_1 \qquad \Delta V_2$$

Die Beiträge an der Grenzfläche zwischen ΔV_1 und ΔV_2 heben sich heraus da

$$d\vec{A}_1 = -d\vec{A}_2$$

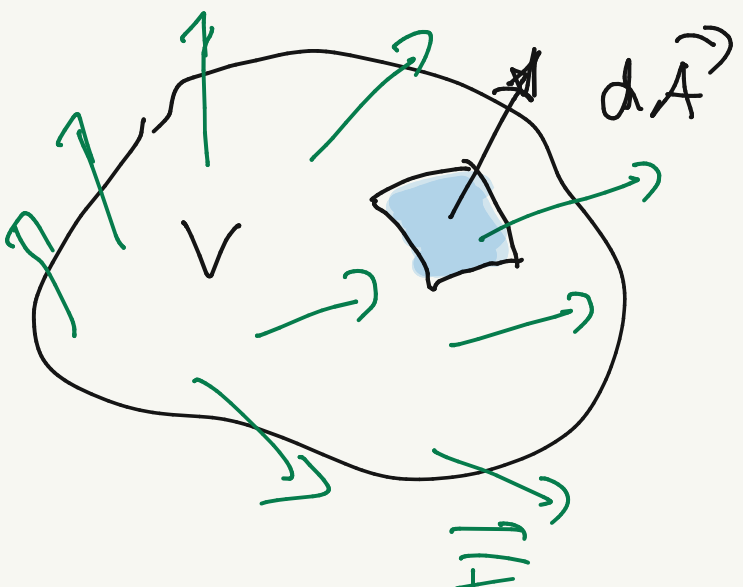
$$\Rightarrow$$

$$\oiint_{\partial(V)} d\vec{A}\cdot\vec{F} = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta V_i \cdot \frac{1}{\Delta V_i} \oiint_{\partial(\Delta V_i)} d\vec{A}\cdot\vec{F}$$

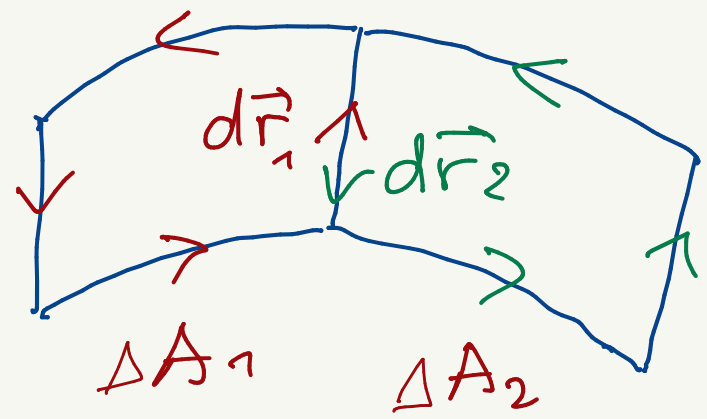
$$= \iiint_V dV \; \mathrm{div}\,\vec{F}$$

$$\boxed{\oiint_{\partial(V)} d\vec{A}\cdot\vec{F} = \iiint_V dV \; \mathrm{div}\,\vec{F}}$$

Satz von Gauß



- Betrachten zwei zusammengesetzte Flächenelemente



$$\Delta A = \Delta A_1 + \Delta A_2$$

$$\oint_{(\Delta A_1)} d\vec{A}\cdot\vec{F} + \oint_{(\Delta A_2)} d\vec{A}\cdot\vec{F} = \oint_{(\Delta A)} d\vec{A}\cdot\vec{F}$$

Die Beiträge von der Integration entlang der gemeinsamen Verbindungslinie heben sich gegenseitig auf, da

$$d\vec{r}_1 = -d\vec{r}_2$$

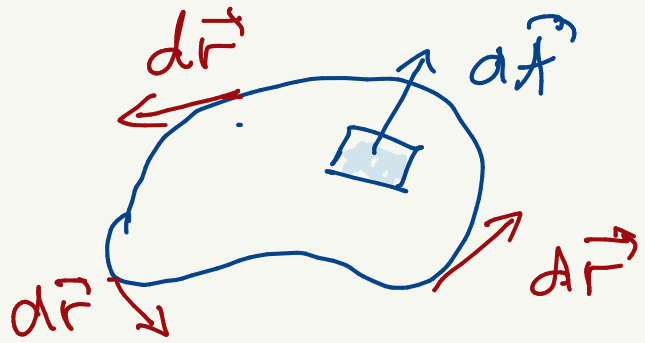
$$\Rightarrow$$

$$\oint_{O(A)} d\vec{r}\cdot\vec{F} = \lim_{N\to\infty} \sum_{i=1}^{N} \oint_{O(\Delta A_i)} d\vec{r}\cdot\vec{F}\,\frac{\Delta A_i}{\Delta A_i}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \Delta A_i\ \mathrm{rot}\vec{F} = \iint_A d\vec{A}\cdot\mathrm{rot}\vec{A}$$

$$\oint_{O(A)} d\vec{r}\cdot\vec{F} = \iint_A d\vec{A}\cdot\mathrm{rot}\vec{A}$$

Satz von Stokes

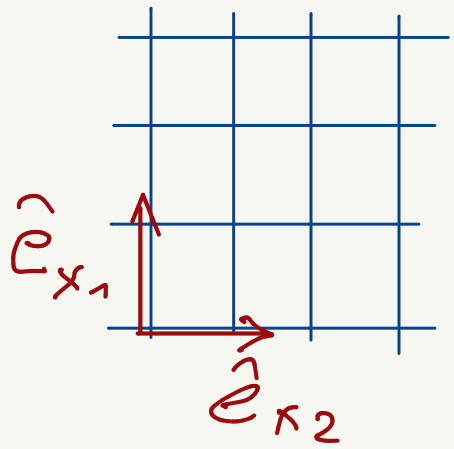


XI.4 Vektoranalysis in krummlinigen Koordinaten

Zur Erinnerung:

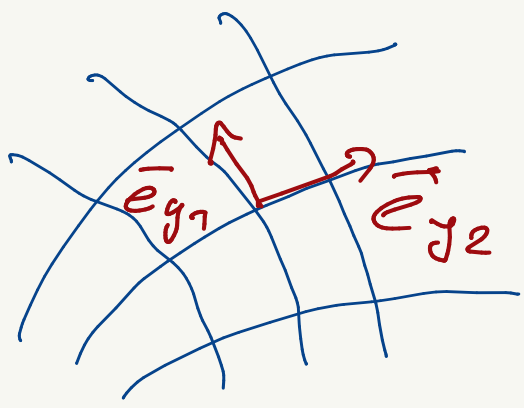
- kartesische Koordinaten

$$\vec{r} \hat{=} (x_1, x_2, x_3)$$



- krummlinige (lokal) orthogonale Koordinaten

$$\vec{r} \hat{=} (y_1, y_2, y_3)$$



Einheitsvektoren

$$\hat{e}_{y_i} = \frac{1}{b_{y_i}} \frac{\partial \vec{r}}{\partial y_i} \qquad b_{y_i} \equiv \left| \frac{\partial \vec{r}}{\partial y_i} \right|$$

(A) Gradient

$$f(\vec{r}) \qquad df(\vec{r}) = \vec{\nabla} f \cdot d\vec{r}$$

in krummlinigen Koordinaten

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial y_i} dy_i$$

$$\vec{\nabla} f = \sum_i \left(\vec{\nabla} f\right)_{y_i} \hat{e}_{y_i} \qquad d\vec{r} = \sum_k b_k \, dy_k \, \hat{e}_{y_k}$$

$$\Rightarrow \quad \vec{\nabla} f \cdot d\vec{r} = \sum_j \left(\vec{\nabla} f\right)_{y_j} b_j \, dy_j$$

auf der anderen Seite

$$\vec{\nabla} f \cdot d\vec{r} = df = \sum_j \frac{\partial f}{\partial y_j} dy_j$$

$$\Rightarrow \boxed{\left(\vec{\nabla} f\right)_{y_j} = \frac{1}{b_j} \frac{\partial f}{\partial y_j}}$$

Komponenten des Gradienten in (y_1, y_2, y_3)

Damit erhält man für die Darstellung des Nabla Operators in krummlinigen Koordinaten:

$$\boxed{\vec{\nabla} = \sum_j \vec{e}_j \frac{1}{b_j} \frac{\partial}{\partial y_j}}$$

wobei $\vec{e}_j \equiv \vec{e}_{y_j}$

- **Besonderheit:** sowohl b_j als auch $\vec{e}_j$ hängen von den y_i ab !

Das ist bei kartesischen Koordinaten nicht der Fall

(B) Divergenz

$$\operatorname{div}\vec{A} = \vec{\nabla}\cdot\vec{A} = \sum_{kj}\left(\hat{e}_k \frac{1}{b_k}\frac{\partial}{\partial y_k}\right)\cdot\left(A_j \hat{e}_j\right)$$

$$(*) \quad = \sum_{kj}\left(\frac{1}{b_k}\frac{\partial A_j}{\partial y_k}\underbrace{\hat{e}_k\cdot\hat{e}_j}_{\delta_{kj}} + \frac{1}{b_k} A_j \hat{e}_k \cdot \frac{\partial \hat{e}_j}{\partial y_k}\right)$$

neuer Term in krummlinigen Koord.

Was ist $\hat{e}_k \cdot \dfrac{\partial \hat{e}_j}{\partial y_k} = ?$

- $\boxed{j=k}$ $\quad \hat{e}_k \cdot \dfrac{\partial \hat{e}_k}{\partial y_k} = \dfrac{1}{2}\dfrac{\partial}{\partial y_k}\left(\hat{e}_k\cdot\hat{e}_k\right) = 0$

- $\boxed{j\neq k}$ $\quad \dfrac{\partial^2 \vec{r}}{\partial y_j \partial y_k} = \dfrac{\partial^2 \vec{r}}{\partial y_k \partial y_j}$

$$\Rightarrow \frac{\partial}{\partial y_j}\left(b_k \hat{e}_k\right) = \frac{\partial}{\partial y_k}\left(b_j \hat{e}_j\right)$$

$\cdot \hat{e}_k$ $\quad b_k \dfrac{\partial \hat{e}_k}{\partial y_j} + \dfrac{\partial b_k}{\partial y_j}\hat{e}_k = b_j \dfrac{\partial \hat{e}_j}{\partial y_k} + \dfrac{\partial b_j}{\partial y_k}\hat{e}_j$

$$\underbrace{b_k \hat{e}_k \cdot \frac{\partial \hat{e}_k}{\partial y_j}}_{=0 \text{ s.o.}} + \frac{\partial b_k}{\partial y_j} = b_j \hat{e}_k \frac{\partial \hat{e}_j}{\partial y_k}$$

$$\Rightarrow \quad b_j \vec{e}_k \frac{\partial \vec{e}_j}{\partial y_k} = \frac{\partial b_k}{\partial y_j}$$

Einsetzen in (*)

$$\operatorname{div} \vec{A} = \sum_j \left[\frac{1}{b_j} \frac{\partial A_j}{\partial y_j} + \sum_{k \neq j} \frac{A_j}{b_k b_j} \frac{\partial \vec{e}_j}{\partial y_k} \right]$$

$$= \frac{1}{b_1} \left[\frac{\partial A_1}{\partial y_1} + \frac{A_1}{b_2} \frac{\partial b_2}{\partial y_1} + \frac{A_1}{b_3} \frac{\partial b_3}{\partial y_1} \right]$$

$$+ \frac{1}{b_2} \left[\frac{\partial A_2}{\partial y_2} + \dots \right] + \frac{1}{b_3} \left[\frac{\partial A_3}{\partial y_3} + \dots \right]$$

$$\operatorname{div} \vec{A} = \frac{1}{b_1 b_2 b_3} \left[\frac{\partial}{\partial y_1} (A_1 b_2 b_3) + \frac{\partial}{\partial y_2} (A_2 b_1 b_3) + \frac{\partial}{\partial y_3} (A_3 b_1 b_2) \right]$$

Divergenz in krummlinigen Koordinaten

(c) Rotation

Analog zu den Rechnungen oben lässt sich zeigen

$$\operatorname{rot}\vec{A} = \vec{\nabla} \times \vec{A} = \sum_{kj} \left(\frac{1}{b_k} \frac{\partial}{\partial y_k} \vec{e}_k \right) \times \left(A_j \vec{e}_j \right)$$

$$\operatorname{rot}\vec{A} = \frac{1}{b_1 b_2 b_3} \begin{vmatrix} b_1\vec{e}_1 & b_2\vec{e}_2 & b_3\vec{e}_3 \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ b_1 A_1 & b_2 A_2 & b_3 A_3 \end{vmatrix}$$

Rotation in krummlinigen Koordinaten

(D) Laplace Operator

Für den Laplace Operator

$$\Delta = \vec{\nabla}^2 = \sum_{ik} \left(\frac{1}{b_k} \frac{\partial}{\partial y_k} \vec{e}_k \right) \left(\frac{1}{b_j} \frac{\partial}{\partial y_j} \vec{e}_j \right)$$

findet man:

$$\Delta = \frac{1}{b_1 b_2 b_3} \left[\frac{\partial}{\partial y_1} \left(\frac{b_2 b_3}{b_1} \frac{\partial}{\partial y_1} \right) + \frac{\partial}{\partial y_2} \left(\frac{b_1 b_3}{b_2} \frac{\partial}{\partial y_2} \right) + \frac{\partial}{\partial y_3} \left(\frac{b_1 b_2}{b_3} \frac{\partial}{\partial y_3} \right) \right]$$