Vektoranalysis in krummlinigen Koordinaten

Koordinaten: (q_1, q_2, q_3)

Linienelement: $\mathrm{d}\vec{r} = \sum_i \mathrm{d}q_i \vec{\mathrm{e}}_i$

Einheitsvektoren:

$$\vec{\mathrm{e}}_i = \frac{1}{b_i}\frac{\partial \vec{r}}{\partial q_i}, \qquad b_i \equiv \left|\frac{\partial \vec{r}}{\partial q_i}\right|$$

Im Allgemeinen gilt: $\vec{\mathrm{e}}_i = \vec{\mathrm{e}}_i\,(q_1, q_2, q_3)$ (Ausnahme: karthesische Koordinaten).

Orthogonalität:

$$\vec{\mathrm{e}}_i \cdot \vec{\mathrm{e}}_j \; = \; \delta_{ij}$$

Nabla-Operator:

$$\vec{\nabla} = \sum_i \vec{\mathrm{e}}_i \,\frac{1}{b_i}\,\frac{\partial}{\partial q_i}$$

Gradient:

$$\mathrm{grad} f = \vec{\nabla}\, f = \sum_i \vec{\mathrm{e}}_i \,\frac{1}{b_i}\,\frac{\partial f}{\partial q_i}$$

Divergenz:

$$\mathrm{div}\vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{1}{b_1 b_2 b_3}\left[\frac{\partial(A_1 b_2 b_3)}{\partial q_1} + \frac{\partial(b_1 A_2 b_3)}{\partial q_2} + \frac{\partial(b_1 b_2 A_3)}{\partial q_3}\right]$$

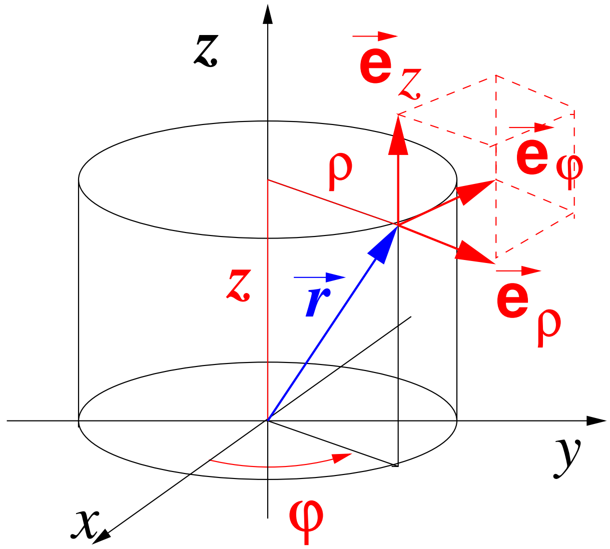
Rotation:

$$\mathrm{rot}\vec{A} = \vec{\nabla}\times\vec{A} = \frac{1}{b_2 b_3}\left[\frac{\partial(b_3 A_3)}{\partial q_2} - \frac{\partial(b_2 A_2)}{\partial q_3}\right]\vec{\mathrm{e}}_1 + \text{zykl. Vert.}$$

Laplace-Operator:

$$\Delta = \frac{1}{b_1 b_2 b_3}\left[\frac{\partial}{\partial q_1}\left(\frac{b_2 b_3}{b_1}\frac{\partial}{\partial q_1}\right) + \text{zykl. Vert.}\right]$$

Zylinderkoordinaten



$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$
$$\rho = \sqrt{x^2+y^2} \qquad \phi = \arctan\left(\frac{y}{x}\right)$$

$$b_\rho = 1, \quad b_\phi = \rho, \quad b_z = 1$$

$$\begin{aligned}
\vec{\mathrm{e}}_\rho &= \vec{\mathrm{e}}_x\,\cos\phi + \vec{\mathrm{e}}_y\,\sin\phi,\\
\vec{\mathrm{e}}_\phi &= -\vec{\mathrm{e}}_x\,\sin\phi + \vec{\mathrm{e}}_y\,\cos\phi,\\
\vec{\mathrm{e}}_z &= \vec{\mathrm{e}}_z.
\end{aligned}$$

Gradient:

$$\mathrm{grad}\,f = \vec{\mathrm{e}}_\rho\frac{\partial f}{\partial\rho} + \vec{\mathrm{e}}_\phi\frac{1}{\rho}\frac{\partial f}{\partial\phi} + \vec{\mathrm{e}}_z\frac{\partial f}{\partial z}$$

Divergenz:

$$\mathrm{div}\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

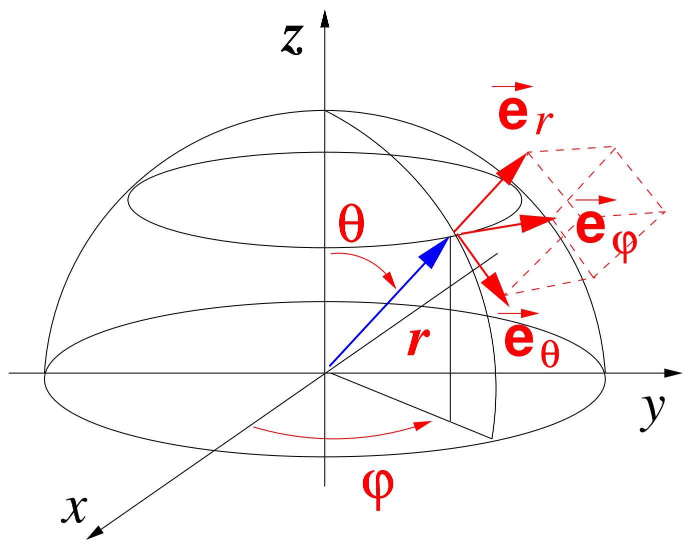
Rotation:

$$\mathrm{rot}\vec{A} \;=\; \vec{\mathrm{e}}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \vec{\mathrm{e}}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \vec{\mathrm{e}}_z\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)$$

Laplace:

$$\Delta\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

<u>**sphärische Polarkoordinaten**</u>



$$x = r\cos\phi\sin\theta \qquad y = r\sin\phi\sin\theta \qquad z = r\cos\theta$$
$$r = \sqrt{x^2+y^2+z^2} \qquad \phi = \arctan\left(\frac{y}{x}\right) \qquad \theta = \arccos\left(\frac{z}{r}\right)$$

$$b_r = 1, \quad b_\theta = r, \quad b_\phi = r\,\sin\theta$$

$$\begin{aligned}\vec{\mathrm{e}}_r &= \vec{\mathrm{e}}_x\,\cos\phi\sin\theta + \vec{\mathrm{e}}_y\,\sin\phi\sin\theta + \vec{\mathrm{e}}_z\,\cos\theta,\\ \vec{\mathrm{e}}_\theta &= \vec{\mathrm{e}}_x\,\cos\phi\cos\theta + \vec{\mathrm{e}}_y\,\sin\phi\cos\theta - \vec{\mathrm{e}}_z\,\sin\theta,\\ \vec{\mathrm{e}}_\phi &= -\vec{\mathrm{e}}_x\,\sin\phi + \vec{\mathrm{e}}_y\,\cos\phi\end{aligned}$$

Gradient:

$$\mathrm{grad} f = \vec{\mathrm{e}}_r\frac{\partial f}{\partial r} + \vec{\mathrm{e}}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \vec{\mathrm{e}}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

Divergenz:

$$\mathrm{div}\vec{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Rotation:

$$\begin{aligned}\mathrm{rot}\vec{A} &= \vec{\mathrm{e}}_r\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \vec{\mathrm{e}}_\theta\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\\ &\quad +\vec{\mathrm{e}}_\phi\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\end{aligned}$$

Laplace:

$$\Delta\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$