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Dedicated to the 100th anniversary of M.L.Ter-Mikaelian birth

Effect of Bloch-Band Dispersion on the Quantized Transport in a Topological Thouless Pump

R. G. Unanyan[a], * and M. Fleischhauer[a]

[a] *Fachbereich Physik, University of Kaiserslautern-Landau, Kaiserslautern, D-67663 Germany*

**e-mail: unanyan@physik.uni-kl.de*

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Abstract—We study the spreading of an initially localized wave packet of a particle hopping on a one-dimensional superlattice during a cycle of a topological Thouless pump. Two contributions to the dispersion of the adiabatic pumping process are identified: a dynamical part and a geometrical part. The magnitude of the dynamical contribution to the spreading depends on the dispersion of the adiabatic transfer state and the cycle time. Unlike the dynamical one, the geometrical contribution does not depend on the duration of the adiabatic process and can be made much smaller than the lattice spacing. We show that as the adiabaticity is enhanced by prolonging the period of the pumping process, the uncertainty in coordinate space is increased linearly with the adiabaticity parameter. We propose a mechanism to smoothen the energy surface of the adiabatic transfer state to reduce the spreading of the spatial distribution of the transported particle. This diminishes or even eliminates (up to the geometric contribution) the dispersion of the coordinate during the transport process.

Keywords: Thouless pumping, Berry phase, adiabatic process

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1. INTRODUCTION

Forty years ago, David Thouless [1] showed that an adiabatic cyclic variation of a periodic potential in space and time can cause a quantized motion of particles. The average displacement of the particle does not depend on details of the time dependence of the parameters and, if an initial state with equal amplitude in all momentum states of a Bloch band is chosen, can be expressed solely in terms of a topological invariant, the Chern number [2], defined over a periodic Brillouin zone formed by quasi-momentum and time. The same holds in an insulating many-body state, were one Bloch band is fully occupied. Adiabatic pumping of single particles has attracted considerable theoretical and experimental interest in recent years. It has received much attention mainly due to its robustness against external parameter uncertainties. These studies focused mainly on the average displacement of the particle during a pump cycle. For the case of non-interacting particles, quantized pumping has been demonstrated with ultracold atoms [3, 4] by measuring the average shift (displacement) of the center of mass of an atomic cloud with in situ imaging. A Thouless pump was experimentally realized also in several other synthetic topological systems [5, 6] and including systems with interactions [7]. When interactions (i.e. nonlinearities) are added to the systems, the pumping can become fractional [8].

While irrelevant in insulating many-body states, the band dispersion plays an essential role during adiabatic pumping of a single particle and can lead to a spreading of the wave packet over the entire system. Potential applications of adiabatic pumps make it necessary to minimize this spreading. The present paper investigates the effects of the energy-band dispersion on quantized particle transport. We show that in the adiabatic limit, two contributions to the uncertainty of the particle coordinate arize: a dynamical and a geometrical part. The dominant contribution comes in general from the dynamical part, which depends explicitly upon the dispersion of the band Hamiltonian and the cycle time. Unlike the dynamical one, the geometrical contribution does not depend on the duration of the process, and it is much smaller than the lattice spacing. We show that the problem associated with the undesired spreading of the probability distribution can be minimized by judicious change of the initial band state so that the energy as a function of

lattice quasi-momentum is smooth. This offers a means of diminishing, or even eliminating (up to the geometric contribution), the dispersion of the coordinate during the transport process.

Concretely, we consider the Rice-Mele model [9], i.e. a quantum particle in a one-dimensional lattice with variable, alternating hopping amplitudes subject to a staggered onsite energy offset in the tight-binding limit. In principle, such a model can be implemented in many different physical systems, e.g., for neutral cold atoms in an optical superlattice potential [10], two-dimensional photonic crystals [11], or in waveguide structures where time is replaced by the propagation coordinate [12].

2. PARTICLE TRANSPORT IN A TIME-DEPENDENT BLOCH HAMILTONIAN

We discuss a single particle on one-dimensional lattice with lattice constant $a = 1$, and an number of lattice sites L (even integer) with periodic boundary conditions. The system is described by the Hamiltonian

$$\sum_{k=-L/2}^{L/2} |\Psi_k\rangle\langle\Psi_k| \otimes h_k, \tag{1}$$

where h_k is the Hamiltonian band matrix describing the dynamics of a single cell and $|\Psi_k\rangle$ are the orthonormalized Bloch vectors with components

$$\langle l|\Psi_k\rangle = \frac{1}{\sqrt{L}}\exp\left(\frac{2\pi i}{L}kl\right),$$
$$\langle\Psi_{k'}|\Psi_k\rangle = \delta_{kk'}.$$

$2\pi k/L$ is the quasi-momentum of the particle and l is the coordinate of the cell on the lattice.

The solution of the Schrödinger equation ($\hbar = 1$)

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H|\Phi(t)\rangle, \tag{2}$$

with the initial condition $|\Phi(t=0)\rangle = |\Phi_0\rangle$, can be represented as

$$|\Phi(t)\rangle = \frac{1}{\sqrt{L}}\sum_{k=-L/2}^{L/2} C_k|\Psi_k\rangle \otimes |\phi_k(t)\rangle,$$

where $|C_k|^2$ gives the initial quasi-momentum probability distribution. The evolution of a cell is governed by

$$i\frac{\partial}{\partial t}|\phi_k(t)\rangle = h_k(t)|\phi_k(t)\rangle. \tag{3}$$

Now we assume that $h_k = h_k(t)$ is explicitly time-dependent and periodic i.e., $h_k(t) = h_k(t+T)$. The momentum dependence of $h_k(t)$ causes tunneling transitions between different superlattice sites. In the following, we will only analyze the case when the initial state $|\phi_k(0)\rangle$ coincides with one of the adiabatic states of the Hamiltonian $h_k(0)$.

In order to describe transport processes it is convenient to work in the Wannier basis [13]

$$C_k|\phi_k(t)\rangle = \sum_{l=-L/2}^{L/2}\exp\left(\frac{2\pi i}{L}kl\right)|u_l(t)\rangle. \tag{4}$$

Then, the probability to find a particle in the cell l at time t, without specification of "internal states" of the cell, is

$$P_l(t) = \langle u_l(t)|u_l(t)\rangle, \tag{5}$$

where

$$|u_l(t)\rangle = \frac{1}{L}\sum_{k=-L/2}^{L/2} C_k\exp\left(-\frac{2\pi i}{L}kl\right)|\phi_k(t)\rangle.$$

In the following we will examine the case of large system size $L \to \infty$. In this limit, this sum can be replaced by an integral

$$|u_l(t)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(k)\exp(ilk)|\phi(k,t)\rangle\, dk, \tag{6}$$

and the expression (4) transforms into

$$C(k)|\phi(k,t)\rangle = \sum_{l=-\infty}^{\infty} \exp(ikl)|u_l(t)\rangle, \tag{7}$$

where

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |C(k)|^2\, dk = 1. \tag{8}$$

Then, the center-of-mass coordinate can be expressed as

$$R(t) = \sum_{l=-\infty}^{\infty} l\langle u_l(t)|u_l(t)\rangle \frac{1}{2\pi i}\int_{-\pi}^{\pi} C^*(k)\frac{dC(k)}{dk}dk + \frac{1}{2\pi i}\int_{-\pi}^{\pi}|C(k)|^2\left\langle \phi(k,t)\left|\frac{d}{dk}\phi(k,t)\right.\right\rangle dk.$$

The relative shift of the center of mass in a period T reads

$$R(T) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}|C(k)|^2\left\langle \phi(k,T)\left|\frac{d\phi(k,T)}{dk}\right.\right\rangle dk. \tag{9}$$

Throughout this paper, we assume that $C(k) = e^{ikl_0}$, where l_0 is the coordinate of the initially occupied cell. In other words, we assume that the system starts in an initial state where all quasi-momentum states are equally occupied.

Analogously, we may calculate the uncertainty of the coordinate (the second momentum of the particle distribution) after one pumping cycle. After a simple calculation, we eventually obtained the following form

$$\Delta R^2(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\langle\frac{d\phi(k,T)}{dk}\middle|\frac{d\phi(k,T)}{dk}\right\rangle dk - \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} dk\left\langle\frac{d\phi(k,T)}{dk}\middle|\phi(k,T)\right\rangle\right|^2 = B(T) + A(T),$$

where

$$B(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\left\langle\frac{d\phi(k,T)}{dk}\middle|\Pi(k,T)\middle|\frac{d\phi(k,T)}{dk}\right\rangle \geq 0, \tag{10}$$

$$A(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left\langle\phi(k,T)\left|\frac{d\phi(k,T)}{dk}\right.\right\rangle\right|^2 dk - R^2(T) \geq 0 \tag{11}$$

and

$$\Pi(k,t) = 1\!\!1 - |\phi(k,t)\rangle\langle\phi(k,t)| \tag{12}$$

is the projection operator onto an orthogonal state to $|\phi(k,t)\rangle$.

The second term, $A(T)$ in Eq. (11), generally includes significant dynamical and small geometric contributions. These geometric dispersions do not depend on the adiabatic parameter, i.e. on period T. We show that smoothing the energy surface of the adiabatic transfer state can reduce the wave packet spreading. In contrast, geometric contributions are unavoidable during the pumping process. In the next section, we evaluate the spreading of the wave packet based on adiabatic approximation.

3. ADIABATIC EVOLUTION AND NON-ADIABATIC CORRECTIONS

For simplicity we now consider a two-band system i.e. each elementary cell has two internal states. To be specific we discuss the two band Rice-Mele model [9]

$$h_k(t) = \sigma_z J_3(t) + \sigma_x (J_1(t) + J_2(t)\cos k) + \sigma_y J_2(t)\sin k, \tag{13}$$

where $\sigma_{x,y,z}$ are Pauli matrices. This model describes a particle moving along a one-dimensional superlattice with alternating hopping amplitudes J_1, J_2 and a staggered onsite energy offset $J_3(t)$. The generalization for arbitrary number of bands is straightforward.

In the limit when the Hamiltonian varies arbitrarily slowly (adiabatically), the solutions $|\phi_{1,2}(k,t)\rangle$ of the Schrödinger Eq. (3) corresponding to different initial instantaneous eigenstates of $h_k(t)$

$$|\phi_{1,2}(k,t=0)\rangle = |\varphi_{1,2}(k,0)\rangle \tag{14}$$

can be written in the form (see [2, 15, 16])

$$|\phi_1(k,t)\rangle \approx \exp\left(i\int_0^t E_0(k,t')\,dt'\right)|\varphi_1(k,t)\rangle - \alpha(k,t)\exp\left(-i\int_0^t E_0(k,t')\,dt\right)|\varphi_2(k,t)\rangle, \tag{15}$$

$$|\phi_2(k,t)\rangle \approx \exp\left(-i\int_0^t E_0(k,t')\,dt'\right)|\varphi_2(k,t)\rangle + \alpha^*(k,t)\exp\left(i\int_0^t E_0(k,t')\,dt'\right)|\varphi_1(k,t)\rangle, \tag{16}$$

where

$$\alpha(k,t)\int_0^t \exp\left(i\int_0^{t'}\Delta(k,t'')\,dt''\right)\left\langle\varphi_2(k,t')\left|\frac{\partial\varphi_1(k,t')}{\partial t'}\right.\right\rangle dt', \tag{17}$$

and

$$\Delta(k,t) = 2E_0(k,t), \tag{18}$$

is the spectral gap of the Hamiltonian. In the derivations of these expressions, we have imposed the parallel transport condition on instantaneous eigenstates

$$h_k(t)|\varphi_{1,2}(k,t)\rangle = \mp E_0(k,t)|\varphi_{1,2}(k,t)\rangle \tag{19}$$

i.e.

$$\left\langle\varphi_1(k,t)\left|\frac{\partial\varphi_1(k,t)}{\partial t}\right.\right\rangle = \left\langle\varphi_2(k,t)\left|\frac{\partial\varphi_2(k,t)}{\partial t}\right.\right\rangle = 0. \tag{20}$$

When the exponential factor in the integrand of Eq. (17) oscillates (adiabatic limit), the calculation of the integral (17) leads to

$$\alpha(k,T) \approx \frac{\exp\left(i\int_0^T\Delta(k,t)\,dt\right)}{i\Delta(k,T)}\left\langle\varphi_2(k,T)\left|\frac{\partial\varphi_1(k,t)}{\partial t}\right|_{t=T}\right\rangle - \frac{1}{i\Delta(k,0)}\left\langle\varphi_2(k,0)\left|\frac{\partial\varphi_1(k,t)}{\partial t}\right|_{t=0}\right\rangle. \tag{21}$$

Substituting this into Eq. (15) gives for the integrand of $B(T)$

$$\left|\left\langle\frac{\partial\phi_1(k,T)}{\partial k}\right|\phi_2(k,T)\right\rangle\right|^2 \approx \left|\frac{\left\langle\varphi_2(k,0)\left|\frac{\partial\varphi_1(k,t)}{\partial t}\right|_{t=0}\right\rangle}{\Delta(k,0)}\int_0^T\frac{\partial\Delta(k,t)}{\partial k}\,dt\right|^2, \tag{22}$$

which does not depend on T and Ω_0.

As already mentioned in the introduction in the limit when the Hamiltonian $h_k(t)$ varies arbitrarily slowly, the average displacement of the particle per cycle is related to a 2D topological invariant, the Chern number

$$R(T) \approx \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^T\Omega(k,t)\,dk\,dt, \tag{23}$$

where $\Omega(k,t)$ is the Berry curvature [14]

$$\Omega(k,t) = 2\mathrm{Im}\left\langle \frac{\partial\varphi_1(k,t)}{\partial k}\middle|\frac{\partial\varphi_1(k,t)}{\partial t}\right\rangle.$$

The Chern numbers of the two bands of Hamiltonian (13) are ±1 (see e.g. [2]).

It is remarkable that for any trajectory (the closed path encircles the singularity point, where the gap $\Delta(k,t)$ vanishes) in the parameter space the average displacement $R(T)$ is the same. On the contrary, as we will see in the following, the second momentum, $\Delta R(T)$, does depend on the path of the pumping process.

After a simple calculation, we arrive at the following expression for the spreading of the particle

$$\Delta R^2(T) \approx \Delta R_{G_1}^2(T) + \Delta R_D^2(T) + \Delta R_{G_2}^2(T), \tag{24}$$

where

$$\Delta R_{G_1}^2(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\left(\int_0^T \Omega(k,t)\,dt\right)^2 - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^T dk\Omega(k,t)\,dt\right)^2, \tag{25}$$

$$\Delta R_{G_2}^2(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\left|\frac{\left\langle\varphi_2(k,0)\left|\frac{\partial\varphi_1(k,t)}{\partial t}\right|_{t=0}\right\rangle}{\Delta(k,0)}\int_0^T\frac{\partial\Delta(k,t)}{\partial k}dt\right|^2, \tag{26}$$

$$\Delta R_D^2(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\left(\int_0^T\frac{\partial E_0(k,t)}{\partial k}dt\right)^2. \tag{27}$$

In deriving these expressions, i.e. Eqs. (25)–(27), we have assumed that spatial inversion symmetry is present, i.e., the band has the properties [2]

$$E_0(k,t) = E_0(-k,t),\ \ \Omega(k,t) = \Omega(-k,t). \tag{28}$$

We see that the first and second terms $\Delta R_{G_1}(T)$ and $\Delta R_{G_2}(T)$ do not depend on Ω_0 nor on T, i.e., they have a geometric origin and depend on the shape of the trajectory in the parameter space. Thus the two first (25) and (26) have a geometric origin rather than a topological one. Usually, these geometrical contributions are much smaller than the dynamical one. It is easy to see that the last term $\Delta R_D(T)$ is proportional to the accumulated dynamical phase that is the adiabaticity parameter $\Omega_0 T \gg 1$. To overcome this large spreading of the probability distribution, we propose in the following to smooth the energy surface of the adiabatic transfer state $|\varphi_1(k,t)\rangle$.

There is a simple way to understand the physical meaning of $\Delta R_{G_1}^2(T) + \Delta R_D^2(T)$. The centre of mass of the particle moves along the lattice with a total velocity

$$V = \frac{\partial E_0(k,t)}{\partial k} + \Omega(k,t), \tag{29}$$

where the first term is the usual group velocity. The second one is the anomalous velocity due to the Berry curvature $\Omega(k,t)$. And therefore the coordinate of the particle at time $t = T$ is

$$x(k,T) = \int_0^T V dt = \int_0^T\left[\frac{\partial E_0(k,t)}{\partial k} + \Omega(k,t)\right]dt.$$

In the adiabatic limit the quasi-momentum k of the particle is uniformly distributed over interval $[-\pi,\pi]$ and therefore the spreading of the coordinate is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2(k,T)\,dk - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} x(k,T)\,dk\right)^2,$$

which yields $\Delta R_{G_1}^2(T) + \Delta R_D^2(T)$. The physical meaning of Eq. (26) is less transparent. The first factor of the integrand in Eq. (27) is simply the non-adiabatic transition amplitude from the initial adiabatic state $|\varphi_1\rangle$ to the second adiabatic state $|\varphi_2\rangle$. The probability for such non-adiabatic excitation is essentially negligible. However, the second adiabatic state has a large relative group velocity $2\frac{\partial E_0(k,t)}{\partial k}$ (even though the transfer adiabatic state $|\varphi_1\rangle$ has a flat energy band) and therefore the product of these two factors yields a non-vanishing non-adiabatic contribution.

To confirm these observations, we have performed a numerical simulation of the Rice-Meme model, eq.(13) with the path for the pumping process parameterized as

$$J_1(t) = \Omega_0 \cos^2\frac{\pi t}{T}, \quad J_2(t) = \Omega_0 \sin^2\frac{\pi t}{T}, \tag{30}$$

and

$$J_3(t) = \Omega_0 \sin\left[2\pi\left(\frac{t}{T}\right)\right]. \tag{31}$$

During the adiabatic pump cycle, the system moves along a path $\mathcal{C}$ in the $J_1 - J_2$ and Δ parameter space.

By assuming that the particle evolution starts from the adiabatic eigenstate $|\varphi_1(k,t=0)\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$, where $|L\rangle$ and $|R\rangle$ describe states of the particle being inside the left and right potential wells, respectively. It is instructive to see how the distribution evolves in space and time. Figure 1 obtained directly from the numerical solution of the Schrödinger equation with the band Hamiltonian (13). As we can see from Fig. 1a, for small ($\Omega_0 T = 6$), the initially localized particle at middle of the chain, after one period nearly ends up in the next to the initial one with the average displacement $R(T) \approx 0.93$. While, for large $\Omega_0 T = 40$ (see Fig. 1b), the distribution spreads over all sites leading to low efficiency of transport towards the first neighbor site. Although, in this case, the average position is shifted by 0.99 (almost one) unit cell to the right in one pumping cycle, the visibility of the topological transport is abysmal. Thus, from a practical point of view, it may be questioned whether it would be really worthwhile to achieve $R(T) = 1$ by improving the adiabaticity of the pumping process. From the above qualitative observations, it is clear that the adiabaticity enhanced by the prolonged duration will smear the visibility of the feature of the Thouless pumping process.

The dynamical part of the spreading can be easily computed, which yields $\Delta R_D(T) \approx 0.07\Omega_0 T$. Although the numerical factor in front of $\Omega_0 T$ is small, it is large enough to smear out the spatial distribution in a deeply adiabatic regime. From this expression, we see that for $\Omega_0 T \lesssim 6$, the uncertainty in coordinate $\Delta R(T) \lesssim 0.5$. In contrast the average displacement $R(T)$ is about 0.93. In the following, we show how, in principle, to reduce the numerical factor by globally changing tunneling couplings between lattice cells.

4. OVERCOMING THE SPREADING

The problem associated with the undesired spreading $\Delta R(T)$ can be minimized by judicious "engineering" of the initial energy band $E_0(k,t) = \text{const}$. We have reexamined the model (13) with the addition of an energy shift, i.e.

$$h_k = J_3\sigma_z + (J_1 + J_2\cos k)\sigma_x + J_2\sigma_y \sin k - E_0(k,t)\cdot 1 \tag{32}$$

to produce an adiabatic state with zero group velocity (flat band).The term $E_0(k,t)\cdot 1$ does not change the adiabatic eigenstates of the Hamiltonian. In other words, the topological and geometrical features remain unchanged in the adiabatic limit. While the dynamical contribution $\Delta R_D(T)$, for the flat band vanishes. The remainder part of $\Delta R(T)$, involves only the geometrical contribution.

Although this compensation of the band energy of the transfer state is ideal for observing the particle's topological transport, it is not so simple to realize experimentally. The problem is that in order to compensate the large energy $E_0(k,t)$, a very complicated space-time dependent inter-cell tunneling process should be added to the hoppings with amplitudes J_1, J_2. Alternatively, a flat band model can be con-

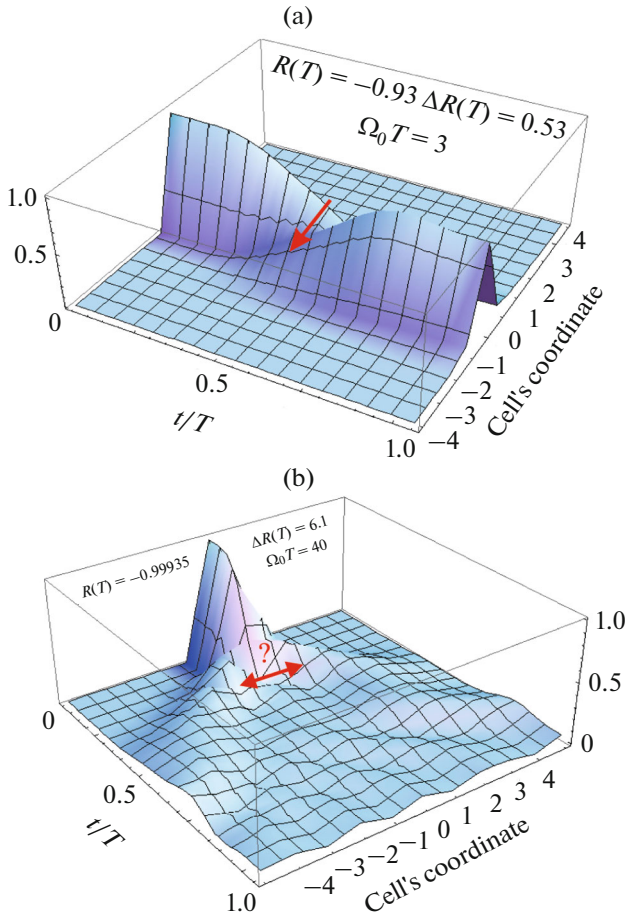

Fig. 1. Probability distribution $P_p(t)$ in space over one period T for different values of the adiabatic parameter $\Omega_0 T$: (a) $\Omega_0 T = 6$ and (b) $\Omega_0 T = 40$. The path for the pump is parameterized as: $J_1 = \Omega_0 \cos^2 \frac{\pi t}{T}$, $J_2 = \Omega_0 \sin^2 \frac{\pi t}{T}$ and $J_3 = \Omega_0 \sin \frac{2\pi t}{T}$.

structed by the replacement $h_k \to h_k / E_0(k,t)$, (without loss of generality, one may assume that $E_0(k,t) \neq 0$). However, in real space, the flattened Hamiltonian $h_k / E_0(k,t)$ includes arbitrary long-range hoppings.

However it is not necessary to fine-tune the inter-cell global tunneling terms. We show now, that keeping only a few short-range tunneling terms gives almost flat band. To this end, let us consider the following Hamiltonian

$$\tilde{h}_k = \chi \cos k \cdot \mathbb{1} + J_3 \sigma_z + (J_1 + J_2 \cos k)\sigma_x + J_2 \sigma_y \sin k. \tag{33}$$

It differs from the original Rice-Mele model by the additional term $\chi \cos k \cdot \mathbb{1}$, which induces the global tunneling between the nearest neighbor superlattice cells with the rate χ.

In the following, we show that the physical mechanism of the reduction ΔR is related to partial compensation of the dispersion of the transfer adiabatic state. The adiabatic energy of the transfer state of the Hamiltonian (33)

$$\widetilde{E}_0(k,t) = \chi \cos k + E_0(k,t) \tag{34}$$

can be expanded in a Fourier series with respect to k

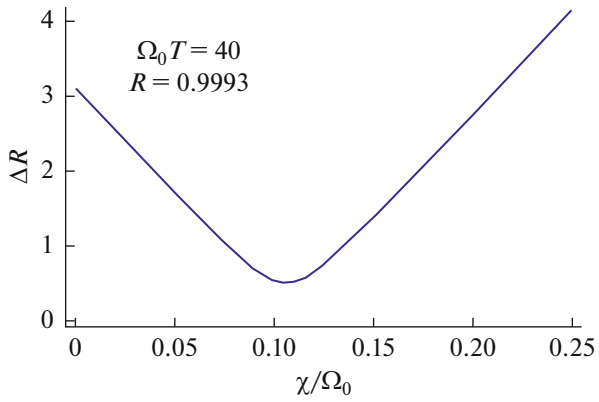

Fig. 2. Variation of ΔR as a function of the parameter χ. The parameters of the pumping process are the same as in Fig. 1.

$$\widetilde{E}_0(k,t) = \chi\cos k + \lambda_0(t) + 2\sum_{n=1}^{\infty}\lambda_n(t)\cos(nk), \tag{35}$$

where

$$\lambda_n(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikn}E_0(k,t)\,dk. \tag{36}$$

Let us remark that, due to the analytic properties of $E_0(k,t)$, the amplitudes of high harmonics $\lambda_n(t)$ are exponentially small in n. So in a good approximation, $\widetilde{E}_0(k,t)$ can be described by the first three terms

$$\widetilde{E}_0(k,t) \approx \lambda_0(t) + (\chi + 2\lambda_1(t))\cos k + 2\lambda_2(t)\cos 2k.$$

Hence, for the dynamical part of ΔR, one obtains the following expression

$$\Delta R_D(T) = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} dk\left(\int_0^T \frac{\partial\widetilde{E}_0(k,t)}{\partial k}\,dt\right)^2},$$

where

$$\frac{\partial\widetilde{E}_0(k,t)}{\partial k} \approx (\chi - 2\lambda_1(t))\sin k - 4\lambda_2(t)\sin 2k.$$

We may choose the free parameter χ, in such a way that the integral $\int_0^T(\chi - 2\lambda_1(t))\,dt$ disappeared i.e., $\chi = 2\int_0^T \lambda_1(t)\,dt$. A simple calculation yields $\chi \approx 0.1\Omega_0$ in good agreement with the numerical results. In Fig. 2 we show the variation of the wave packet spreading as a function of χ, which is measured in units of Ω_0.

5. CONCLUSIONS

We have shown that the quantized displacement of the center-of-mass of a particle in an adiabatic Thouless pump is in general associated with considerable and fast growing uncertainty of the coordinate, which increases as the adiabaticity is enhanced by the prolonged duration of the pumping process. This spreading of the wave packet can lead to a smeared distribution of particles over extended regions of the unit cell. The only exceptional cases are those in which numerical factors make the dispersion of the coordinate small in comparison with the average displacement. The minimum uncertainty of the coordinate may be achieved by the pumping strategy, namely by judicious "engineering" of the initial energy band such that it becomes a smooth function of the quasi-momentum. We have shown here that there is in addition an unavoidable geometrical spreading of the probability distribution, which depends on the details of realizations of the pumping process but does not depend on the period of the adiabatic process.

Our analysis shows that it should be possible to overcome these difficulties and for a smooth, adiabatic transfer state to achieve high visibility in the quantized transport. A modified Rice-Mele tight-binding lattice model is taken as an example to show that the shift per cycle can be larger than the dispersion and the same time, very close to the ideal value.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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