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Quantum fluctuations in the optical parametric oscillator in the limit of a fast decaying subharmonic mode

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The quantum fluctuations in a nondegenerate optical parametric oscillator are calculated without the standard linearization assumption. Analytical results are obtained with the help of a mean-field approximation, which is exact in the limit of a fast decaying subharmonic mode. This approach is equivalent to a linearization around a self-consistently determined pump field amplitude that takes into account fluctuations of the subharmonic mode. The range of validity of the mean-field approximation is analyzed, and the influence of the system size on field amplitudes and squeezing is discussed.

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The degenerate optical parametric oscillator (OPO) plays an important role in the generation of squeezed light [1] with quantum fluctuations below the standard quantum limit. At the same time the OPO is a simple prototype of a dissipative nonlinear quantum system. It is therefore of much interest for practical applications such as gravitational wave detectors [2–4], interferometers [5], and spectroscopical measurements [6], as well as for studying fundamental concepts.

It has been experimentally demonstrated that close to the threshold of parametric oscillation an almost perfect squeezed vacuum state can be generated [7]. The most common approach to theoretically describe the quantum noise in this device is to linearize around the semiclassical amplitudes of the fields [8–11]. This approach predicts perfect squeezing in one quadrature component at the classical threshold. Since correspondingly the orthogonal component shows infinite fluctuations, the linearization breaks down near the point of largest interest. An exact solution of the nonlinear problem can be found only in the adiabatic limit of a fast decaying pump mode. In this case one can obtain the steady-state solutions of the Fokker-Planck equations for the complex [8,12] or positive P representation [13]. The problem of nonlinear quantum noise in the OPO near threshold recently became again the focus of much interest. Stochastic simulations based on the positive P representation have been performed [14] and most recently many-body techniques have been applied [15–17].

In the present paper we discuss the case of a fast decaying subharmonic mode, in which a simple analytical derivation of the steady-state and spectral properties of the OPO at and also above the classical threshold is possible. As we will show, this limit permits a mean-field approximation. In this approximation the pump-mode operator in the equation of motion of the subharmonic mode is replaced by its mean value, which deviates substantially from the semiclassical value, when the threshold is approached and needs to be determined self-consistently. We discuss the limits of the approach and analyze the field amplitudes and squeezing spectra near and above threshold.

The OPO is a system of two coupled optical cavity modes of frequencies ω and 2ω, where the upper mode is driven by a classical external field. The coupling is due to a nonlinear crystal that splits a pump photon in two subharmonic photons and vice versa. The interaction Hamiltonian reads

$$V_{\rm OPO}=i\hbar\frac{K}{2}(a_2a_1^{\dagger 2}-\text{H.c.}), \tag{1}$$

where K is a real and positive coupling constant. Both modes are damped due to cavity losses with rates γ_1 and γ_2. The Langevin equations of this system read in a rotating frame

$$\frac{d}{dt}a_1=-\gamma_1a_1+Ka_2a_1^\dagger+\sqrt{2\gamma_1}F_1(t), \tag{2}$$

$$\frac{d}{dt}a_2=-\gamma_2a_2-\frac{K}{2}a_1^2+\epsilon+\sqrt{2\gamma_2}F_2(t), \tag{3}$$

where ϵ is the pumping rate and F_1 and F_2 are δ-correlated fluctuating forces. Note that these equations are invariant under the transformation $a_1\rightarrow -a_1$. This implies a vanishing mean value of the operator a_1 in steady state. In the semiclassical limit, where a_1 and a_2 can be treated as complex numbers with vanishing fluctuations, we recover the well known behavior for the semiclassical amplitudes of the two modes, as shown in Figs. 1 and 2. Below threshold the subharmonic field is zero and the pump-mode amplitude

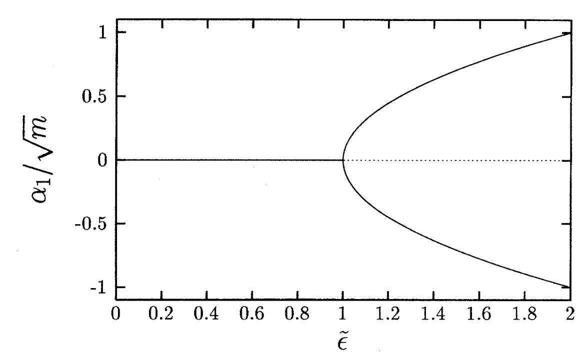

FIG. 1. Coherent amplitude of the subharmonic in units of $(2n_2^{\rm th}\gamma_2/\gamma_1)^{1/2}$. $\tilde{\epsilon}$ is the pump rate normalized by its threshold value $\epsilon^{\rm th}=\gamma_1\gamma_2/K$.

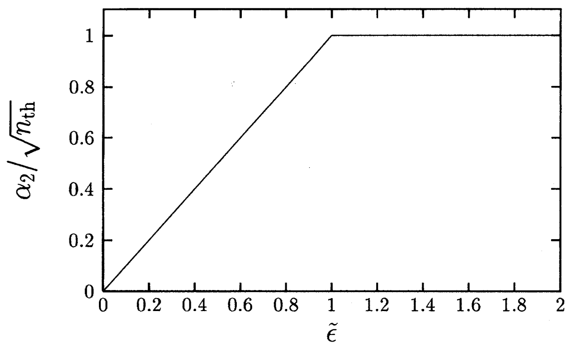

FIG. 2. Coherent amplitude of the pump mode in units of $\gamma_1/K=\alpha_2^{th}$.

increases linearly with ϵ. At threshold the subharmonic undergoes a Hopf bifurcation and the pump-mode amplitude stays constant thereafter.

For the following discussion we define the ratio of the cavity decay rates $\gamma\equiv\gamma_2/\gamma_1$, the normalized pump rate $\tilde{\epsilon}\equiv\epsilon/\epsilon^{th}=\epsilon/(\gamma_1\gamma_2/K)$, and the threshold photon number of the pump mode in the semiclassical limit $n_{th}\equiv\gamma_1^2/K^2=m/(2\gamma)$. Here $m\equiv 2\gamma_1\gamma_2/K^2$ describes the slope of the subharmonic photon number above threshold. We furthermore normalize the operators by their semiclassical scale $\tilde{a}_1\equiv a_1/\sqrt{m}$, $\tilde{a}_2\equiv a_2/\sqrt{n_{th}}$ and measure time in units of the free decay time of the pump mode $\tau\equiv\gamma_2 t$. Expressed in terms of the new variables the Langevin equations simplify to

$$\gamma\frac{d}{d\tau}\tilde{a}_1=-\tilde{a}_1+\tilde{a}_2\tilde{a}_1^{+}+\sqrt{\frac{2}{m}}F_1(\tau/\gamma), \tag{4}$$

$$\frac{d}{d\tau}\tilde{a}_2=-\tilde{a}_2+\tilde{\epsilon}-\tilde{a}_1^2+\sqrt{\frac{2}{n_{th}}}F_2(\tau). \tag{5}$$

In the standard linearization approach [10] the pump mode is replaced by its semiclassical steady-state value $\tilde{a}_2\rightarrow\tilde{\alpha}_2^{sc}$ ($=\tilde{\epsilon}-1$ below threshold; $=1$ above threshold). In this approach one quadrature component of the subharmonic mode displays perfect squeezing at threshold ($\tilde{\epsilon}=1$). At the same time the fluctuations in the orthogonal component diverge. This critical behavior of the subharmonic mode, which indicates the breakdown of the approach, has two origins: First, the depletion of the pump field amplitude due to the energy transfer into fluctuations of the subharmonic field is neglected, and second, quantum fluctuations of the pump mode are disregarded.

In the present paper we discuss the case $\gamma\ll 1$. Here the second of the implicit assumptions of the linearized theory is still valid, and only the first one needs to be relaxed if the threshold is approached. Note that $\gamma\ll 1$ or $\gamma_1\gg\gamma_2$ is actually the favorable case in order to obtain an intense output of broadband squeezed vacuum. Since $\gamma=m/2n_{th}$, $\gamma\rightarrow 0$ implies $n_{th}\rightarrow\infty$ for a nonzero value of m. The intracavity intensity of the subharmonic mode above threshold scales like $m(\tilde{\epsilon}-1)$ and hence a sufficiently large value of m is desirable. As can be seen from Eq. (5), the fluctuation force F_2 of the (noncritical) pump mode becomes irrelevant in the limit $n_{th}\rightarrow\infty$. The effective fluctuation force $\langle\tilde{a}_1^2\rangle-\tilde{a}_1^2$, which results from the quantum noise of the subharmonic mode, decays on the time scale of $\tilde{a}_1$ and is effectively proportional to $\gamma\delta(\tau-\tau')$. This fluctuation force therefore vanishes in the considered limit as well. For these reasons we can treat the pump mode as a classical quantity obeying the equation of motion

$$\frac{d}{d\tau}\tilde{\alpha}_2=-\tilde{\alpha}_2+\tilde{\epsilon}-\langle\tilde{a}_1^2\rangle, \tag{6}$$

with the steady-state solution

$$\tilde{\alpha}_2=\tilde{\epsilon}-\langle\tilde{a}_1^2\rangle. \tag{7}$$

We note the presence of the second term, which is absent in the standard linearized theory. This term describes the depletion of the pump mode, which becomes important when the threshold is approached.

The mean-field approximation outlined above turns the quantum Langevin equation of the subharmonic mode into a linear equation, and hence the problem can be solved in a simple and straightforward way. The unphysical results of the standard linearization at threshold are avoided by taking into account the depletion of the pump-mode amplitude. This allows one to calculate the true stationary dynamics at and to a certain extent, even above threshold.

The solution of the linear Langevin equation obtained from Eq. (4) by $\tilde{a}_2\rightarrow\tilde{\alpha}_2$ yields

$$\langle\tilde{a}_1^2\rangle=\frac{\tilde{\alpha}_2}{2m(1-|\tilde{\alpha}_2|^2)}. \tag{8}$$

The pump-mode amplitude is obtained by a "self-consistent" solution of Eqs. (7) and (8), i.e.,

$$\tilde{\alpha}_2=\tilde{\epsilon}-\frac{\tilde{\alpha}_2}{2m(1-|\tilde{\alpha}_2|^2)}. \tag{9}$$

It should be noted at this point that the pump field depletion discussed here is implicitly included in the recent approaches of Plimak and Walls [16] and Mertens *et al.* [17], which are based on Greens-function techniques. In the language of Feynman diagrams the bubble diagrams in the Dyson equations for the subharmonic Greens functions are responsible for the reduction of the pump field amplitude. We will show in a forthcoming presentation [18] that the so-called one-loop approximation in a Greens-function approach is a consistent extension of the present mean-field theory for $\gamma\ll 1$. In the opposite (adiabatic) case of a slowly decaying subharmonic mode, as discussed by Plimak and Walls, the one-loop approach will be shown to break down before the classical threshold is reached.

As shown in Fig. 3 the dressed amplitude $\tilde{\alpha}_2$ stays well below the semiclassical value and approaches unity only for an infinite pump rate. Thus, unlike in the standard linearized theory, the subharmonic photon number

$$\langle n_1\rangle\equiv\langle a_1^{\dagger}a_1\rangle=\frac{|\tilde{\alpha}_2|^2}{2(1-|\tilde{\alpha}_2|^2)} \tag{10}$$

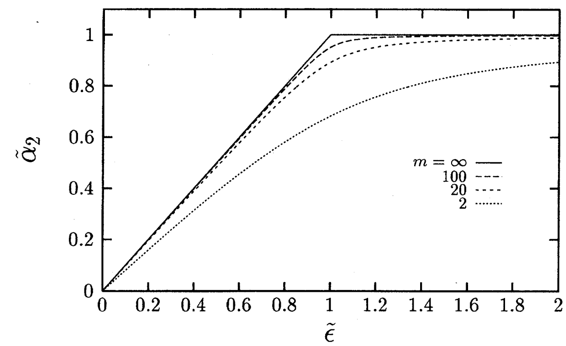

FIG. 3. Pump-mode amplitude for various slope parameters $m=2\gamma n_2^{\rm th}$.

diverges only for $\tilde{\epsilon}\to\infty$. More precisely, n_1/m approaches its semiclassical value $\tilde{\epsilon}-1$ in this case. This is illustrated in Fig. 4.

In order to discuss the squeezing in the subharmonic mode, we introduce quadrature components

$$x_+(t)=\frac{1}{2}[a_1(t)+a_1^\dagger(t)], \tag{11}$$

$$x_-(t)=\frac{1}{2i}[a_1(t)-a_1^\dagger(t)] \tag{12}$$

and their Fourier transforms

$$\check{x}_\pm(\omega)=\int_{-\infty}^{\infty}dt\; e^{i\omega t}x_\pm(t). \tag{13}$$

We obtain for the squeezing spectra [10] of the subharmonic $S_\pm=2\gamma_1\langle\check{x}_\pm(\omega)x_\pm(t=0)\rangle$ the result

$$S_\pm(\omega)=\pm\frac{\tilde{\alpha}_2}{\omega^2/\gamma_1^2+(1\mp\tilde{\alpha}_2)^2}, \tag{14}$$

which deviates from the standard linearization expression only in the fact that $\tilde{\alpha}_2$ is the solution of Eq. (9) instead of representing the semiclassical amplitude. In particular, since $\tilde{\alpha}_2$ approaches unity only for an infinite pump rate, the squeezing in x_- at threshold does not attain its maximum

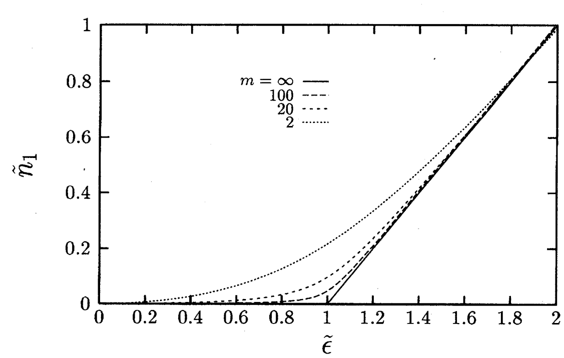

FIG. 4. Subharmonic photon number approaching semiclassical behavior for increasing slope parameter $m=2\gamma n_2^{\rm th}$.

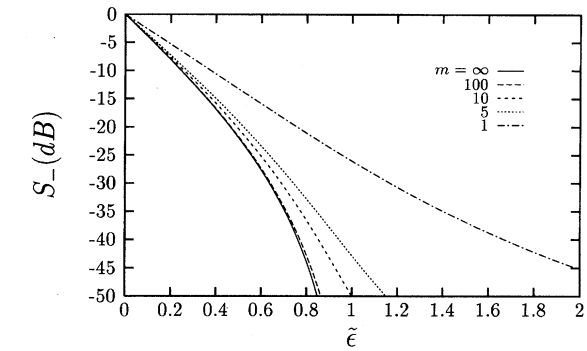

FIG. 5. Steady-state value of squeezing of the x_- quadrature in logarithmic scale.

value of $-\frac{1}{4}$ and the fluctuations in x_+ do not diverge. For increasing slope parameter m, which in our case takes the role of a system-size parameter, the squeezing spectra approach their standard linearized values, as shown in Figs. 5 and 6.

Despite the fast decaying subharmonic mode and the associated depletion of the pump field, a substantial degree of squeezing can be achieved. Furthermore the broadband character of the squeezing spectra in the limit of large γ_1 is of particular interest for practical applications.

We have argued that the mean-field theory is exact in the "diabatic" limit, where γ approaches zero. We now will discuss the limits of this approximation when applied to the case $\gamma>0$. The mean-field approximation $\tilde{a}_2\to\langle\tilde{a}_2\rangle$ holds only for negligible fluctuations of $\tilde{a}_2$. Hence, in order to estimate the range of its validity we now take into account small fluctuations of the pump mode.

The effective fluctuating force F of this mode consists of two parts,

$$F=\sqrt{2/n_{\rm th}}F_2+(\langle\tilde{a}_1^2\rangle-\tilde{a}_1^2). \tag{15}$$

The first part, $\sqrt{2/n_{\rm th}}F_2$, is due to the coupling to the vacuum, and the second one, $(\langle\tilde{a}_1^2\rangle-\tilde{a}_1^2)$, results from the coupling to the subharmonic mode.

Rigorously the second term in Eq. (15) represents a nonwhite Langevin force for nonzero values of γ. According to Eq. (4) the effective decay rates of the subharmonic mode in the mean-field approximation are given by $(1\pm\tilde{\alpha}_2)/\gamma$. How-

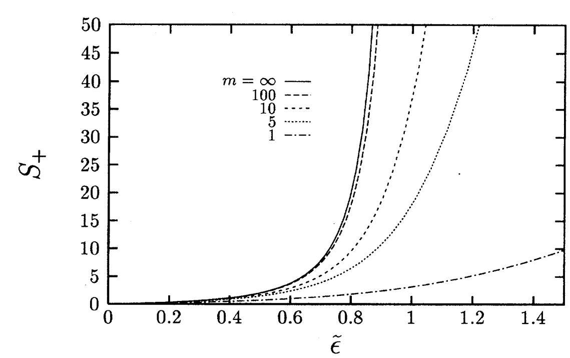

FIG. 6. Steady-state value of squeezing of the x_+ quadrature.

ever, the time scale of the evolution of the pump mode is given by unity. Hence we may treat $(\langle\tilde{a}_1^2\rangle-\tilde{a}_1^2)$ in Markov approximation if

$$\gamma \ll 1-\tilde{\alpha}_2. \tag{16}$$

Since we are interested in small γ we now restrict ourselves to cases where (16) is fulfilled.

For the present discussion only the fluctuations of the in-phase quadrature component of the pump mode $\tilde{y}_+ \equiv \frac{1}{2}(\tilde{a}_2+\tilde{a}_2^+)$ are relevant. Applying the Markov approximation mentioned above we can calculate $\langle\Delta\tilde{y}_+\rangle^2 \equiv \langle(\tilde{y}_+-\langle\tilde{y}_+\rangle)^2\rangle$. The contribution from the pump-mode vacuum fluctuations [the first term in Eq.(15)] is

$$\langle\Delta y_+\rangle^2 = 1/4 \ \text{or}\ \langle\Delta\tilde{y}_+\rangle^2 = \gamma/(2m). \tag{17}$$

For $\tilde{\alpha}_2$ being in the vicinity of unity, the fluctuations due to the coupling to the subharmonic [the second term in Eq. (15)] yield

$$\langle\Delta\tilde{y}_+\rangle^2 = \frac{\gamma}{16m^2}(1-\tilde{\alpha}_2)^{-3}. \tag{18}$$

The quantity most sensitive to fluctuations in $\tilde{a}_2$ is the photon number n_1 of the subharmonic mode. Its dependence on $\langle\tilde{y}_+\rangle$ can be estimated from mean-field theory Eq. (10) to obey

$$\frac{\langle\Delta n_1\rangle}{\langle n_1\rangle} = \frac{\langle\Delta\tilde{y}_+\rangle}{1-\langle\tilde{y}_+\rangle}, \tag{19}$$

when $\tilde{\alpha}_2 \equiv \langle\tilde{y}_+\rangle$ is close to unity. Thus we expect the properties of the subharmonic mode to be described accurately by the mean-field approach if

$$\langle\Delta\tilde{y}_+\rangle \ll 1-\langle\tilde{y}_+\rangle \equiv 1-\tilde{\alpha}_2. \tag{20}$$

With Eqs. (17) and (18) this gives the two conditions

$$\gamma/m \ll 2(1-\tilde{\alpha}_2)^2 \tag{21}$$

and

$$\gamma/m^2 \ll 16(1-\tilde{\alpha}_2)^5. \tag{22}$$

Conditions (21) and (22) set an upper limit for the pump rate up to which the mean-field approximation holds. At threshold $(1-\tilde{\alpha}_2)$ is roughly $1/(2\sqrt{m})$ and hence γ needs to be much smaller than unity for the approach to be valid. Above threshold $(1-\tilde{\alpha}_2)$ scales like $1/m$. Thus the approach works better for smaller values of m, in which case the bifurcation of, for example, the Q function into two separate peaks is shifted to higher pump-rate values.

In the present paper we have shown that the quantum properties of the degenerate optical parametric oscillator at and above threshold can be calculated analytically in the diabatic limit $\gamma \to 0$. In this case a mean-field approximation turns the problem into a linear one, where only one parameter—the mean amplitude of the pump mode—needs to be determined from a nonlinear equation. As opposed to predictions of standard linearized theory at threshold, the fluctuations in the nonsqueezed component do not diverge. Nevertheless a substantial amount of squeezing can be achieved over a broad range of Fourier frequencies. In the thermodynamic limit, where the system-size parameter m tends to infinity, the results of the standard linearized theory are recovered.

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