

Relation between the N -atom laser and the one-atom laser

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The interaction of a quantized field mode with N identical atoms coupled to individual reservoirs and classical driving fields is studied. An effective-time evolution operator for the radiation mode is derived. The exponent of this operator can be written as the number of atoms times an infinite sum containing field operators and correlation functions of freely evolving (no atom-field coupling) *single-atom* operators. In the adiabatic limit of fast decaying atomic correlations one can derive a master equation for the field density matrix. It is shown that the Liouvillian describing the interaction of the field mode with the N atoms is N times the single-atom Liouvillian.

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In quantum optics the interaction of N identical atoms with the quantized radiation field is usually analyzed for the limiting cases of either very many or very few atoms. The first case is that of the laser theories of Haken, Risken, and Weidlich [1,2] and Lax and Louisell [3]. In this limit, the dynamical behavior of the radiation field can be described by quasiprobability distributions which obey a Fokker-Planck equation. On the other hand, an integration of the equation of motion of the total density matrix containing field and atomic degrees of freedom is possible only for a single or very few atoms [4], which is also the basis of the Scully-Lamb laser model [5]. Only very few attempts have been made to solve the Laser equations for the case of intermediate atom numbers [6]. In the present report a relation between the master equation for the field density operator for the interaction with N identical atoms and a single atom is derived. This relation provides a simple tool to investigate effects of the finite atom number by studying the single-atom problem.

We consider the interaction of a single quantized radiation mode, described by the creation and annihilation operators a^\dagger and a , with the transitions $|a\rangle_j - |b\rangle_j$ ($j=1, \dots, N$), of N independent atoms. The corresponding interaction potential reads as

$$V_0 = -\hbar g \sum_{j=1}^N A \Sigma_j, \quad (1)$$

where $A = a + a^\dagger$ and $\Sigma_j = \sigma_j + \sigma_j^\dagger$, σ_j being the flip operator of the j th atom, $\sigma_j = |b\rangle_j \langle a|$. Here we do not specify the interaction of the atoms with other systems, but we assume that the atoms interact only with *individual* reservoirs and allow for an interaction with undepleted classical driving fields. The interaction of the radiation mode with other systems such as reservoirs or driving fields is also not specified. We assume that the total density operator W_0 factorizes into subsystem operators at some initial time t_0 and that there are no atom-atom correlations at this time. As a consequence, operators of different atoms are statistically independent in the absence of coupling to the field mode, and correlation functions of these operators factorize.

We now introduce an interaction picture in which the state vector or density matrix evolve with V_0 and the operators evolve according to their free Hamiltonians plus the interaction with reservoirs and classical driving fields. At this point, the reservoir interaction is still described by the full Hamiltonian dynamics and the average over the reservoir degrees of freedom is done at a later point. To distinguish this picture from the Heisenberg picture we denote the operators in the latter by the subscript H .

The properties of the radiation field are, in general, described by normal- and time-ordered correlation functions $\langle \Phi[a_H, a_H^\dagger] \rangle$ of field operators in the Heisenberg picture, where the average is taken with W_0 . Following standard techniques of nonequilibrium quantum statistics [7,8], we express $\langle \Phi \rangle$ in terms of operators in the interaction picture. This can be done in a formally simple way with the help of a time-ordered evolution operator S_C defined on a time contour C as shown in Fig. 1 [8]:

$$\begin{aligned} \langle \Phi[a_H, a_H^\dagger] \rangle &= \langle T_C S_C \Phi[a, a^\dagger] \rangle \\ &= \text{Tr} \{ W_0 T_C (S_C \Phi[a, a^\dagger]) \}, \end{aligned} \quad (2)$$

where

$$S_C = T_C \exp \left\{ -\frac{i}{\hbar} \int_C d\bar{\tau} V_0(\bar{\tau}) \right\}. \quad (3)$$

Here the subscript C denotes integration along the time contour and the bar distinguishes times on the contour from physical times. T_C is the time-ordering operator on C , which corresponds to causal time ordering on the upper branch and to anticausal time ordering on the lower branch of C . In addition, it orders all operators on the upper branch to the right of all operators on the

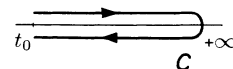


FIG. 1. Keldysh time-contour C . To each physical time correspond two time arguments on the upper and lower branches of C . Time ordering is indicated by arrows.

lower branch. We emphasize that the operators in Eqs. (2) and (3) evolve according to the free Hamiltonians and the interaction with the reservoirs and driving fields, but do not contain atom-field interaction V_0 .

Since we are interested in field properties only, we may

$$S_C^{\text{eff}} = \text{Tr}_{\text{atom}} \{ S_C \rho_0^{\text{atom}} \} = \left\langle T_C \exp \left\{ ig \int_C d\bar{\tau} \sum_j A(\bar{\tau}) \Sigma_j(\bar{\tau}) \right\} \right\rangle_{\text{atom}} \quad (4)$$

which is a function of a and a^\dagger .

We introduce the so-called ordered cumulants [9,10] by rewriting S_C^{eff} in the form

$$\begin{aligned} S_C^{\text{eff}} &= \left\langle T_C \exp \left\{ ig \int_C d\bar{\tau} \sum_j A(\bar{\tau}) \Sigma_j(\bar{\tau}) \right\} \right\rangle_{\text{atom}} \\ &= T_C \exp \left\{ \sum_{m=1}^{\infty} \frac{(ig)^m}{m!} \int_C d\bar{\tau}_1 \cdots \int_C d\bar{\tau}_m \sum_{j_\mu} \langle\langle \Sigma_{j_1}(\bar{\tau}_1) \cdots \Sigma_{j_m}(\bar{\tau}_m) \rangle\rangle A(\bar{\tau}_1) \cdots A(\bar{\tau}_m) \right\}. \end{aligned} \quad (5)$$

The ordered cumulants $\langle\langle m \rangle\rangle$ are functions of T_C -ordered correlation functions with m operators Σ_j . The cumulant expansion of the exponential is thereby the same as if Σ_j were treated as c numbers, which is allowed under the time-ordering symbol [9,10]. Hence, the relation between the ordered cumulants and correlation functions is the same as in the case of classical variables, with the exception that the time-ordered prescription has to be applied. The explicit relation between the cumulants and the correlation functions of atomic operators is not of interest here, but will be discussed in more detail for specific laser systems at a different place [11]. In principle, it can be obtained from an expansion of the original time-evolution operator S_C and of S_C^{eff} in powers of the coupling strength. The advantage of the representation of S_C^{eff} as given in Eq. (5) for the present discussion lies in

take the average of S_C in Eq. (2) with respect to the atomic degrees of freedom and the degrees of freedom of the atomic reservoirs and define an effective time evolution operator

the important property of the cumulants to vanish, if any two of the operators involved are statistically independent [9,10]. Since we have assumed that the atoms interact with individual reservoirs and have disregarded initial atom-atom correlations, operators in the interaction picture belonging to different atoms are statistically independent. Consequently, in the sum over the atomic indices in Eq. (5), only those terms which have the same atomic index survive. We therefore may replace \sum_{j_μ} in Eq. (5) by a factor N and obtain eventually an effective time-evolution operator which contains only *single-atom* correlation functions in the interaction picture. These correlation functions can be calculated from, in most cases, linear quantum Langevin equations with the help of the regression theorem [12],

$$S_C^{\text{eff}} = T_C \exp \left\{ N \sum_{m=1}^{\infty} \frac{(ig)^m}{m!} \int_C d\bar{\tau}_1 \cdots \int_C d\bar{\tau}_m S_m(\bar{\tau}_1, \dots, \bar{\tau}_m) A(\bar{\tau}_1) \cdots A(\bar{\tau}_m) \right\}. \quad (6)$$

Here $S_m(\bar{\tau}_1, \dots, \bar{\tau}_m)$ is used as an abbreviation for the single-atom cumulants, which are symmetric in all time arguments. The separation of a factor N in the exponent of S_C^{eff} can easily be understood in the language of Feynman graphs. The exponent in Eq. (6) yields a perturbation series for the evolution of Green's functions of the radiation field [10]. The cumulants S_m thereby represent the connected graphs of the atomic system with m end points including the atomic reservoirs. Since we have assumed individual atomic reservoirs, there are only connected graphs belonging to the same atom, and the overall prefactor N in the exponent of S_C^{eff} is simply the multiplicity of each of them. The effective time-evolution operator describes the dynamics of the quantized field mode under general conditions. The only assumption we have used so far is that the atoms are statistically independent in the absence of the interaction with the field mode. In particular, Eq. (6) holds for arbitrary initial

states of the radiation field and can therefore be used to derive master equations for the field density operator.

In order to derive a master equation for the field density operator ρ we note that the dynamical evolution of the matrix element $\rho_{nm} = \langle n | \rho | m \rangle$ in the Schrödinger picture, where $|n\rangle$ and $|m\rangle$ are Fock states, is determined by the time evolution of the operator $P_{mn}^H = |m\rangle \langle n|$ in the Heisenberg picture,

$$\begin{aligned} \frac{d}{dt} \rho_{nm}(t) &= \frac{d}{dt} \langle P_{mn}^H(t) \rangle \\ &= \frac{d}{dt} \langle T_C P_{mn}(t) S_C^{\text{eff}} \rangle \\ &= \left\langle T_C \left[\frac{d}{dt} P_{mn}(t) \right] S_C^{\text{eff}} \right\rangle \\ &\quad + \left\langle T_C P_{mn}(t) \frac{d}{dt} S_C^{\text{eff}} \right\rangle. \end{aligned} \quad (7)$$

The first term in the third line describes the free evolution of the field and the interaction with field reservoirs and classical driving fields. The second term accounts for the interaction with the atomic system and will be analyzed in the following. To do so we first note that it is sufficient to extend the time contour C in Fig. 1 to t instead of $+\infty$, since we consider only a single time expect-

tation value. With the help of

$$\int_C^t d\bar{\tau} f(\bar{\tau}) = \left[\int_{t_0}^t d\tau^{(+)} - \int_{t_0}^t d\tau^{(-)} \right] f(\bar{\tau}), \quad (8)$$

where the superscripts (\pm) denote the branch on the time contour, we can calculate the derivative of S_C^{eff} and find for the second term in Eq. (7),

$$\begin{aligned} & igN \left\langle T_C P_{mn} A \sum_{m=1}^{\infty} \frac{(ig)^m}{m!} \int_C^t d\bar{\tau}_1 \cdots \int_C^t d\bar{\tau}_m S_{m+1}(t^{(+)}, \bar{\tau}_1, \dots, \bar{\tau}_m) A(\bar{\tau}_1) \cdots A(\bar{\tau}_m) S_C^{\text{eff}} \right\rangle \\ & + igN \left\langle T_C A P_{mn} \sum_{m=1}^{\infty} \frac{(ig)^m}{m!} \int_C^t d\bar{\tau}_1 \cdots \int_C^t d\bar{\tau}_m S_{m+1}(t^{(-)}, \bar{\tau}_1, \dots, \bar{\tau}_m) A(\bar{\tau}_1) \cdots A(\bar{\tau}_m) S_C^{\text{eff}} \right\rangle. \end{aligned} \quad (9)$$

This expression can be written in terms of a single-time expectation value, if the cumulants $S_m(t^{(\pm)}, \bar{\tau}_1, \dots, \bar{\tau}_m)$ are essentially zero for $\tau_\mu \neq t$. This is the case if the atomic correlations decay on a time scale much faster than the time scale of the field evolution, that is in the adiabatic limit. In this case we can write

$$\int_C^t d\bar{\tau}_1 \cdots \int_C^t d\bar{\tau}_m S_{m+1}(t^{(\pm)}, \bar{\tau}_1, \dots, \bar{\tau}_m) A(\bar{\tau}_1) \cdots A(\bar{\tau}_m) \approx \sum_{\alpha_\mu = \pm} f_{m+1}^{[\alpha_\mu]}(t) A(t^{(\alpha_1)}) \cdots A(t^{(\alpha_m)}). \quad (10)$$

One recognizes from Eq. (10) that in the adiabatic limit all field operators A in expression (9) are operators at the physical time t on the upper or lower branch of C . The time-ordering operator T_C moves all field operators with a contour index “-” (corresponding to the lower branch of the contour) to the left of $P_{mn} A$ or $A P_{mn}$ and all operators with a superscript “+” to the right. Note that the field operators A commute with each other for equal times. Making use of the cyclic property of the trace and expressing the field operators A in terms of annihilation and creation operators, we can thus write expression (9) in terms of matrix elements of the field density operator at time t . A typical term would read, for example, as

$$\begin{aligned} \dot{\rho}_{nm} &= \cdots + (ig)^4 N f_4^{++++} \langle a_H(t) P_{mn}^H(t) a_H^+(t) a_H(t) a_H^+(t) \rangle + \cdots \\ &= \cdots + (ig)^4 N f_4^{++++} \langle n | a^+ a a^+ \rho(t) a | m \rangle + \cdots \\ &= \cdots + (ig)^4 N f_4^{++++} n \sqrt{n-1} \sqrt{m-1} \rho_{n-1, m-1} + \cdots \end{aligned} \quad (11)$$

It is clear, of course, that a practical derivation of a master equation with this method is only possible if the infinite sum in expression (9) and hence in the exponent of S_C^{eff} may be truncated, which requires that a perturbation expansion in the atom-field coupling is permissible. Nevertheless, the factorization of the exponent into the number of atoms N times a single-atom expression leads to a simple relation between the N -atom and one-atom master equations for the field density operator in the adiabatic limit: If we denote the Liouvillian describing the interaction of the radiation field with a single atom by \mathcal{L}_{AF} and the Liouvillian describing both the free evolution, the dissipation into reservoirs, and the interaction with some classical driving fields by \mathcal{L}_0 , the master equation for the interaction with N atoms reads as

$$\frac{d}{dt} \rho(t) = \mathcal{L}_0 \rho(t) + N \mathcal{L}_{AF} \rho(t). \quad (12)$$

Equation (12) is the main result of this paper. It shows

that the dynamical evolution of the radiation field interacting with N atoms can be obtained from the solution of the single-atom problem by rescaling the time unit according to $T^{(N)} = T^{(1)}/N$ and by replacing $\mathcal{L}_0^{(N)} = \mathcal{L}_0^{(1)}N$. This result enables us to deduce the properties of any many-atom laser system in the adiabatic regime from a solution of the corresponding single-atom problem. It also verifies the implicit assumption of the Scully-Lamb laser theory, that the change of the field due to the action of N atoms is N times the single-atom action. It should be noted at this point that although the separation of a factor N in the exponent of the effective time-evolution operator, Eq. (6), is also true in the nonadiabatic case, Eq. (12) is not. For typical nonadiabatic problems such as superfluorescence the N -atom case cannot be reduced to the single-atom problem in such a simple way as in Eq. (12).

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