Nonadiabatic linewidth of a Λ-type noninversion laser

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The linewidth of a Λ-type noninversion laser is calculated on the basis of a c-number Langevin approach in lowest order of the laser field. It is shown that the system behaves in general—that is, also in the good-cavity limit—nonadiabatically. In the nonadiabatic regime a substantial deviation of the laser linewidth from the Schawlow-Townes formula is found and a simple interpretation is given.

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I. INTRODUCTION

One of the important results of the quantum theory of the laser is the explanation of the laser line shape as a result of the random fluctuations imposed on the field by spontaneous emissions of the atoms.

If the phase change due to a single spontaneous emission event takes place on a much shorter time scale than the overall evolution of the field, the laser line is a Lorentzian with a width [full width at half maximum (FWHM)] given by the well-known Schawlow-Townes formula [1]

$$\Delta\nu=\frac{A}{2\bar{n}}\ . \tag{1}$$

Here A is the linear gain coefficient of the laser, and $\bar{n}$ is the mean number of photons in the cavity. In many cases the active laser medium can be described adequately by two-level systems [2]. In a two-level atom the time scale of spontaneous emission is determined by the decay rate γ of the atomic polarization. If γ is large compared to the damping rate κ_0 of the cavity mode, the field evolution follows the atomic polarization adiabatically, and the linewidth is given by Eq. (1).

On the other hand, if κ_0 becomes comparable to γ, a non-Lorentzian line shape is observed with full width at half maximum [3]

$$\Delta\nu=\frac{A}{2\bar{n}}\left[\frac{\gamma}{\kappa_0+\gamma}\right]^2 . \tag{2}$$

The deviation of the linewidth from the Schawlow-Townes formula by a factor $[\gamma/(\kappa_0+\gamma)]^2$ is due to the nonadiabatic evolution of the atoms. A single atom does not have enough time to decay in the lifetime of the cavity photon, and hence its spontaneous decay can only partially contribute to the phase fluctuations of the laser field. It should be noted at this point that in general larger cavity losses do not result in a narrower laser line, as Eqs. (1) and (2) might suggest, since the mean number of photons decreases with increasing cavity losses as well. There are, however, parameter regions where an increase of the cavity losses leads to a narrowing of the linewidth.

In common laser systems one can apply the adiabatic approximation if the cavity damping rate is small compared to the decay rates of atomic populations and polarizations [4], and nonadiabatic effects occur when low-Q cavities are used [5]. In the present paper we show that in systems where atomic coherence is established, profound nonadiabatic effects can occur even for high-Q cavities. In particular, we demonstrate a substantial deviation of the linewidth from the Schawlow-Townes result in a Λ-type noninversion laser [6].

The atomic system studied is depicted in Fig. 1. The upper level a of a dipole-allowed optical transition is driven by a strong field to a metastable level c. Due to quantum interference induced by the driving field, the absorption of a weak probe field resonant with the a-b transition is strongly suppressed. A weak incoherent pump process populating level c or a can then lead to laser gain even if the total population in levels c and a is smaller than the population in b; that is, to lasing without inversion [6].

In the present paper we analyze the phase fluctuations of such a laser system by applying a c-number Langevin approach without adiabatic elimination of the atomic variables. We thereby take effects of the finite linewidth of the driving field into account as well. For the sake of simplicity, we restrict ourselves to a perturbative treatment of the laser field, but take into account the driving field to all orders.

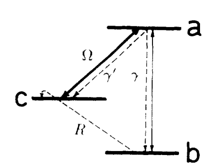

FIG. 1. Active medium of a Λ-type noninversion laser. A strong field of Rabi frequency Ω drives the transition from the upper lasing level a to a metastable level c and generates transparency on the noninverted transition a-b. An (indirect) pump process from b to c with rate R leads to laser gain.

II. LANGEVIN EQUATIONS AND SEMICLASSICAL DESCRIPTION OF LASER OPERATION

In this section we derive equations of motion for the atomic and field variables and analyze the semiclassical steady state of the laser in lowest nonvanishing order of the field strength. The active medium consists of three-level atoms with the level structure shown in Fig. 1. A strong driving field of Rabi frequency Ω couples the upper level a to a metastable level c and induces transparency on the a-b transition [6]. A small indirect[1] pumping mechanism populates level c which eventually leads to amplification of the laser mode. The energy of level c is perturbed by collisions, leading to a dephasing of the c-b and a-c polarizations with rate $\gamma_c \ll \gamma, \gamma'$, where γ and γ' are the radiative decay rates from level a.

The interaction of the laser mode $\hat{a}$ and the driving field with the atoms is described by the interaction Hamiltonian

$$H^{\rm int} = -\hbar g \sum_{j=1}^{N} (\hat{a}^\dagger \hat{\sigma}_0^j + \hat{a}\hat{\sigma}_0^{j\dagger}) - \hbar \sum_{j=1}^{N} (\Omega^* e^{i\nu' t}\hat{\sigma}_2^j + \Omega e^{-i\nu' t}\hat{\sigma}_2^{j\dagger}) , \tag{3}$$

where $g = \wp/\hbar\sqrt{\hbar\nu/2\varepsilon_0 V}$ is the vacuum Rabi frequency associated with the atom-field interaction, in which $\wp$ is the dipole moment, ν the laser frequency, and V the mode volume. N is the total number of atoms and ν' is the frequency of the driving field. The atomic variables in Eq. (3) are defined as

$$\begin{aligned} &\hat{\sigma}_0^j = |b\rangle\langle a|_j , \quad \hat{\sigma}_a^j = |a\rangle\langle a|_j , \\ &\hat{\sigma}_1^j = |b\rangle\langle c|_j , \quad \hat{\sigma}_b^j = |b\rangle\langle b|_j , \\ &\hat{\sigma}_2^j = |c\rangle\langle a|_j , \quad \hat{\sigma}_c^j = |c\rangle\langle c|_j . \end{aligned} \tag{4}$$

If we consider a large number of independent atoms we can describe the system using collective variables

$$\hat{\sigma}_\mu = \frac{1}{N}\sum_j \hat{\sigma}_\mu^j . \tag{5}$$

Since we want to focus here on small laser field intensities, the average polarization $\langle\hat{\sigma}_2\rangle$, which couples to the driving field, is much smaller than the average polarization $\langle\hat{\sigma}_0\rangle$ of the laser transition. We therefore may assume an undepleted driving field.

From the Liouville equation for the density operator $\hat{\rho}$, one can derive a generalized Fokker-Planck equation for a quasidistribution $P(\alpha,\{\sigma_\mu\})$, defined by the relation

$$\begin{aligned} \hat{\rho} = \int d\beta\, P(\alpha,\{\sigma_\mu\}) &\delta(\hat{a}^\dagger-\alpha^*)\delta(\hat{\sigma}_2^\dagger-\sigma_2^*) \\ &\times\delta(\hat{\sigma}_1^\dagger-\sigma_1^*)\delta(\hat{\sigma}_0^\dagger-\sigma_0^*) \\ &\times\delta(\hat{\sigma}_a-\sigma_a)\delta(\hat{\sigma}_b-\sigma_b)\delta(\hat{\sigma}_c-\sigma_c)\delta(\hat{\sigma}_0-\sigma_0) \\ &\times\delta(\hat{\sigma}_1-\sigma_1)\delta(\hat{\sigma}_2-\sigma_2)\delta(\hat{a}-\alpha) , \end{aligned} \tag{6}$$

where $d\beta = d^2\alpha d^2\sigma_0 d^2\sigma_1 \ldots$. Neglecting higher-order derivatives—they decrease with powers of $1/N$—one can transform the Fokker-Planck equation into a set of c-number Langevin equations [4,7], which in a rotating frame read

$$\dot{\sigma}_a = -\Gamma\sigma_a - ig(\alpha^*\sigma_0 - \text{c.c.}) - i(\Omega^*\sigma_2 - \text{c.c.}) + F_a , \tag{7a}$$

$$\dot{\sigma}_b = -R\sigma_b + \gamma\sigma_a + ig(\alpha^*\sigma_0 - \text{c.c.}) + F_b , \tag{7b}$$

$$\dot{\sigma}_0 = -[i\Delta + \tfrac{1}{2}(\Gamma+R)]\sigma_0 + ig(\sigma_b-\sigma_a)\alpha + i\Omega\sigma_1 + F_{\sigma_0} , \tag{7c}$$

$$\dot{\sigma}_1 = -[i(\Delta-\Delta') + \tfrac{1}{2}(R+\gamma_c)]\sigma_1 - ig\alpha\sigma_2^* + i\Omega^*\sigma_0 + F_{\sigma_1} , \tag{7d}$$

$$\dot{\sigma}_2 = -[i\Delta' + \tfrac{1}{2}(\Gamma+\gamma_c)]\sigma_2 + ig\alpha\sigma_1^* + i\Omega(\sigma_c-\sigma_a) + F_{\sigma_2} , \tag{7e}$$

where $\Gamma = \gamma+\gamma'$ and $\Delta = \omega_{ab}-\nu$, and $\Delta' = \omega_{ac}-\nu'$. The fluctuation operators F_x in Eqs. (7) have zero mean value and are δ correlated. Assuming individual atomic reservoirs, their diffusion coefficients can be obtained with the help of the generalized Einstein relations [2,4,7] and Eq. (5). In lowest order of the laser field the only relevant noise contributions, however, come from F_{σ_0} and F_{σ_1}:

$$N\langle F_{\sigma_0}^* F_{\sigma_0}\rangle = R\langle\sigma_a^{(0)}\rangle , \tag{8a}$$

$$N\langle F_{\sigma_1}^* F_{\sigma_1}\rangle = R(\langle\sigma_b^{(0)}\rangle + \langle\sigma_c^{(0)}\rangle) + \gamma'\langle\sigma_a^{(0)}\rangle + \gamma_c\langle\sigma_c^{(0)}\rangle , \tag{8b}$$

$$N\langle F_{\sigma_0}^* F_{\sigma_1}\rangle = R\langle\sigma_2^{(0)}\rangle^* . \tag{8c}$$

It has been shown that phase fluctuations of the driving field can strongly influence the optical properties of Λ resonances [8]. Taking them into account within a phase-diffusion model leads to a modification of the equations of motion [Eqs. (7)] and the diffusion coefficients [Eqs. (8)]: The collisional dephasing rate γ_c has to be replaced by the sum of γ_c, and the linewidth of the driving laser γ_L, i.e., $\gamma_c \to \Gamma_c = \gamma_c + \gamma_L$ [8].

In order to describe the semiclassical steady-state properties of the laser medium and, in particular, the linear gain spectrum, we neglect all fluctuation operators in Eqs. (7) and set the time derivatives equal to zero. For simplicity, in the following we assume a resonant driving field, i.e., $\Delta'=0$, and set the phase of the Rabi frequency of the driving field equal to zero, i.e., $\Omega^*=\Omega$. Solving Eqs. (7a)–(7c) in zeroth order of α, we obtain the semiclassical steady-state values

[1]Here "indirect" means that the lower level b is coupled to an auxiliary level, for example, by an optical field, from which the population rapidly decays into level c.

$$\bar{\sigma}_a^{(0)}=R\frac{4\Omega^2}{D}\ ,\quad \bar{\sigma}_b^{(0)}=\gamma\frac{4\Omega^2}{D}\ ,$$
$$\bar{\sigma}_c^{(0)}=R\frac{4\Omega^2+\Gamma(\Gamma+\Gamma_c)}{D}\ ,\quad \bar{\sigma}_2^{(0)}=R\frac{2i\Omega\Gamma}{D}\ , \tag{9}$$

where $D=R\Gamma(\Gamma+\Gamma_c)+(2R+\gamma)4\Omega^2$. Similarly, in first order we find

$$\bar{\sigma}_1^{(1)}=-\frac{g\alpha\Omega}{DD'}\{4\Omega^2(\gamma-R)+2R\Gamma[i\Delta+\tfrac{1}{2}(R+\Gamma)]\}\ , \tag{10a}$$

$$\bar{\sigma}_0^{(1)}=\frac{ig\alpha(\bar{\sigma}_b^{(0)}-\bar{\sigma}_a^{(0)})}{[i\Delta+\frac{1}{2}(R+\Gamma)]}+\frac{i\Omega\bar{\sigma}_1^{(1)}}{[i\Delta+\frac{1}{2}(R+\Gamma)]}\ . \tag{10b}$$

where

$$D'=\Omega^2-\Delta^2+(i/2)\Delta(\Gamma+\Gamma_c+2R)+\tfrac{1}{4}(R+\Gamma)(R+\Gamma_c)\ .$$

The linear gain spectrum of the laser medium is given by the imaginary part χ'' of the susceptibility $\chi=2gN\bar{\sigma}_0^{(1)}/\nu\alpha$. We find

$$\chi''=-\frac{4g^2N}{\nu}\frac{\Omega^2}{R\Gamma(\Gamma+\Gamma_c)+(2R+\gamma)4\Omega^2}\times\frac{[\Omega^2-\Delta^2+\frac{1}{4}(R+\Gamma)(R+\Gamma_c)][R(\gamma'+\Gamma_c+R)-\Gamma_c\gamma]-\Delta^2(\gamma-R)(\Gamma+\Gamma_c+2R)}{[\Omega^2-\Delta^2+\frac{1}{4}(R+\Gamma)(R+\Gamma_c)]^2+\frac{1}{4}\Delta^2(\Gamma+\Gamma_c+2R)^2}\ . \tag{11}$$

In Fig. 2 we have plotted χ'' in arbitrary units as a function of the normalized detuning Δ/Γ for different values of Ω with $\Gamma_c=10^{-3}\Gamma$ and $\gamma=\gamma'$. The pump rate is chosen such that $\bar{\sigma}_a^{(0)}+\bar{\sigma}_c^{(0)}=\bar{\sigma}_b^{(0)}$; that is, $R=R_0=\gamma\Omega^2/[2\Omega^2+\frac{1}{4}\Gamma(\Gamma+\Gamma_c)]$. We recognize a narrow gain peak in the center of the spectrum which broadens with increasing Ω and is superimposed on a broad absorption line. The broad absorption line in the susceptibility spectrum corresponds to the first term in Eq. (10b); that is, to the contribution of the population difference $\bar{\sigma}_b^{(0)}-\bar{\sigma}_a^{(0)}$ which leads to absorption as long as $R<\gamma$. The sharp gain peak described by the second term in Eq. (10b) results from the coherence $\bar{\sigma}_1^{(1)}$. Figure 3 shows the decomposition of the susceptibility spectrum (dashed curve) into its contributions from the population difference (upper solid curve) and the b-c coherence (lower solid curve). For sufficiently small Ω and pump rate R—that is, for $\Omega^2+\frac{1}{4}(R+\Gamma)(R+\Gamma_c)\ll\Gamma^2$—the width of the gain peak is

$$\Gamma_g=\frac{4\Omega^2}{(R+\Gamma)}+R+\Gamma_c\ . \tag{12}$$

The evolution of the complex amplitude of the laser field is governed by the equation

$$\dot{\alpha}=-\left[i\Delta_c+\frac{\kappa_0}{2}\right]\alpha+igN\sigma_0\ , \tag{13}$$

where κ_0 is the cavity damping rate and $\Delta_c=\nu_c-\nu$ is the detuning of the laser frequency ν from the resonance frequency ν_c of the cavity. Since here we disregard the thermal excitation of the laser field, there is no Langevin noise operator in Eq. (13). In the following we assume that the cavity is tuned to the atomic transition frequency, such that $\Delta=\Delta_c=0$. If we insert the first-order expression for $\bar{\sigma}_0$ [Eq. (10b)] into the field equation of motion of the laser photon number

$$\dot{n}=-\kappa_0 n-2gN\,\mathrm{Im}\left[\frac{\sigma_0}{\alpha}\right]n\ , \tag{14}$$

we find the linear gain coefficient of the laser

$$A=-2gN\,\mathrm{Im}\left[\frac{\sigma_0^{(1)}}{\alpha}\right]=\frac{4g^2N\Omega^2}{DD''}[R(\gamma'+\Gamma_c+R)-\gamma\Gamma_c]\ , \tag{15}$$

where $D''=\Omega^2+\frac{1}{4}(R+\Gamma)(R+\Gamma_c)$. A becomes positive,

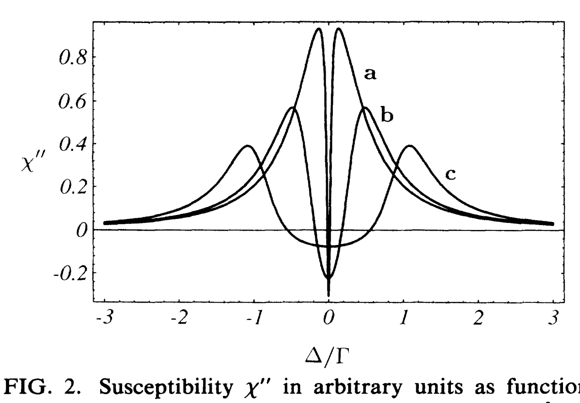

FIG. 2. Susceptibility χ'' in arbitrary units as function of normalized detuning Δ/Γ for $R=R_0$, $\gamma=\gamma'$, $\Gamma_c=10^{-3}\Gamma$, and $\Omega=0.1\Gamma$ (a), 0.4Γ (b), and 1Γ (c).

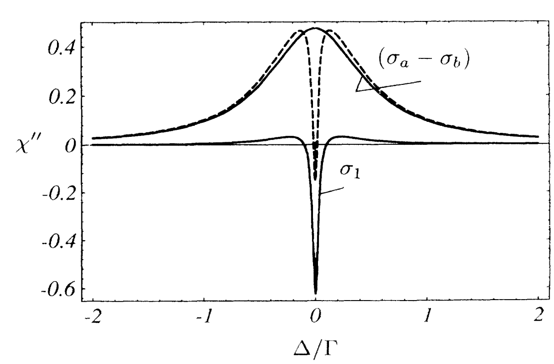

FIG. 3. Decomposition of the linear susceptibility spectrum of Fig. 2 for $\Omega=0.1\Gamma$ (dashed curve) into contributions of the population difference $\bar{\sigma}_b^{(0)}-\bar{\sigma}_a^{(0)}$ (upper solid curve) and the coherence $\bar{\sigma}_1^{(1)}$ (lower solid curve).

which indicates gain, if $R(\gamma'+\Gamma_c+R)>\gamma\Gamma_c$. We emphasize that in the limit $\Gamma_c\to 0$, amplification is possible for any nonvanishing pump rate R. In particular, as can be seen from Eqs. (9), amplification *without inversion* (in the presence of the driving field) is possible if $R<R_0$.

In order to find an approximate value for the mean number of photons $\bar{n}$, a third-order approach is necessary. Solving Eqs. (7) for $\bar{\sigma}_0^{(3)}$ and substituting the result into Eq. (14), we find

$$\bar{n}=\frac{A-\kappa_0}{B}\,, \tag{16}$$

where

$$B=g^2\frac{A}{DD''}[4\Omega^2(\Gamma+3\Gamma_c)+R\Gamma(R+\Gamma)+(R+\gamma')(\Gamma+\Gamma_c)(R+\Gamma_c)]\,. \tag{17}$$

III. SPECTRUM OF PHASE FLUCTUATIONS AND LASER LINEWIDTH

In this section, we calculate the spectrum of the phase fluctuations and the laser linewidth under resonance conditions ($\Delta=\Delta'=\Delta_c=0$) by linearizing the c-number Langevin equations for the collective atomic variables, Eqs. (7), and the field mode Eq. (13) around the semiclassical steady state solutions. To this end it is convenient to transform the Langevin equations into intensity and phase variables according to the definitions

$$\sigma_\mu=\sqrt{m_\mu}\,e^{i\Theta_\mu}\,,\qquad \alpha=\sqrt{n}\,e^{i\phi}\,, \tag{18}$$

where $\mu=0$, 1, and 2. Transforming the zeroth-order equation of motion for σ_2 into intensity-phase variables, we find

$$\dot{\Theta}_2=\frac{\Omega}{\sqrt{m_2}}(\sigma_c-\sigma_a)\cos\Theta_2\,. \tag{19}$$

Equation (19) has the linearly stable solution $\Theta_2=\bar{\Theta}_2=\pi/2$. [Note that, according to Eqs. (9), $\bar{\sigma}_c>\bar{\sigma}_a$.] Similarly we find, in first order of g,

$$\dot{\Theta}_0=g(\sigma_b-\sigma_a)\left[\frac{n}{m_0}\right]^{1/2}\cos(\Theta_0-\phi)+\Omega\left[\frac{m_1}{m_0}\right]^{1/2}\cos(\Theta_0-\Theta_1)+F_{\Theta_0}\,, \tag{20a}$$

$$\dot{\Theta}_1=g\left[\frac{nm_2}{m_1}\right]^{1/2}\sin(\Theta_1-\phi)+\Omega\left[\frac{m_0}{m_1}\right]^{1/2}\cos(\Theta_0-\Theta_1)+F_{\Theta_1}\,, \tag{20b}$$

$$\dot{\phi}=gN\left[\frac{m_0}{n}\right]^{1/2}\cos(\Theta_0-\phi)\,, \tag{20c}$$

where we have disregarded all fluctuation operators except F_{σ_0} and F_{σ_1}. The noise operators in Eqs. (20a) and (20b) are defined as $F_{\Theta_\mu}=\mathrm{Im}[F_{\sigma_\mu}/\sigma_\mu]$. The intensity equations read

$$\dot{m}_0=-(R+\Gamma)m_0+2g(\sigma_b-\sigma_a)\sqrt{nm_0}\sin(\Theta_0-\phi)+2\Omega\sqrt{m_0m_1}\sin(\Theta_0-\Theta_1)+F_{m_0}\,, \tag{21a}$$

$$\dot{m}_1=-(R+\Gamma_c)m_1-2g\sqrt{nm_1m_2}\cos(\phi-\Theta_1)-2\Omega\sqrt{m_0m_1}\sin(\Theta_0-\Theta_1)+F_{m_1}\,, \tag{21b}$$

$$\dot{m}_2=-(\Gamma+\Gamma_c)m_2+2\Omega(\sigma_c-\sigma_a)\sqrt{m_2}\,, \tag{21c}$$

$$\dot{n}=-\kappa_0 n-2gN\sqrt{nm_0}\sin(\Theta_0-\phi)\,. \tag{21d}$$

As can be recognized from Eqs. (20), there is no semiclassical steady-state solution for the phases ϕ, Θ_0, and Θ_1. This is due to the well-known phase symmetry of the laser [2–4]. In order to perform a linearization, we introduce a reduced set of variables which have well-defined steady-state values. We here chose $\Phi=\Theta_0-\phi$ and $\Psi=\Theta_0-\Theta_1$. Rewriting Eqs. (20) in terms of these variables, we have

$$\dot{\Phi}=\left[g(\sigma_b-\sigma_a)\left[\frac{n}{m_0}\right]^{1/2}-gN\left[\frac{m_0}{n}\right]^{1/2}\right]\cos\Phi+\Omega\left[\frac{m_1}{m_0}\right]^{1/2}\cos\Psi+F_{\Theta_0}\,, \tag{22a}$$

$$\dot{\Psi}=g(\sigma_b-\sigma_a)\left[\frac{n}{m_0}\right]^{1/2}\cos\Phi-g\left[\frac{nm_2}{m_1}\right]^{1/2}\sin(\Phi-\Psi)+\Omega\left[\left[\frac{m_1}{m_0}\right]^{1/2}-\left[\frac{m_0}{m_1}\right]^{1/2}\right]\cos\Psi+F_{\Theta_0}-F_{\Theta_1}\,. \tag{22b}$$

The stable semiclassical steady-state solutions of these equations above threshold and under noninversion conditions (i.e., $\bar{\sigma}_b^{(0)}>\bar{\sigma}_a^{(0)}+\bar{\sigma}_c^{(0)}$) are

$$\bar{\Phi}=-\frac{\pi}{2}\,,\qquad \bar{\Psi}=\frac{\pi}{2}\,. \tag{23}$$

We can now linearize Eqs. (22) by assuming small fluctuations around the semiclassical steady-state values:

$$\Phi(t)=\bar{\Phi}+\delta\Phi(t)\,,\qquad \Psi(t)=\bar{\Psi}+\delta\Psi(t)\,. \tag{24}$$

Using these expressions together with Eqs. (22), neglecting higher-order terms in $\delta\Phi$ and $\delta\Psi$, and replacing all remaining variables by their semiclassical steady-state values, we obtain

$$\frac{d}{dt}\delta\Phi=\left[g(\bar{\sigma}_b-\bar{\sigma}_a)\left[\frac{\bar{n}}{\bar{m}_0}\right]^{1/2}-gN\left[\frac{\bar{m}_0}{\bar{n}}\right]^{1/2}\right]\delta\Phi-\Omega\left[\frac{\bar{m}_1}{\bar{m}_0}\right]^{1/2}\delta\Psi+\bar{F}_{\Theta_0}\ , \tag{25a}$$

$$\frac{d}{dt}\delta\Psi=\left[g(\bar{\sigma}_b-\bar{\sigma}_a)\left[\frac{\bar{n}}{\bar{m}_0}\right]^{1/2}+g\left[\frac{\bar{n}\,\bar{m}_2}{\bar{m}_1}\right]^{1/2}\right]\delta\Phi-\left\{g\left[\frac{\bar{n}\,\bar{m}_2}{\bar{m}_1}\right]^{1/2}+\Omega\left[\left[\frac{\bar{m}_1}{\bar{m}_0}\right]^{1/2}-\left[\frac{\bar{m}_0}{\bar{m}_1}\right]^{1/2}\right]\right\}\delta\Psi+\bar{F}_{\Theta_0}-\bar{F}_{\Theta_1}\ . \tag{25b}$$

We transform Eqs. (25) into algebraic equations by Fourier transformation according to $\tilde{x}(\omega)=\int dt\,\exp(i\omega t)x(t)$. Furthermore, we make use of Eqs. (21) to substitute $gN\sqrt{\bar{m}_0/\bar{n}}$, $g(\bar{\sigma}_b-\bar{\sigma}_a)\sqrt{\bar{n}/\bar{m}_0}$, and $g\sqrt{\bar{n}\,\bar{m}_2/\bar{m}_1}$, and find the solution

$$\delta\tilde{\Phi}(\omega)=\frac{1}{M}\left[[\tfrac{1}{2}(R+\Gamma_c)-i\omega]\tilde{F}_{\Theta_0}+\Omega\left[\frac{\bar{m}_1}{\bar{m}_0}\right]^{1/2}F_{\Theta_1}\right]\ , \tag{26a}$$

where

$$M=\Omega^2+\frac{\kappa_0}{2}\Omega\left[\frac{\bar{m}_1}{\bar{m}_0}\right]^{1/2}+\tfrac{1}{4}(R+\Gamma+\kappa_0)(R+\Gamma_c)-\omega^2-\frac{i\omega}{2}(2R+\Gamma+\Gamma_c+\kappa_0)\ . \tag{26b}$$

Making use of $\langle\tilde{F}_x(\omega)\tilde{F}_y(\omega')\rangle=2\pi\delta(\omega+\omega')\langle F_xF_y\rangle$ and Eqs. (8), we obtain from Eqs. (26) the spectral correlation function of the fluctuations

$$\langle\delta\tilde{\Phi}(\omega)\delta\tilde{\Phi}(\omega')\rangle=\frac{2\pi\delta(\omega+\omega')}{|M|^2}\frac{\Omega^2R}{2N\bar{m}_0D}\times\{4R\omega^2+[4\Omega^2+(R+\Gamma)(R+\Gamma_c)](R+\Gamma+\Gamma_c)\}\ . \tag{27}$$

In order to obtain the laser linewidth, we need to find a relation between the fluctuation spectrum of the laser phase ϕ and that of Φ. For small fluctuations we obtain, from Eq. (20c),

$$\dot{\phi}(t)=gN\left[\frac{\bar{m}_0}{\bar{n}}\right]^{1/2}\cos\Phi(t)\ ,$$

$$\approx gN\left[\frac{\bar{m}_0}{\bar{n}}\right]^{1/2}\delta\Phi(t)\ . \tag{28}$$

A Fourier transformation immediately yields

$$\langle\tilde{\phi}(\omega)\tilde{\phi}(\omega')\rangle=-\frac{1}{\omega\omega'}g^2N^2\frac{\bar{m}_0}{\bar{n}}\langle\delta\tilde{\Phi}(\omega)\delta\tilde{\Phi}(\omega')\rangle\ . \tag{29}$$

Therefore with Eq. (27) we find the spectral correlation function of the laser phase

$$\langle\tilde{\phi}(\omega)\tilde{\phi}(\omega')\rangle=\frac{2\pi\delta(\omega+\omega')}{\omega^2}s(\omega)\ , \tag{30a}$$

where

$$s(\omega)=\frac{g^2N}{\bar{n}}\frac{2\Omega^2R}{D(R+\Gamma)^2}\frac{4[4R\omega^2+\Gamma_g(R+\Gamma)(R+\Gamma+\Gamma_c)]}{\left[\Gamma_g+\kappa_0\left[1+\frac{R+\Gamma_c}{R+\Gamma}\right]+\kappa_1-\frac{4\omega^2}{R+\Gamma}\right]^2+4\omega^2\left[1+\frac{R+\Gamma_c+\kappa_0}{R+\Gamma}\right]^2}\ . \tag{30b}$$

Here we have introduced

$$\kappa_1=\frac{4}{R+\Gamma}\frac{\kappa_0}{2}\Omega\left[\frac{\bar{m}_1}{\bar{m}_0}\right]^{1/2}-\kappa_0\ . \tag{31}$$

The physical meaning of κ_1 becomes clear if we make use of Eqs. (21a) and (21d), and rewrite κ_1 in the form

$$\kappa_1=\frac{4g^2N(\bar{\sigma}_b^{(0)}-\bar{\sigma}_a^{(0)})}{R+\Gamma}\ . \tag{32}$$

κ_1 can thus be identified with the absorption rate of the medium corresponding to the population-difference term in the total susceptibility, Eq. (10b).

As discussed in the Appendix, the laser linewidth $\Delta\nu$ is to a good approximation given by $s(\omega=0)$. We thus obtain

$$\Delta\nu\simeq s(0)=\frac{A}{2\bar{n}}\frac{\gamma+\gamma'+R+\Gamma_c}{\gamma'+R+\Gamma_c-\gamma\frac{\Gamma_c}{R}}\times\left[\frac{\Gamma_g}{\Gamma_g+\kappa_0\left[1+\frac{R+\Gamma_c}{R+\Gamma}\right]+\kappa_1}\right]^2\ , \tag{33}$$

where A is the linear gain coefficient defined in Eq. (15). Equation (33) is the main result of this paper. Comparing this result with Eq. (1), we recognize a deviation of the laser linewidth from the Schawlow-Townes result by the second and third terms. Whereas well above threshold the second term is of order unity, since usually $\gamma\sim\gamma'\gg\Gamma_c$, R, the third term can lead to an essential reduction of the linewidth below the Schawlow-Townes limit. This factor is a result of the nonadiabatic behavior of the laser system. In order to see this, we now calculate the laser linewidth under the assumption of an adiabatic evolution of the laser field.

If we assume that the atoms follow the time evolution of the laser field adiabatically, we may set all time derivatives in the equations of motion (7) for the atomic system equal to zero. Taking into account again only the relevant noise operators and applying perturbation theory with respect to the laser field in lowest order, we find in the resonance case ($\Delta=0$)

$$\sigma_0^{(1)}=-\frac{2ig\alpha\Omega^2}{DD''}[R(\gamma'+\Gamma_c+R)-\gamma\Gamma_c] + \frac{1}{D''}[\tfrac{1}{2}(R+\Gamma_c)F_{\sigma_0}+i\Omega F_{\sigma_1}] . \tag{34}$$

Inserting this result into the field equation (13), we find an effective noise operator for the field

$$F_\alpha=\frac{igN}{D''}[\tfrac{1}{2}(R+\Gamma_c)F_{\sigma_0}+i\Omega F_{\sigma_1}] , \tag{35}$$

yielding the steady-state phase diffusion coefficient

$$\begin{aligned} D_{\phi\phi}&=\frac{g^2N\Omega^2R}{2DD''^2\bar{n}}[4\Omega^2(\Gamma+\Gamma_c+R) \\ &\qquad +(R+\Gamma_c)(R+\Gamma)(\Gamma+\Gamma_c+R)] \\ &=\frac{A}{2\bar{n}}\frac{\gamma+\gamma'+\Gamma_c+R}{\gamma'+R+\Gamma_c-\gamma\dfrac{\Gamma_c}{R}} , \end{aligned} \tag{36}$$

which is the adiabatic linewidth $\Delta\nu_{\text{ad}}$. We therefore have

$$\Delta\nu=\Delta\nu_{\text{ad}}\left[\frac{\Gamma_g}{\Gamma_g+\kappa_0\left[1+\dfrac{R+\Gamma_c}{R+\Gamma}\right]+\kappa_1}\right]^2 . \tag{37}$$

A surprising conclusion from Eq. (37) is that, even in the limit of a high-Q cavity—that is, for $\kappa_0\ll\Gamma_g$—the adiabatic result is generally not correct.

Expression (37) for the laser linewidth can be given a simple physical interpretation if we consider the case of a small Rabi frequency Ω and small pump rate R. Under these conditions we have $(R+\Gamma_c)\ll(R+\Gamma)$, and $\kappa_0+\kappa_1$ represents the total photon loss rate which governs the lifetime of a photon inside the cavity. On the other hand, Γ_g is the width of the gain peak, which determines the time scale of spontaneous emission events. If the time of a single spontaneous emission is long compared to the lifetime of a photon, it contributes only partially to the phase noise of the laser photon, and leads to a nonadiabatic line narrowing in full analogy to Eq. (2).

IV. SUMMARY

We have analyzed the phase fluctuations of a Λ-type noninversion laser by using standard Fokker-Planck and linearization techniques [4,7]. We have observed a nonadiabatic narrowing of the linewidth below the Schawlow-Townes limit.

Nonadiabatic effects occur if the decay time of a cavity photon is much shorter than the spontaneous emission time of the atoms. The time scale of spontaneous emission in the noninversion laser discussed here is given essentially by the width of the gain peak of the medium. For sufficiently small Rabi frequencies of the driving field, this width is $\Gamma_g=4\Omega^2/\Gamma+R+\Gamma_c$, and can be much smaller than the natural width Γ of the laser transition. On the other hand, the time scale of the photon decay is given by the sum of the cavity decay rate κ_0 and the effective damping rate κ_1 resulting from the absorptive part in the susceptibility spectrum. If $\kappa_0+\kappa_1$ is larger than Γ_g, we observe a nonadiabatic line shape.

The line narrowing studied in the present paper is fundamentally different from the linewidth reduction in these systems due to nonlinear saturation effects, as discussed in Refs. [9,10] for a similar Raman laser in the adiabatic limit. High above threshold the phase diffusion decreases with $1/\bar{n}^3$ in a Raman laser (in an ordinary three-level laser it decreases with $1/\bar{n}^2$ [4]), as opposed to the $1/\bar{n}$ dependence in the Schawlow-Townes regime. However, the nonadiabatic effects discussed here are already present in the linear regime.

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APPENDIX

Since the fluctuations of the laser intensity take place on a much slower time scale than the phase fluctuations, one can approximate the laser linewidth $\Delta\nu$ by the long-time diffusion coefficient $D_{\phi\phi}=(d/dt)\langle\phi(t)^2\rangle$ of the phase [2]. This diffusion coefficient can in turn be obtained from the spectral correlation function [Eq. (30)] with the help of relation (28):

$$\begin{aligned} D_{\phi\phi}&=2\langle\dot{\phi}(t)\phi(t)\rangle=2\frac{g^2N^2\overline{m}_0}{\bar{n}}\int_{-\infty}^{t}dt'\langle\delta\Phi(t')\delta\Phi(t)\rangle \\ &=\frac{2}{(2\pi)^2}\frac{g^2N^2\overline{m}_0}{\bar{n}}\int_{-\infty}^{t}dt'\int d\omega\int d\omega' e^{-i\omega t}e^{-i\omega' t'}\langle\delta\tilde{\Phi}(\omega')\delta\tilde{\Phi}(\omega)\rangle \\ &=\frac{2}{2\pi}\int_{-\infty}^{t}dt'\int d\omega e^{-i\omega(t-t')}s(\omega) . \end{aligned} \tag{A1}$$

With the help of

$$\frac{1}{2\pi}\int_{-\infty}^{t} dt' e^{-i\omega(t-t')}=\tfrac{1}{2}\delta(\omega)+\frac{i}{\pi}\mathrm{P}\left[\frac{1}{\omega}\right] , \tag{A2}$$

where $\mathrm{P}[1/\omega]$ denotes the principle value, (A1) can be written as

$$D_{\phi\phi}=s(0)+\frac{2i}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\omega \frac{s(\omega)}{\omega} . \tag{A3}$$

Since $s(\omega)$ is a symmetric function of ω, the last integral vanishes and we find

$$\Delta\nu \simeq \frac{d}{dt}\langle \phi(t)\phi(t)\rangle = s(0) . \tag{A4}$$

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