PHYSICAL REVIEW A VOLUME 47, NUMBER 5 MAY 1993

Revivals made simple: Poisson summation formula as a key to the revivals in the Jaynes-Cummings model

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(Received 23 September 1992)

We investigate the phenomenon of quantum revivals in the Jaynes-Cummings model for an arbitrary quantized field mode. With the help of the Poisson summation formula, we cast the infinite sum determining the atomic inversion into an infinite sum of integrals. Each integral, when evaluated using the method of stationary phase, yields under appropriate conditions one revival. We present simple approximate analytical expressions for these revivals and illustrate this general technique by the examples of a coherent and a highly squeezed state. The oscillatory photon distribution of the latter creates slightly different Rabi frequencies which give rise to a beat note; that is, echos in the revivals. We obtain the photon statistics of the quantized field by "measuring" the atomic collapse of a single revival—a technique which might be applicable in the realm of the one-atom maser.

PACS number(s): 42.50.Dv, 42.52.+x

I. POISSON SUMMATION FORMULA: THE KEY TO QUANTUM REVIVALS

A two-level atom in the presence of a quantized mode of the radiation field—the well-known Jaynes-Cummings model [1]—has been explored theoretically [2–4] and tested in its predictions experimentally [5] in a wealth of articles. This model displays fascinating quantum features: the squeezing [6] of the fluctuations in the electromagnetic field [7], the generation of sub-Poissonian photon statistics [8,9], and the so-called quantum revivals [2,10], to name only a few effects arising in this paradigm of quantum optics. The impressive phenomenon of the revival—the recovery of population at a time long after the Cummings collapse has taken place—has been investigated extensively for the case of the field mode being in a coherent state [2]. But what are the necessary conditions to obtain such a revival in the presence of an arbitrary state? What is the fine structure of each revival in this general case? Is it possible to recover the photon distribution W_m of the electromagnetic field from the atomic inversion? These are three questions put forward, discussed, and answered in the present article.

The photon distribution W_m determines the atomic inversion $w=w(t)$ at a time $t>0$, given that the atom is in its ground state at $t=0$, via the Jaynes-Cummings sum [1]

$$w(t)=-\frac{1}{2}\sum_{m=0}^{\infty} W_m \cos(2\lambda t\sqrt{m}\,)\ , \tag{1.1}$$

where λ denotes the interaction strength. Unfortunately this sum involves the summation index m as the square root. This complication has prevented so far the derivation of a simple and explicit expression for w in the presence of an arbitrary field. For the case of a coherent-state field mode, approximate expressions have been derived and have been compared to a numerical evaluation of the relevant sum [2]. However, this treatment is limited to this particular example of the coherent state and is rather difficult to generalize. The Jaynes-Cummings sum, Eq. (1.1), can be converted to a complex integral. Its explicit evaluation is possible, however, only for special quantum states of the field mode, such as a coherent or a weakly squeezed state [11]. In the present article we make use of the Poisson summation formula [12] and the method of stationary phase [13]—two techniques already successfully applied to rainbow scattering [14], the renormalization of curlicues [15], and Rydberg wave packets [16].

The article is organized as follows: With the help of the Poisson summation formula we, in Sec. II, convert the original Jaynes-Cummings sum over m, Eq. (1.1), into an infinite sum of integrals. Under appropriate conditions each term of this sum represents one revival which we evaluate with the method of stationary phase. In particular, for a slowly varying photon distribution we can derive a *general* expression for the revivals which identifies the shape of the photon distribution as the determiner of the revival envelope. The $\nu=0$ term of this expression corresponds then to the Cummings collapse. We illustrate this technique for the case of a coherent state and find excellent agreement with the exact numerical evaluation of the sum, Eq. (1.1).

In Sec. III we demonstrate the power of this approach for the example of a highly squeezed state. It has been shown numerically that also in this case the atomic inversion exhibits revival. More importantly, however, each revival is accompanied by little echoes [17]. The physical origin of these echos stands out most clearly when we recall the discussion of Sec. II together with the fact that an appropriately squeezed state displays an *oscillatory* photon distribution [18]. Since the shape of each revival is governed by the shape of the photon distribution, the modulation in W_m must lead to a modulation of the envelope of the revival. This explains the qualitative but not the quantitative features. A more appropriate analysis has to include the modulation in W_m in the stationary-phase analysis as well. This is made possible by a simple asymptotic expression for W_m [18] that allows us to separate the rapid oscillations in m from the slowly varying part. This approach yields results which are again in good agreement with the numerical calculation.

How to extract from the time dependence of the atomic inversion the initial photon distribution is the question investigated in Sec. IV. Under appropriate conditions this distribution follows by a Fourier transformation of either the collapse or of one of the revivals. We illustrate this technique by a special example. Section V is devoted to a summary and an outlook.

II. REVIVAL AS READOUT OF THE PHOTON DISTRIBUTION

In the present section we rewrite the Jaynes-Cummings sum, Eq. (1.1), with the help of the Poisson summation formula [12]

$$\sum_{m=0}^{\infty} f_m = \sum_{\nu=-\infty}^{\infty} \int_0^{\infty} dm\, f(m) e^{2\pi i \nu m} + \tfrac{1}{2} f_0 , \tag{2.1}$$

where $f(m)$ is a continuous version of f_m. We evaluate the resulting integrals with the method of stationary phase for the case of a slowly varying photon distribution $W(m)$. The example of a coherent state serves as an illustration of these results.

A. General formalism

When we substitute the Poisson summation formula, Eq. (2.1), into the Jaynes-Cummings sum, Eq. (1.1), we arrive at

$$w(t) = -\tfrac{1}{2} \sum_{\nu=-\infty}^{\infty} \int_0^{\infty} dm\, W(m) e^{2\pi i \nu m} \cos(2\lambda t \sqrt{m}\,) - \tfrac{1}{4} W_0$$

$$= \sum_{\nu=-\infty}^{\infty} w_\nu(t) - \tfrac{1}{4} W_0 . \tag{2.2a}$$

Here we have introduced the definition

$$w_\nu(t) = -\tfrac{1}{2} \mathrm{Re}\left\{ \int_0^{\infty} dm\, W(m) \exp[2iS_\nu(m)] \right\} , \tag{2.2b}$$

with a phase

$$S_\nu(m) \equiv \pi \nu m - \lambda t \sqrt{m} . \tag{2.2c}$$

When the variation of the photon distribution $W(m)$ is slow compared to the m variation of $\exp[2iS_\nu(m)]$, we can evaluate the integral w_ν, Eq. (2.2b), by expanding the phase $S_\nu(m)$, Eq. (2.2c),

$$S_\nu(m) \cong S_\nu(m=m_\nu) + \frac{1}{2} \frac{\partial^2 S_\nu(m)}{\partial m^2} \bigg|_{m=m_\nu} (m-m_\nu)^2 \tag{2.3}$$

around the point of stationary phase m_ν defined by

$$\frac{\partial S_\nu(m)}{\partial m} \bigg|_{m=m_\nu} = 0 . \tag{2.4}$$

The resulting integral

$$w_\nu(t) \cong -\tfrac{1}{2} W(m=m_\nu) \mathrm{Re}\left\{ e^{2iS_\nu(m=m_\nu)} \int_0^{\infty} dm \exp\left[i \frac{\partial^2 S_\nu}{\partial m^2} \bigg|_{m=m_\nu} (m-m_\nu)^2 \right] \right\}$$

yields [19]

$$w_\nu(t) \cong -\tfrac{1}{2} W(m=m_\nu) \left[\pi \Big/ \left| \frac{\partial^2 S_\nu}{\partial m^2} \right|_{m=m_\nu} \right]^{1/2} \times \cos\left[2S_\nu(m=m_\nu) + \eta \frac{\pi}{4} \right] , \tag{2.5}$$

where

$$\eta = \mathrm{sgn}\left[\frac{\partial^2 S_\nu}{\partial m^2} \bigg|_{m=m_\nu} \right] .$$

The explicit form of $S_\nu(m)$, Eq. (2.2c), transforms Eq. (2.4) into

$$0 = \frac{\partial S_\nu(m)}{\partial m} \bigg|_{m=m_\nu} = \left[\pi \nu - \frac{1}{2} \frac{\lambda t}{\sqrt{m}} \right] \bigg|_{m=m_\nu} , \tag{2.6}$$

that is,

$$\sqrt{m_\nu} = \frac{\lambda t}{2\pi\nu} . \tag{2.7}$$

Since we are only interested in positive times $t>0$, only positive ν values will provide a point of stationary phase. Note that the term $\nu=0$ exhibits only a stationary-phase point for $t=0$ and we have to evaluate the corresponding integral w_0 by different means. We therefore express the approximate atomic inversion, Eq. (2.2a), in the form

$$w(t) \cong -\tfrac{1}{4} W_0 + w_0(t) + \sum_{\nu=1}^{\infty} w_\nu(t) , \tag{2.8a}$$

where we have taken out the term $\nu=0$ from the sum. We now discuss the explicit form of w_ν for the case $\nu>0$. From Eq. (2.2c) we find

$$S_\nu(m=m_\nu) = -\frac{\lambda^2 t^2}{4\pi\nu} ,$$

and Eqs. (2.6) and (2.7) yield

$$\left.\frac{\partial^2 S_\nu}{\partial m^2}\right|_{m=m_\nu}=\left.\tfrac{1}{4}\lambda t m^{-3/2}\right|_{m=m_\nu}=\frac{2\pi^3\nu^3}{\lambda^2 t^2} .$$

Therefore the final expression for w_ν, when $\nu>0$ reads

$$w_\nu(t)=-\tfrac{1}{2}W\left[m=\frac{\lambda^2t^2}{4\pi^2\nu^2}\right]\frac{\lambda t}{\pi\sqrt{2\nu^3}}\cos\left[\frac{\lambda^2t^2}{2\pi\nu}-\frac{\pi}{4}\right] . \tag{2.8b}$$

The time dependence of $w_\nu(t)$ is hence governed by three factors: (i) a rapidly oscillating cosine function, (ii) a slowly varying amplitude which decays with increasing ν, and (iii) an envelope translating photon number into time according to the simple substitution rule

$$m\leftrightarrow\frac{\lambda^2t^2}{4\pi^2\nu^2} \quad \text{or} \quad t\leftrightarrow\frac{2\pi\nu}{\lambda}\sqrt{m} . \tag{2.9}$$

From this rule we immediately recognize that the envelope of $w_\nu(t)$ is approximately centered at the times

$$t_\nu\equiv\frac{2\pi\nu}{\lambda}\sqrt{\overline{m}} , \tag{2.10}$$

where $\overline{m}=\sum_{m=0}^{\infty}mW_m$ denotes the average number of photons. The substitution rule of Eq. (2.9) translates the width

$$\sigma=\left[\sum_{m=0}^{\infty}[m-\overline{m}]^2W_m\right]^{1/2}$$

of the photon distribution W_m into a measure for the width of w_ν in time, denoted by Δt_ν. Indeed we find

$$\frac{\lambda^2}{4\pi^2\nu^2}\left[\left[t_\nu+\frac{\Delta t_\nu}{2}\right]^2-t_\nu^2\right]=\sigma \tag{2.11}$$

or

$$\Delta t_\nu=\frac{4\pi^2\nu^2}{\lambda^2 t_\nu}\sigma=\frac{2\pi\nu}{\lambda}\frac{\sigma}{\sqrt{\overline{m}}} . \tag{2.12}$$

Here we have made use of Eq. (2.10) and have neglected the quadratic contribution of Δt_ν. Two consecutive terms of the sum Eq. (2.8a) separate in time when their temporal separation $\delta t_\nu\equiv t_{\nu+1}-t_\nu=(2\pi/\lambda)\sqrt{\overline{m}}$ is larger than their width Δt_ν, that is, when $\delta t_\nu>\Delta t_\nu$. With Eq. (2.12) this condition reads

$$\nu<\frac{\overline{m}}{\sigma} . \tag{2.13}$$

In this case only a single term w_ν [$\nu\cong\lambda t/(2\pi\sqrt{\overline{m}})$] contributes at a given instance of time t. This term w_ν represents therefore the νth revival.

We emphasize that for a given photon distribution the inequality (2.13) is always violated for appropriately large ν, and hence the corresponding revivals cannot be resolved. Since according to Eq. (2.9) ν corresponds to time (note that $m\sim\overline{m}$), many terms w_ν in the sum (2.8a) determine the atomic inversion in the large-time limit. When $\sigma>\overline{m}$, as is true for thermal light, not even the first revival is well defined [20].

B. Example of a coherent state

We now illustrate the results of Sec. II A by the example of a coherent state of average photon number $\overline{m}\equiv|\alpha|^2$. The familiar Poisson photon distribution

$$W_m=\frac{|\alpha|^{2m}}{m!}e^{-|\alpha|^2}$$

reduces in the large $|\alpha|^2$ limit to the Gaussian approximation [21]

$$W(m)=\frac{1}{\sqrt{2\pi\overline{m}}}\exp[-2(\sqrt{m}-\sqrt{\overline{m}})^2] , \tag{2.14}$$

and $W_0=e^{-|\alpha|^2}\approx 0$. When we substitute this expression into Eq. (2.8b) we immediately arrive at

$$w_\nu(t)=-\frac{1}{2\sqrt{\pi\nu}}\frac{\lambda t}{2\pi\nu|\alpha|}\exp\left[-\frac{\lambda^2}{2\pi^2\nu^2}(t-t_\nu)^2\right]\times\cos\left[\frac{\lambda^2t^2}{2\pi\nu}-\frac{\pi}{4}\right] , \tag{2.15a}$$

where the revival times following from Eq. (2.14) are

$$t_\nu=\frac{2\pi\nu}{\lambda}|\alpha| , \tag{2.15b}$$

in agreement with Eq. (2.10). When we recognize that w_ν assumes appreciable values only at times in the neighborhood of the revival time, that is, $t\approx t_\nu$, the expression Eq. (2.15a) simplifies even further

$$w_\nu(t)=-\frac{1}{\sqrt{4\pi\nu}}\exp\left[-\frac{\lambda^2}{2\pi^2\nu^2}(t-t_\nu)^2\right]\times\cos\left[\frac{\lambda^2t^2}{2\pi\nu}-\frac{\pi}{4}\right] . \tag{2.15c}$$

This formula is in agreement with the well-known results obtained by different methods [2]. In Fig. 1 we compare and contrast this approximate expression, Eq. (2.15c), for the first revival w_1, shown in (a), to the numerical evaluation of the Jaynes-Cummings sum, displayed in (b). We conclude by discussing the term $\nu=0$ in Eq. (2.8a), that is,

$$\begin{aligned}w_0(t)&=-\tfrac{1}{2}\int_0^\infty dm\, W(m)\cos(2\lambda t\sqrt{m})\\&=-\tfrac{1}{2}(2\pi|\alpha|^2)^{-1/2}\int_0^\infty dm\,\exp[-2(\sqrt{m}-|\alpha|)^2]\cos(2\lambda t\sqrt{m})\\&=-(2\pi)^{-1/2}\int_0^\infty dy\left[\frac{y}{|\alpha|}\right]\exp[-2(y-|\alpha|)^2]\cos(2\lambda t y) .\end{aligned}$$

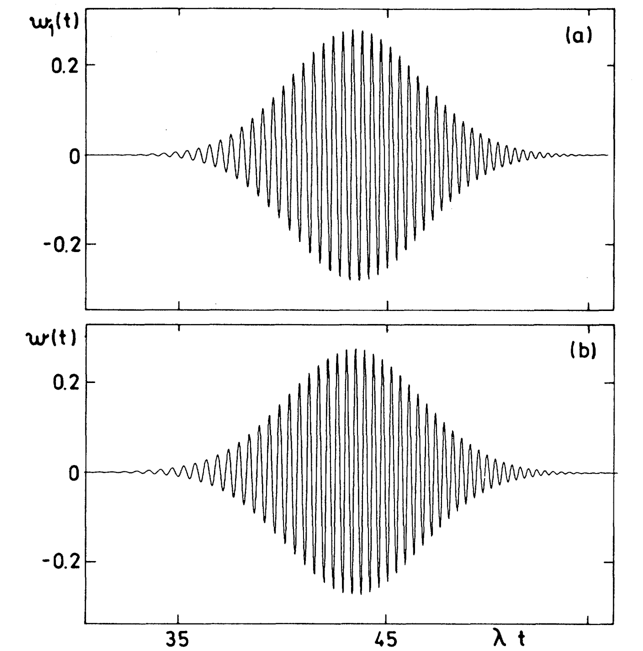

FIG. 1. First quantum revival in the Jaynes-Cummings model for a coherent state of average number of photons $\bar{m}=\alpha^2=49$. The approximate analytical result based on Eq. (2.15c) and shown in (a) is in good agreement with the numerical evaluation of the Jaynes-Cummings sum Eq. (1.1) depicted in (b).

Here we have applied the approximate photon-number distribution, Eq. (2.14), and have made the substitution $y\equiv\sqrt{m}$. Since the Gaussian distribution $W(m)$ in the above integral is sharply centered at $\sqrt{m}\approx|\alpha|$, we can approximate the term $(y/|\alpha|)$ by unity and extend the lower limit of integration to minus infinity. When we substitute $x=y-|\alpha|$ and perform the remaining integral [19]

$$w_0(t)=-(2\pi)^{-1/2}\int_{-\infty}^{\infty}dx\, e^{-2x^2}\cos[2\lambda t x+2\lambda t|\alpha|]$$

we arrive at

$$w_0(t)=-\tfrac{1}{2}\exp\left[-\frac{\lambda^2t^2}{2}\right]\cos(2\lambda t|\alpha|)\ . \qquad (2.16)$$

Equation (2.16) describes the Gaussian decay of the Rabi oscillations: the Cummings collapse. Although we have derived this result only for the special case of a coherent state it may serve as an indication for the more general feature: The contribution w_0 describes the collapse even for an arbitrary field state, provided the first revival separates from the collapse.

III. REVIVALS AND ECHOS IN THE PRESENCE OF A HIGHLY SQUEEZED STATE

In the present section we apply the techniques of Sec. II A and extend them to gain a deeper understanding of the revivals in the atomic inversion in the presence of a highly squeezed state. This problem has been investigated in Ref. [17] by numerical means. In this case revivals with an oscillatory envelope, the so-called echos, make their appearance as exemplified in Fig. 2(a). Moreover, the collapse time may exceed that of a coherent state of identical photon number [17,22]. To obtain an analytical result for the atomic inversion $w=w(t)$, Eq. (1.1), in the

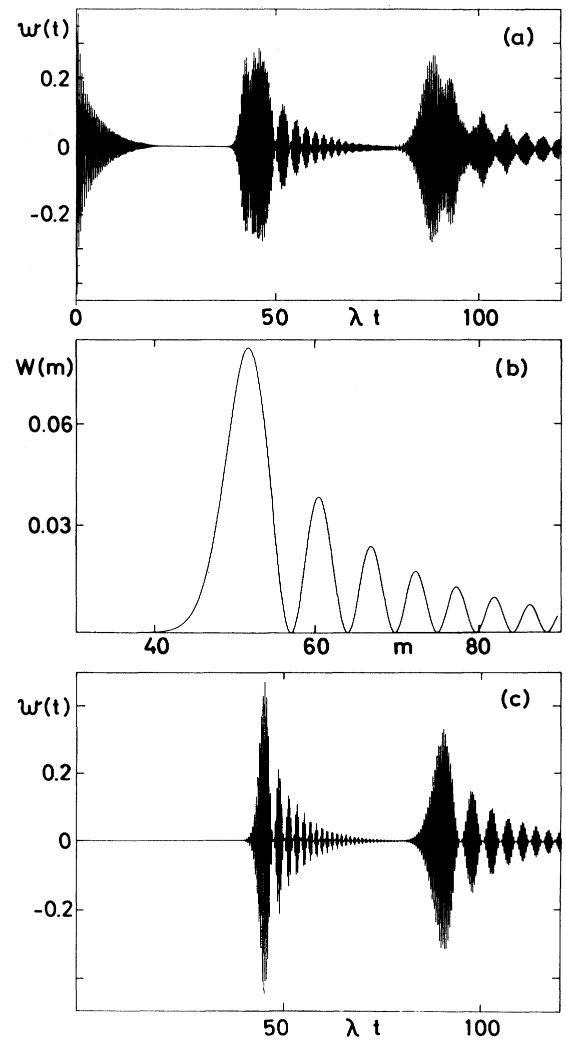

FIG. 2. The atomic inversion $w=w(t)$ in the presence of a highly squeezed state exhibits revivals accompanied by little echos when numerically evaluated via Eq. (1.1) as depicted in (a). Here we have used the exact photon distribution derived in Ref. [18], which is shown in (b) for a coherent amplitude $\alpha=7$ and a squeeze parameter $s=21$. The most elementary attempt at understanding the echos of (a) consists of substituting the oscillatory photon distribution of (b) into the approximate analytical expression for the atomic inversion, Eq. (2.8b). According to this result, the envelope of each revival is a rescaled readout of the photon distribution $W(m)$: The oscillations in $W(m)$ translate into oscillations of the revival envelope as shown in (c). However, when we compare this approximation with the exact numerical result of (a), we recognize that the position and the shape of the echos are not properly described by this approach.

presence of a highly squeezed state is our goal in this section.

A. Oscillations in photon-number distribution modify stationary-phase analysis

It is tempting to make use of the approximation Eq. (2.8b) for w_ν and substitute the photon-number distribution of a highly squeezed state, shown in Fig. 2(b) into this expression. The approximation of $W(m)$ most suitable for this purpose reads [23]

$$W(m)=\sqrt{4\pi\epsilon}[\mathrm{Ai}(\mu_m)]^2\exp[-\epsilon(m+\tfrac{1}{2}-\alpha^2)] \times\left[\frac{\mu_m}{\alpha^2-m-\frac{1}{2}}\right]^{1/2} \quad (3.1a)$$

with

$$\mu_m=\begin{cases} +\left[\frac{3}{2}\int_{\sqrt{2m+1}}^{\sqrt{2\alpha^2}}dx(x^2-2m-1)^{1/2}\right]^{2/3} & \text{for } m+\frac{1}{2}\le\alpha^2 \\ -\left[\frac{3}{2}\int_{\sqrt{2\alpha^2}}^{\sqrt{2m+1}}dx(2m+1-x^2)^{1/2}\right]^{2/3} & \text{for } m+\frac{1}{2}\ge\alpha^2 . \end{cases} \quad (3.1b)$$

Here Ai denotes the Airy function. This formula is valid in the limit of strong squeezing, $0<2/s\ll 1$, and large coherent amplitude, $\alpha>0$. Equation (2.8) together with the photon distribution Eq. (3.1) indeed creates revivals as well as *echos* as displayed in Fig. 2(c). Why? Equation (2.8b) and remark (iii) in Sec. II A provide an immediate answer: The envelope of each revival is a readout of the photon distribution, which in this case is oscillating, as indicated in Fig. 2(b). Hence each maximum in $W(m)$ corresponds to one echo. A closer comparison between Figs. 2(a) and 2(c), however, reveals that this argument is quantitatively not correct: It cannot provide the precise location, the correct amplitude, and the shape of each echo. This is, however, not surprising when we recall that in the derivation of Eq. (2.8) we have assumed that $W(m)$ is slowly varying compared to the oscillations in $\exp[2iS_\nu(m)]$. Obviously the oscillatory photon distribution Eq. (3.1) violates this condition. But a slightly improved version of the approach of Sec. II A gets a grip on this problem: Include the oscillations of $W(m)$ in the stationary-phase treatment; this is the strategy we pursue in the remainder of the present section.

We achieve a decomposition of $W(m)$ into a slowly varying and an oscillatory part when we recall the asymptotic approximation of Eq. (3.1) [18,23]

$$W(m)\cong 4\mathcal{A}_m\cos^2\phi_m , \quad (3.2a)$$

where

$$\mathcal{A}_m=\left[\frac{\epsilon}{4\pi}\right]^{1/2}\frac{\exp[-\epsilon(m+\frac{1}{2}-\alpha^2)]}{(m+\frac{1}{2}-\alpha^2)^{1/2}} \quad (3.2b)$$

and

$$\phi_m=(m+\tfrac{1}{2})\arctan[(m+\tfrac{1}{2}-\alpha^2)^{1/2}/\alpha] -\alpha(m+\tfrac{1}{2}-\alpha^2)^{1/2}-\frac{\pi}{4} . \quad (3.2c)$$

This formula is valid for numbers m appropriately larger than α^2, that is, in the oscillatory regime of $W(m)$. We obtain an expression for $W(m)$ which holds for arbitrary m values by introducing the appropriate Heaviside step functions denoted by $\Theta=\Theta(y)$. We thus rewrite $W(m)$ in the form

$$W(m)=A(m)+\Theta(m+\tfrac{1}{2}-\alpha^2)\frac{W(m)}{2\cos^2\phi_m}\cos(2\phi_m) , \quad (3.3a)$$

where

$$A(m)=\Theta(\alpha^2-m-\tfrac{1}{2})W(m)+\Theta(m+\tfrac{1}{2}-\alpha^2)\frac{W(m)}{2\cos^2\phi_m} . \quad (3.3b)$$

Equations (3.3) give a decomposition of the photon-number distribution in slowly varying amplitudes $A(m)$ and $W(m)/2\cos^2\phi_m$ and in an oscillating function $\cos 2\phi_m$. We substitute Eq. (3.3a) into Eq. (2.2b) and arrive at

$$w_\nu=a_\nu+b_\nu^{(+)}+b_\nu^{(-)} , \quad (3.4a)$$

where

$$a_\nu=-\tfrac{1}{2}\mathrm{Re}\left\{\int_0^\infty dm\ A(m)\exp[2iS_\nu(m)]\right\} , \quad (3.4b)$$

$$b_\nu^{(+)}=-\tfrac{1}{2}\mathrm{Re}\left\{\int_{\alpha^2-1/2}^\infty dm\frac{W(m)}{4\cos^2\phi_m}\exp[2iS_\nu^{(+)}(m)]\right\} , \quad (3.4c)$$

$$b_\nu^{(-)}=-\tfrac{1}{2}\mathrm{Re}\left\{\int_{\alpha^2-1/2}^\infty dm\frac{W(m)}{4\cos^2\phi_m}\exp[2iS_\nu^{(-)}(m)]\right\} . \quad (3.4d)$$

Here we have introduced the new phases

$$S_\nu^{(\pm)}(m)\equiv S_\nu(m)\pm\phi_m . \quad (3.4e)$$

The decomposition of $W(m)$ through Eq. (3.3) has created integrals Eqs. (3.4), which now consist of a slowly varying amplitude $A(m)$ or $W(m)/(4\cos^2\phi_m)$, and a rapidly oscillating phase factor $\exp[2iS_\nu(m)]$ or $\exp[2iS_\nu^{(\pm)}(m)]$. Hence we are in the position to apply again the techniques of Sec. II A.

The point of stationary phase for a_ν, determined by $S_\nu(m)$, is identical to that calculated in Sec. II A, that is,

$$\sqrt{m_\nu}=\frac{\lambda t}{2\pi\nu} . \quad (3.5)$$

The presence of the oscillations in $W(m)$, represented by the phase ϕ_m, modifies this point when we investigate the integrals $b_\nu^{(\pm)}$. Here the stationary-phase points following from Eqs. (3.4e) and (3.2c) are given by

$$0=\frac{\partial S_\nu^{(\pm)}(m)}{\partial m}=\frac{\partial S_\nu(m)}{\partial m}\pm\frac{\partial \phi_m}{\partial m}$$
$$=\pi\nu-\tfrac{1}{2}\frac{\lambda t}{\sqrt{m}}\pm\arctan[(m+\tfrac{1}{2}-\alpha^2)^{1/2}/\alpha]\ , \tag{3.6}$$

that is

$$\arctan[(m_\nu^{(+)}+\tfrac{1}{2}-\alpha^2)^{1/2}/\alpha]+\pi\nu=\frac{\lambda t}{2(m_\nu^{(+)})^{1/2}}\ , \tag{3.7a}$$

$$-\arctan[(m_\nu^{(-)}+\tfrac{1}{2}-\alpha^2)^{1/2}/\alpha]+\pi\nu=\frac{\lambda t}{2(m_\nu^{(-)})^{1/2}}\ . \tag{3.7b}$$

We gain considerable insight into the solutions of these transcendental equations and their dependence on the parameters of interest, such as the time t, by a graphical solution. In Fig. 3 we depict the time-independent left-hand side of Eq. (3.7) in its functional dependence on m by dashed and dotted lines, Eq. (3.7a) and (3.7b), respectively. The solid curves (a)–(c) represent the right-hand side of Eq. (3.7) for different times. The m coordinates of the crossings between the solid and dashed or dotted curves correspond to points of stationary phase $m_\nu^{(\pm)}$. In this picture the point of stationary phase m_ν, Eq. (3.5), for a_ν is then the m coordinate of the crossings of the curves (a), (b), or (c), with the solid straight lines $\pi\nu$.

Figure 3 reveals three striking features of the solutions:

(i) No crossings and hence no points of stationary phase exist for negative ν values. This is a result of the right-hand side of Eq. (3.7) always being positive.

(ii) For a given positive value of ν there exist points of stationary phase for $b_\nu^{(+)}$ as well as $b_\nu^{(-)}$ provided that

$$\left.\frac{\lambda t}{2\sqrt{m}}\right|_{m=\alpha^2-1/2}\geq\pi\nu\ ,$$

that is,

$$t\geq\tau_\nu\equiv\frac{2\pi\nu}{\lambda}(\alpha^2-\tfrac{1}{2})^{1/2}\ ,$$

as exemplified by curve (b). For $t\leq\tau_\nu$ there is no real-valued phase point $m_\nu^{(\pm)}$. In the remainder of this article we focus on the echo aspect of the revivals and therefore confine ourselves to times $t\geq\tau_\nu$.

(iii) A point of stationary phase exists even for $\nu=0$ as shown by the trajectory (a). This is in contrast to the discussion of Sec. II A, where no such point occurred.

Motivated by these considerations we now proceed to find an approximate analytical result for $m_\nu^{(\pm)}$. Here the substitution

$$\arctan x\leftrightarrow\frac{x}{(1+x^2)^{1/2}}\ . \tag{3.8}$$

guides us in our quest. This is a quite remarkable substitution when we recognize that the expressions in Eq. (3.8) do not agree at any x value except for $x=0$. Nevertheless the points of stationary phase, now given by

$$\pm\frac{(m_\nu^{(\pm)}+\tfrac{1}{2}-\alpha^2)^{1/2}}{(m_\nu^{(\pm)}+\tfrac{1}{2})^{1/2}}+\pi\nu=\frac{\lambda t}{2(m_\nu^{(\pm)})^{1/2}} \tag{3.9a}$$

rather than Eq. (3.7), are only slightly affected by this "brutal" replacement. We can simplify Eq. (3.9) even further when we recall that according to Fig. 3 and Eqs. (3.4c) and (3.4d) the values of $m_\nu^{(\pm)}$ must be larger than $\alpha^2\gg 1$. We can therefore neglect the contribution $\frac{1}{2}$ in the denominator of Eq. (3.9a) and arrive at the quadratic equation for $(m_\nu^{(\pm)})^{1/2}$,

$$0=\pm\frac{(m_\nu^{(\pm)}+\tfrac{1}{2}-\alpha^2)^{1/2}}{(m_\nu^{(\pm)})^{1/2}}+\pi\nu-\frac{\lambda t}{2(m_\nu^{(\pm)})^{1/2}}\ , \tag{3.9b}$$

which has the solutions

$$(m_\nu^{(\pm)})^{1/2}=\frac{\pi\nu\lambda t}{2(\pi^2\nu^2-1)}$$
$$\pm\frac{1}{2(1-\pi^2\nu^2)}[\lambda^2t^2+4(1-\pi^2\nu^2)(\alpha^2-\tfrac{1}{2})]^{1/2}\ . \tag{3.10}$$

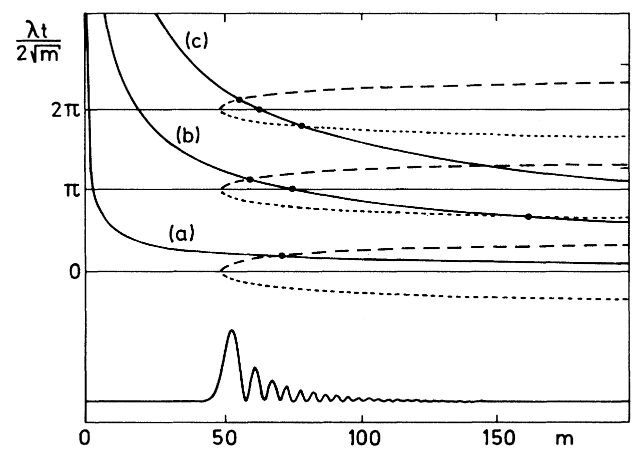

FIG. 3. Graphical solution of the transcendental equation (3.7) determining the points of stationary phase, m_ν and $m_\nu^{(\pm)}$, of the integrals a_ν and $b_\nu^{(\pm)}$, Eq. (3.4). The point of stationary phase m_ν—given by $\lambda t/(2\sqrt{m_\nu})=\pi\nu$—is the crossing between the horizontal lines at $\pi\nu$ and the solid decaying trajectory. The oscillations in $W(m)$ shown in the lower part of the figure modify this picture for the case of $b_\nu^{(\pm)}$ and replace each constant horizontal line by two arctangent functions of different signs indicated here by dashed and dotted lines. We note an intersection between the solid trajectory (a) and the dashed curve at $\nu=0$, which leads to a prolongation of the Cummings collapse. For appropriately large times t we find intersections between the decaying trajectory (b) with both the dashed and dotted lines giving rise to two points of stationary phase $m_\nu^{(\pm)}$. However, only when m_ν and $m_\nu^{(\pm)}$ lie within the dominant region of $W(m)$ do they contribute. In curve (b), the contribution from the value $m_1^{(-)}$ is negligible and the two remaining points of stationary phase, m_1 and $m_1^{(+)}$, *interfere* and lead to the echos of Fig. 4. For higher ν values, as indicated by the curve (c) for $\nu=2$, the point $m_\nu^{(-)}$ moves towards the dominant region of $W(m)$ and gives rise to a third contribution to $w(t)$ and a beat note in the echos, as shown in Fig. 5 for the second revival.

In the next two subsections we discuss the revivals ($\nu>0$) and the Cummings collapse ($\nu=0$) resulting from the points of stationary phase m_ν, Eq. (3.5), and $m_\nu^{(\pm)}$, Eq. (3.10).

B. Beating between stationary phases creates echos

We now study the phenomenon of revivals, that is we calculate the integrals a_ν and $b_\nu^{(\pm)}$ for $\nu \geq 1$. Since these integrals are of the type discussed in Sec. II A—the integrand consists of a slowly varying envelope and a rapidly oscillating function—they follow from Eq. (2.5). We therefore substitute the so-calculated points of stationary phase, Eqs. (3.5) and (3.10), into the phases S_ν and $S_\nu^{(\pm)}$, as well as their second derivatives. The integral a_ν, Eq. (3.4b), follows immediately from Eq. (2.8b) and we find

$$a_\nu(t)=\tilde{a}_\nu(t)\cos[2S_\nu(t)+\pi/4] , \tag{3.11a}$$

where

$$\tilde{a}_\nu(t)=-\tfrac{1}{2}A\left[m=\frac{\lambda^2t^2}{4\pi^2\nu^2}\right]\frac{\lambda t}{\pi(2\nu^3)^{1/2}} \tag{3.11b}$$

and

$$S_\nu(t)=-\frac{\lambda^2t^2}{4\pi\nu} . \tag{3.11c}$$

After some minor algebra, which is shown in Appendix A, the resulting expressions for the integrals $b_\nu^{(\pm)}$ read

$$b_\nu^{(\pm)}(t)=\tilde{b}_\nu^{(\pm)}(t)\cos\{2[S_\nu(t)+\delta S_\nu^{(\pm)}(t)]+\pi/4\} , \tag{3.12a}$$

where the amplitudes $\tilde{b}_\nu^{(\pm)}(t)$ are given by

$$\tilde{b}_\nu^{(\pm)}(t)=-\frac{1}{8}\frac{W(m)}{\cos^2\phi_m}\bigg|_{m=m_\nu^{(\pm)}}\frac{\lambda t}{\pi(2\nu^3)^{1/2}} \times\left[\frac{t\mp\pi\nu\left[t^2-\tau_\nu^2+\frac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}}{(t-\tau_\nu)\mp\pi\nu\left[t^2-\tau_\nu^2+\frac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}}\right]^{1/2} , \tag{3.12b}$$

and the phase corrections due to the oscillations in W_m are

$$\delta S_\nu^{(\pm)}(t)=\frac{\lambda^2}{4\pi^3\nu^3}(t-\tau_\nu) \times\left[t\mp\pi\nu\left[t^2-\tau_\nu^2+\frac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}\right]\pm\frac{\pi}{4} . \tag{3.12c}$$

We are now in the position to discuss the details of the νth revival. According to Eq. (3.4a), w_ν consists of the sum of the three contributions a_ν, $b_\nu^{(+)}$, and $b_\nu^{(-)}$, given by Eqs. (3.11) and (3.12), that is,

$$w_\nu(t)=\tilde{a}_\nu(t)\cos\left[2S_\nu(t)+\frac{\pi}{4}\right]+\tilde{b}_\nu^{(+)}(t)\cos\left[2S_\nu^{(+)}(t)+\frac{\pi}{4}\right]+\tilde{b}_\nu^{(-)}(t)\cos\left[2S_\nu^{(-)}(t)+\frac{\pi}{4}\right] . \tag{3.13}$$

The magnitude of the amplitudes $\tilde{a}_\nu$ and $\tilde{b}_\nu^{(+)}$ are approximately the same for times $t^2 \geq \tau_\nu^2$. Moreover for small values of ν the amplitude $\tilde{b}_\nu^{(-)}$ is small since the point of stationary phase $m_\nu^{(-)}$ is located in the exponential tail of the photon-number distribution, as shown by curve (b) of Fig. 3. We are therefore led to combine the $b_\nu^{(+)}$ oscillation with the a_ν oscillation, that is,

$$w_\nu(t)\simeq 2\tilde{a}_\nu\cos[2S_\nu(t)+\pi/4+\delta S_\nu^{(+)}(t)]\cos[\delta S_\nu^{(+)}(t)] , \tag{3.14}$$

where we have approximated $\tilde{b}_\nu^{(+)}$, by $\tilde{a}_\nu$ and neglected the $b_\nu^{(-)}$ term. Hence the time dependence of w_ν consists mainly of the Rabi oscillations—described by the first cosine term—modulated by the oscillations due to the phase correction from the oscillatory photon statistics. It is this modulation which forms the echos.

In Fig. 4 we compare and contrast the exact numerical treatment [17] of the first revival to the stationary phase result (3.13). For higher revivals the stationary-phase point $m_\nu^{(-)}$ moves towards smaller values of m and hence at this point the amplitude $\tilde{b}_\nu^{(-)}$ gains in importance. This brings in a third frequency, which is apparent in the

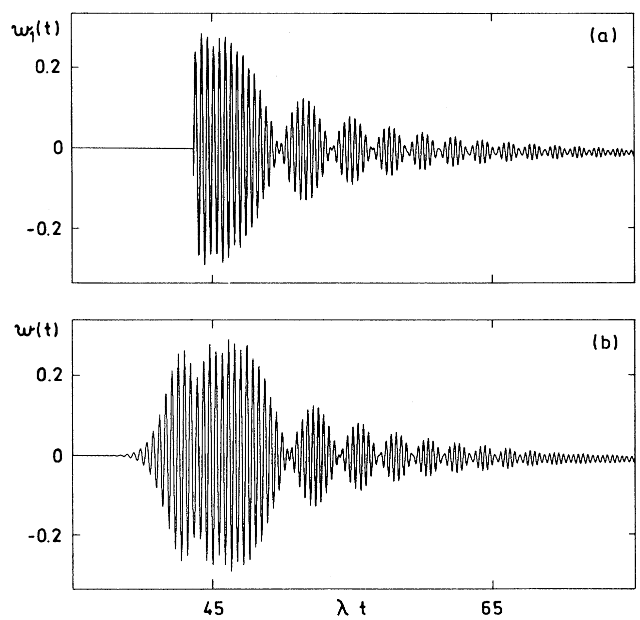

FIG. 4. Comparison between the approximate analytical expression for the first revival, Eq. (3.13), shown in (a) for times $t>\tau_1$, and the exact numerical calculation of the Jaynes-Cummings sum, Eq. (1.1), depicted in (b). Here we have used a squeezed state that has squeeze and displacement parameters $s=21$ and $\alpha=7$, respectively.

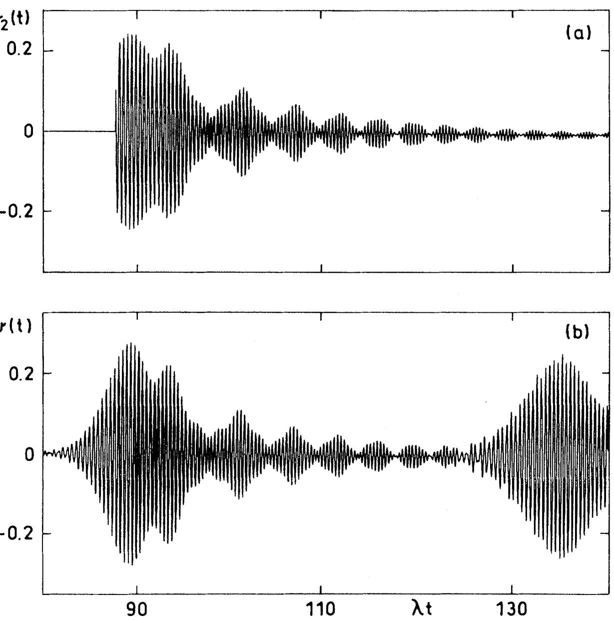

FIG. 5. Comparison between the approximate analytical expression for the *second* revival w_2, shown in (a), and the numerical evaluation of the Jaynes-Cummings sum, Eq. (1.1), depicted in (b). The parameters are identical to those of Fig. 4. Note the appearance of the beat in the echos arising from the third stationary phase point $m_2^{(-)}$. Moreover, the *third* revival already makes its appearance in the numerical representation, shown in (b).

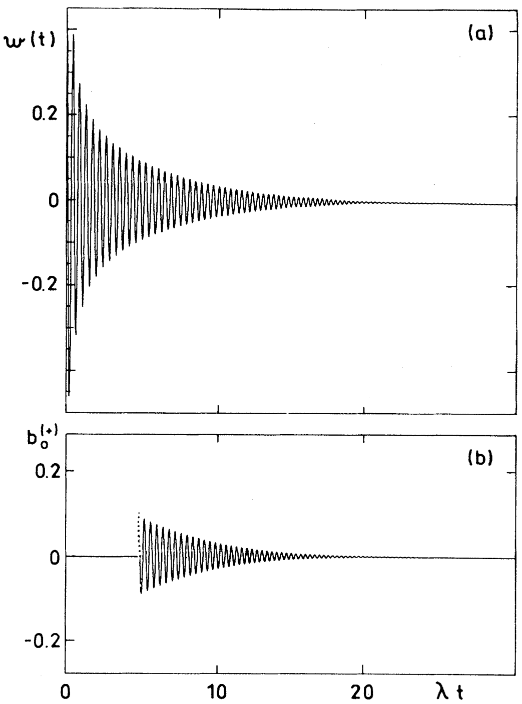

FIG. 6. The enhanced lifetime of the exact collapse, shown in (a), results from the existence of the stationary phase point $m_0^{(+)}$ which creates the curve shown in (b). Here we have chosen parameters identical to those in Figs. 4 and 5.

beating of the Rabi oscillations, that is, in the modulation of the echos shown in Fig. 5 for the case of the second revival.

C. Oscillations in $W(m)$ enhance lifetime

We conclude this section by analyzing the effect of the oscillatory photon distribution on the Cummings collapse. The method of stationary phase as discussed in Sec. III A fails to provide a stationary point for a_0 and $b_0^{(-)}$ [Eqs. (3.5) and (3.10)]. We have to evaluate the integrals directly. However the integral $b_0^{(+)}$ enjoys a point of stationary phase governed by

$$(m_0^{(+)}+\tfrac{1}{2}-\alpha^2)^{1/2}=\frac{\lambda t}{2}\ . \tag{3.15}$$

Since we are only interested in the tail of the Cummings collapse we use in the stationary-phase calculation the asymptotic expression Eq. (3.2a) for $W(m)$ and obtain after minor algebra, shown in Appendix B,

$$\tilde{b}_0^{(+)}(t)=-\frac{1}{2}\left[\frac{\epsilon}{\lambda t}\right]^{1/2}\exp\left[-\epsilon\frac{\lambda^2t^2}{4}\right]\times\frac{[\lambda^2t^2+4\alpha^2-2]^{3/4}}{[2\lambda^2t^2+4\alpha(\lambda^2t^2+4\alpha^2-2)^{1/2}]^{1/2}}\ , \tag{3.16}$$

which is valid for times t appropriately larger than zero. In Fig. 6(a) we show the Cummings collapse as calculated numerically from the sum Eq. (1.1). In Fig. 6(b) we display the quantity $b_0^{(+)}$ from Eq. (3.13), (3.16), which is the origin of the long tail of the collapse. Hence the oscillations of the photon distribution of the highly squeezed state which give rise to the additional stationary-phase point (3.15) enhance the decay time of the Cummings collapse.

IV. PHOTON-NUMBER DISTRIBUTION FROM MEASUREMENT OF ATOMIC INVERSION

Under appropriate conditions the stationary state of the radiation field in a one-atom micromaser can be a nonclassical state of sub-Poissonian photon statistics [8]. But how does one measure the photon distribution of the maser field? Standard photon-count experiments are out of question. Coupling the radiation out the superconducting cavity destroys the high-quality factor and hence the maser action. To deduce from the statistics of the *excited* atoms leaving the cavity the corresponding photon statistics inside the cavity represents one possibility which has recently been experimentally realized [9]. Another way to get a handle on the phenomena taking place inside the cavity without coupling light out consists of realizing the real Jaynes-Cummings model in the one-atom maser [24]: Generate a stationary electromagnetic field in the cavity by pumping it by one strong atomic beam. Probe the corresponding stationary maser photon distribution $W_m^{(\mathrm{st})}$ by a second dilute atomic beam and record the atomic inversion as a function of the interaction time; this is the approach introduced in this section.

Under the conditions described above the dynamical behavior of the probe atom follows that of the Jaynes-Cummings model and the atomic inversion is given by Eq. (1.1),

$$w(t)=-\tfrac{1}{2}\sum_{m=0}^{\infty} W_m^{(\mathrm{st})}\cos(2\lambda t\sqrt{m}\,)\ . \tag{4.1}$$

How does one invert this equation to obtain from $w(t)$ the stationary photon distribution? The first idea of using the Fourier transform in time t fails because it involves an integration over all times, and therefore the measurement of $w(t)$ over an infinite time. But would the collapse or a single revival, described by one term w_ν, be enough to obtain $W_m^{(\mathrm{st})}$? Yes, provided the collapse and the revivals are well separated in time. We will demonstrate the simple calculation for the collapse only and assume that

$$w_0(t)=-\tfrac{1}{2}\int_0^{\infty} dm\ W^{(\mathrm{st})}(m)\cos(2\lambda t\sqrt{m}\,) \tag{4.2}$$

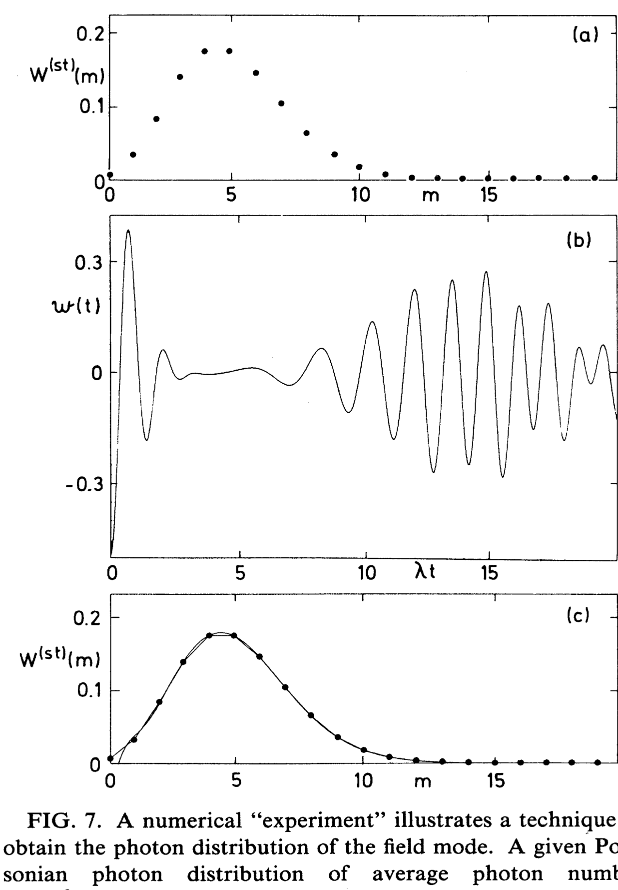

FIG. 7. A numerical "experiment" illustrates a technique to obtain the photon distribution of the field mode. A given Poissonian photon distribution of average photon number $\overline{m}=\alpha^2=5$, shown in (a), creates the collapse and the first revival shown in (b) and is to be measured by experimental means. We obtain from these data, that is, from the collapse alone via Eq. (4.4), the *continuous* photon distribution compared to the exact *discrete* distribution in (c).

can be determined experimentally. When we multiply $w_0(t)$ by e^{iyt} and integrate with respect to time we find

$$\int_{-\infty}^{\infty} dt\ w_0(t)e^{iyt}=-\frac{\pi}{2}\int_0^{\infty} dm\ W^{(\mathrm{st})}(m)\times[\delta(y-2\lambda\sqrt{m}\,)+\delta(y+2\lambda\sqrt{m}\,)]\ . \tag{4.3}$$

For positive y values the second term vanishes and we arrive for $m>0$ at

$$W^{(\mathrm{st})}(m)=-\frac{2\lambda}{\pi\sqrt{m}}\int_{-\infty}^{\infty} dt\ w_0(t)\cos(2\lambda t\sqrt{m}\,)\ . \tag{4.4}$$

Similar calculations with any revival $w_\nu(t)$ lead to different functions $W^{(\mathrm{st})}(m)$, which, however, have the same values for integer m as for the case of the collapse. Hence, if we take into account the discreteness of the photon distribution, we are able to determine the stationary photon-number statistics of the micromaser field by a measurement of a (separable) revival or the collapse. We illustrate this technique in the example of Fig. 7.

V. SUMMARY AND CONCLUSIONS

How does one perform analytically—at least in an approximate way—the Jaynes-Cummings sum, Eq. (1.1), determining the atomic inversion in the presence of an arbitrary quantized field mode? This has been the central question discussed in the present article. But why is it so difficult to calculate this sum analytically? The Jaynes-Cummings sum is not a Fourier sum: the summation index enters as a square root. Why not replace summation by integration? It is a well-established fact [2] that under these conditions we recover the Cummings *collapse* but miss out on the *revivals*. It has been emphasized repeatedly that the revivals are a consequence of the granular structure of the photon distribution and hence of the discreteness of the m values. The Poisson summation formula—the key to an analytical treatment of the revivals—illuminates this statement from a different angle: To replace the discrete summation over m by an integral over m corresponds to neglecting in the Poisson summation formula all terms except the term $\nu=0$. This contribution corresponds to the collapse only, whereas the $\nu>0$ terms represent the revivals. This fact comes to light when we evaluate these integrals with the method of stationary phase. Two examples for a quantized field mode—a coherent state and a highly squeezed state—illustrate the central points of this approach [25]. In the case of a coherent state the photon distribution is slowly varying compared to the oscillations of the term $\exp[2iS_\nu(m)]$. In this case only a single point of stationary phase of S_ν exists, leading to the rapid Rabi oscillations with an envelope given by the rescaled photon distribution $W(m)$. In the case of a highly squeezed state we have to include the oscillations in $W(m)$ in the stationary-phase analysis, giving birth to additional points which *interfere* [18,26]. This creates a beating in the revivals—the echos—beats in the echos and a prolonged collapse.

Does the Poisson summation illumination of the

Jaynes-Cummings sum suggest an idea of how to measure the photon statistics underlying the dynamics of an atom in the one-atom maser? Yes. To observe the collapse or a single revival is enough to deduce the photon distribution.

We could only gain insight into the Jaynes-Cummings sum hiding the quantum phenomenon of revivals by applying the tools of semiclassical quantum mechanics: the method of stationary phase combined with the Poisson summation equation.

ACKNOWLEDGMENTS

We thank M. V. Berry, L. Davidovich, J. P. Dowling, J. Eberly, P. Knight, P. Meystre, G. Milburn, J. M. Raimond, H. Risken, M. O. Scully, and H. Walther for many fruitful and stimulating discussions on this topic. One of us (M.F.) would like to express his appreciation for the hospitality during his stay at the Max-Planck-Institut für Quantenoptik, Garching, Germany.

APPENDIX A: CALCULATION OF PHASES AND ENVELOPES OF INTEGRALS $b_\nu^{(\pm)}$ FOR $\nu>0$

In this appendix we evaluate the integrals $b_\nu^{(\pm)}$ with the help of the points of stationary phase

$$(m_\nu^{(\pm)})^{1/2}=\frac{\pi\nu\lambda t}{2(\pi^2\nu^2-1)} \mp\frac{1}{2(\pi^2\nu^2-1)}[\lambda^2t^2+4(1-\pi^2\nu^2)(\alpha^2-\tfrac{1}{2})]^{1/2} .$$

For $\nu\geq 1$ we can simplify this expression when we neglect in the denominator of the second term the contribution of unity compared to $\pi^2\nu^2$, expand the denominator of the first term, make use of the definition of τ_ν, and arrive at

$$(m_\nu^{(\pm)})^{1/2}=\frac{\lambda t}{2\pi\nu}+\frac{\lambda}{2\pi^3\nu^3}\left[t\mp\pi\nu\left[t^2-\tau_\nu^2+\frac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}\right] . \tag{A1}$$

Hence the oscillations in $W(m)$ have modified the stationary-phase point $\lambda t/(2\pi\nu)$ of a_ν, Eq. (3.5). We now use this expression to evaluate the phases $S_\nu^{(\pm)}(m=m_\nu^{(\pm)})$.

According to Eqs. (3.4e), (3.2c), and (2.2c) these phases $S_\nu^{(\pm)}$ read

$$S_\nu^{(\pm)}(m=m_\nu^{(\pm)})=\pi\nu m_\nu^{(\pm)}-\lambda t(m_\nu^{(\pm)})^{1/2}\pm\{(m_\nu^{(\pm)}+\tfrac{1}{2})\arctan[(m_\nu^{(\pm)}+\tfrac{1}{2}-\alpha^2)^{1/2}/\alpha]-\alpha(m_\nu^{(\pm)}+\tfrac{1}{2}-\alpha^2)^{1/2}-\frac{\pi}{4}\} . \tag{A2}$$

We substitute the arctangent via Eq. (3.7) and neglect the contribution of $\frac{1}{2}$ compared to $m_\nu^{(\pm)}$ in the prefactor of the arctangent term in Eq. (A2) and obtain

$$S_\nu^{(\pm)}=-\frac{\lambda t}{2}(m_\nu^{(\pm)})^{1/2}\mp\alpha(m_\nu^{(\pm)}+\tfrac{1}{2}-\alpha^2)^{1/2}\mp\frac{\pi}{4} .$$

With the help of Eq. (3.9b) we eliminate the square-root expression in the second term.

$$S_\nu^{(\pm)}=-\frac{\lambda t}{2}(m_\nu^{(\pm)})^{1/2}+\alpha\left[\pi\nu(m_\nu^{(\pm)})^{1/2}-\frac{\lambda t}{2}\right]\mp\frac{\pi}{4} . \tag{A3}$$

We now substitute the expression for $m_\nu^{(\pm)}$, Eq. (A1) into this result and find

$$S_\nu^{(\pm)}=-\frac{\lambda^2t^2}{4\pi\nu}-\frac{\lambda^2}{4\pi^3\nu^3}(t-\tau_\nu) \times\left[t\mp\pi\nu\left[t^2-\tau_\nu^2+\frac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}\right]\mp\frac{\pi}{4} . \tag{A4}$$

Let us now calculate the envelope $\tilde{b}_\nu^{(\pm)}$. According to Eqs. (2.5), (3.4c), and (3.4d) the amplitudes $\tilde{b}_\nu^{(\pm)}(t)$ read

$$\tilde{b}_\nu^{(\pm)}(t)=-\frac{1}{8}\left.\frac{W(m)}{\cos^2\phi_m}\right|_{m=m_\nu^{(\pm)}} \times\left[\pi\Big/\left|\left.\frac{\partial^2S_\nu^{(\pm)}(m)}{\partial m^2}\right|_{m=m_\nu^{(\pm)}}\right|\right]^{1/2} . \tag{A5}$$

We obtain the second derivative of the phase $S_\nu^{(\pm)}$ immediately from Eq. (3.6)

$$\left.\frac{\partial^2S_\nu^{(\pm)}(m)}{\partial m^2}\right|_{m=m_\nu^{(\pm)}}=\frac{\lambda t}{4(m_\nu^{(\pm)})^{3/2}}\pm\frac{\alpha}{2(m_\nu^{(\pm)}+\frac{1}{2})(m_\nu^{(\pm)}+\frac{1}{2}-\alpha^2)^{1/2}} . \tag{A6}$$

We eliminate the square-root expression in the second term with the help of Eq. (3.9a), which yields

$$\left.\frac{\partial^2S_\nu^{(\pm)}(m)}{\partial m^2}\right|_{m=m_\nu^{(\pm)}}=\frac{\lambda t}{4(m_\nu^{(\pm)})^{3/2}}+\frac{\alpha(m_\nu^{(\pm)})^{1/2}}{2(m_\nu^{(\pm)}+\frac{1}{2})^{3/2}}\left[\frac{\lambda t}{2}-\pi\nu(m_\nu^{(\pm)})^{1/2}\right]^{-1} . \tag{A7}$$

When we substitute the expression for $m_\nu^{(\pm)}$, Eq. (A1), into this expression and neglect the contribution of $\frac{1}{2}$ in the denominator of the second term, we find

$$\left.\frac{\partial^2 S_\nu^{(\pm)}(m)}{\partial m^2}\right|_{m=m_\nu^{(\pm)}}=\frac{\lambda t}{4(m_\nu^{(\pm)})^{3/2}}-\frac{\alpha}{m_\nu^{(\pm)}\dfrac{\lambda}{\pi^2\nu^2}\left[t\mp\pi\nu\left[t^2-\tau_\nu^2+\dfrac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}\right]}$$

$$=\frac{\lambda t}{4(m_\nu^{(\pm)})^{3/2}}\left[1-\frac{4\alpha\pi^2\nu^2(m_\nu^{(\pm)})^{1/2}}{\lambda^2 t\left[t\mp\pi\nu\left[t^2-\tau_\nu^2+\dfrac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}\right]}\right]. \tag{A8}$$

According to Eq. (A5) the second derivative of $S_\nu^{(\pm)}$ only enters in the denominator of $b_\nu^{(\pm)}$. Therefore these amplitudes do not depend sensitively on the points of stationary phase. We can therefore approximate $m_\nu^{(\pm)}$ in this equation by m_ν, Eq. (2.7), and find

$$\left.\frac{\partial^2 S_\nu^{(\pm)}(m)}{\partial m^2}\right|_{m=m_\nu^{(\pm)}}=\frac{\lambda t}{4m_\nu^{3/2}}\left[\frac{(t-\tau_\nu)\mp\pi\nu\left[t^2-\tau_\nu^2+\dfrac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}}{t\mp\pi\nu\left[t^2-\tau_\nu^2+\dfrac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}}\right]. \tag{A9}$$

Note that for $t>\tau_\nu$ this expression is always positive. With this result we find for the envelopes $\tilde{b}_\nu^{(\pm)}$, Eq. (A5),

$$\tilde{b}_\nu^{(\pm)}(t)=-\frac{1}{8}\left.\frac{W(m)}{\cos^2\phi_m}\right|_{m=m_\nu^{(\pm)}}\frac{\lambda t}{\pi(2\nu^3)^{1/2}}\left[\frac{t\mp\pi\nu\left[t^2-\tau_\nu^2+\dfrac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}}{(t-\tau_\nu)\mp\pi\nu\left[t^2-\tau_\nu^2+\dfrac{\tau_\nu^2}{\pi^2\nu^2}\right]^{1/2}}\right]^{1/2}. \tag{A10}$$

APPENDIX B: CALCULATION OF THE PHASE AND ENVELOPE OF THE INTEGRAL $b_0^{(+)}$

We obtain the stationary phase $S_0^{(+)}$ from Eq. (A3) by setting $\nu=0$ and making use of Eq. (3.15).

$$S_0^{(+)}=-\frac{\lambda t}{2}(m_0^{(+)})^{1/2}-\alpha\frac{\lambda t}{2}-\frac{\pi}{4}$$

$$=-\frac{[\lambda t}{4}\{[\lambda^2t^2+4(\alpha^2-\tfrac{1}{2})]^{1/2}+2\alpha\}-\frac{\pi}{4}. \tag{B1}$$

The second derivative of the phase $S_0^{(+)}$ follows from Eq. (A7) and Eq. (3.15),

$$\left.\frac{\partial^2 S_0^{(+)}(m)}{\partial m^2}\right|_{m=m_0^{(+)}}=\frac{\lambda t}{4(m_0^{(+)})^{3/2}}+\frac{\alpha}{\lambda t m_0^{(+)}}$$

$$=\frac{2\lambda t}{(\lambda^2t^2+4\alpha^2-2)^{3/2}}+\frac{4\alpha}{\lambda t(\lambda^2t^2+4\alpha^2-2)}$$

$$=\frac{2\lambda^2t^2+4\alpha(\lambda^2t^2+4\alpha^2-2)^{1/2}}{\lambda t(\lambda^2t^2+4\alpha^2-2)^{3/2}}. \tag{B2}$$

We are interested in the tail of the Cummings collapse, that is, in times t appropriately larger than zero. When we use the asymptotic expression, Eq. (3.2), for $W(m)$, that is,

$$\left.\frac{W(m)}{4\cos^2\phi_m}\right|_{m=m_0^{(+)}}=\mathcal{A}_{m=m_0^{(+)}}=\left[\frac{\epsilon}{\pi}\right]^{1/2}\frac{\exp\left[-\epsilon\dfrac{\lambda^2t^2}{4}\right]}{\lambda t}, \tag{B3}$$

the amplitude $\tilde{b}_0^{(+)}(t)$, Eq. (A5), reads

$$\tilde{b}_0^{(+)}=-\frac{1}{2}\left[\frac{\epsilon}{\lambda t}\right]^{1/2}\exp\left[-\epsilon\frac{\lambda^2t^2}{4}\right]\frac{[\lambda^2t^2+4\alpha^2-2]^{3/4}}{[2\lambda^2t^2+4\alpha(\lambda^2t^2+4\alpha^2-2)^{1/2}]^{1/2}}. \tag{B4}$$

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