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Entanglement of collectively interacting harmonic chains: An effective two-dimensional system

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We study the ground-state entanglement of one-dimensional harmonic chains that are coupled to each other by a collective interaction as realized, e.g., in an anisotropic ion crystal. Due to the collective type of coupling, where each chain interacts with every other one in the same way, the total system shows critical behavior in the direction orthogonal to the chains, while the isolated harmonic chains can be gapped and noncritical. We derive lower and most importantly upper bounds for the entanglement, quantified by the von Neumann entropy, between a compact block of oscillators and its environment. For sufficiently large size of the subsystems, the bounds coincide and show that the area law for entanglement is violated by a logarithmic correction.

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Presently there is a growing interest in the interrelation between entanglement and ground-state properties of many-body lattice models. For a number of spin systems [1], a strict correspondence between the absence of criticality, the presence of an energy gap, and an area law for the entanglement was established. The latter states that the entanglement of a compact subset of lattice sites with the rest of the system, measured by the von Neumann entropy, scales with the surface area of the subset. For critical spin systems it was shown that an additional logarithmic correction to the area law emerges. A similar relation between criticality and entanglement was suggested for harmonic lattice models [2,3]. In [4,5], an area law was established for harmonic lattice models in arbitrary dimensions with nearest-neighbor coupling which have a gapped spectrum. For finite-range couplings in one dimension a one-to-one correspondence between the validity of the area law and noncriticality was established in [6], and logarithmic corrections were derived for critical systems.

Although the relation between criticality and entropy-area law seems rather universal, there are a number of examples where this relation does not hold [5,7]. Until now there is no general understanding of the conditions for the validity of an entropy-area law in particular in higher dimensions [1,4,5,8]. In the present paper, we discuss a specific gapless oscillator model with dimension larger than 1, for which an exact asymptotic expression for the entropy can be obtained. Due to the collective nature of the interactions in one spatial direction, the system is critical and thus a violation of the area law is expected. We here derive a lower and, most importantly, a tight upper bound for the entropy, and in this way obtain an exact form of the correction term to the area law.

Let us consider a set of parallel harmonic chains (see Fig. 1) each containing n_x oscillators, with $n_x \to \infty$ in the thermodynamic limit. We will refer to the direction parallel to the chains as the x axis, and to the orthogonal direction as the y axis. The number of parallel chains is denoted as n_y, again with $n_y \to \infty$ in the thermodynamic limit. The oscillators are described by the canonical variables (q_i, p_i), where $i = 1, 2, \ldots, N$ $(N = n_x n_y)$ is a collective index that labels the oscillator. We adopt the following notation: $i = 1, \ldots, n_x$ correspond to the oscillators in the first chain with growing x coordinate, $i = n_x + 1, \ldots, 2n_x$ corresponds to oscillators in the second chain, and so on. We consider a quadratic Hamiltonian of the form

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} V_{ij} q_i q_j, \tag{1}$$

with a coupling matrix V. We are interested only in a translationally invariant coupling, i.e., we assume that the matrix elements of V depend only on the difference of the x coordinates and the difference of the y coordinates. Hence V is a block Toeplitz matrix. For oscillator systems with a quadratic coupling of the form of Eq. (1), the ground state

$$\Psi_0(\mathbf{q}) \sim \exp\left(-\frac{1}{2}\langle \mathbf{q}|V^{1/2}|\mathbf{q}\rangle\right) \tag{2}$$

and all its properties, such as, e.g., the correlation length in position or momentum space, are determined by the square root of V, where $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ is the vector of position variables. The ground state can easily be determined if V is the square of another matrix, which we assume to be again a Toeplitz matrix,

$$V = Z^2/n_y. \tag{3}$$

The factor $1/n_y$ is chosen such that the matrix elements of V remain finite in the limit $N \to \infty$. Assuming Z to be a Toeplitz matrix guarantees that the coupling V is a Toeplitz matrix as well. We furthermore consider Z to be of the block-matrix form

$$Z = \begin{bmatrix} \Lambda & Q & Q & . & . & Q \\ Q & \Lambda & Q & . & . & Q \\ Q & Q & \Lambda & . & & \\ & . & & & . & Q \\ & & . & & & Q \\ Q & & & Q & Q & \Lambda \end{bmatrix}. \tag{4}$$

The elements of Z are $n_x \times n_x$ matrices and are characterized according to the correlations of Eq. (2). The diagonal elements of Z describe correlations within one chain, i.e., in the x direction, the off-diagonal elements describe correlations

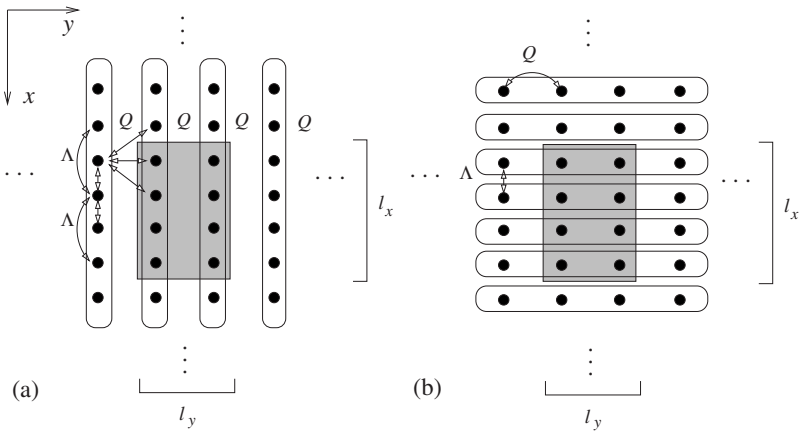

FIG. 1. (a) Collectively interacting strings of harmonic oscillators with finite-range intrachain coupling Λ and collective interchain coupling Q. The gray area indicates the subsystem I of oscillators. (b) Alternative view: interacting strings with collective intrachain coupling Q and finite-range interchain coupling Λ.

between the chains. Λ and Q are both assumed to be Toeplitz matrices of finite range, i.e., their matrix elements Λ_k and Q_k, where $\Lambda_k \equiv \Lambda_{k=|i-j|} = \langle i|\Lambda|j\rangle$, vanish exactly for $k \geqslant R$. The finite range of Λ and Q ensures that the interaction V is of finite range within the chains, while the form of Z implies that V is constant orthogonal to the chains. We assume furthermore that Λ, Q, and $\Lambda - Q$ are positive definite matrices. A simple calculation shows that the ground state of V is degenerate and in the thermodynamic limit $n_x, n_y \to \infty$ has only one nonzero eigenvalue. This means that the total Hamiltonian, Eq. (1), is gapless. It should be noted, however, that the collective nature of the interactions is not sufficient for a gapless spectrum of the Hamiltonian.

Since all off-diagonal elements of Z are identical, correlations between oscillators do not depend on their distance in the y direction, and the total system is critical irrespective of the correlation properties within the chains. Thus one expects that the entropy-area law is broken. In fact one can easily find a lower bound to the entropy by the following simple argument: Let us consider a partition of the set of N oscillators into a compact subsystem I with $N_0 = l_x l_y$ and a subsystem II with $N - N_0$ oscillators (see Fig. 1). If we now consider harmonic chains in the y—rather than the x—direction, the y chains couple to each other with finite-range interaction Λ [see Fig. 1(b)]. We thus have reason to assume that $S \geqslant l_x S_0$, where S_0 is the entropy of a single y chain. Since the coupling within the chain is now collective (Q), the y chain itself is critical and its entropy scales as $S_0 \sim \ln l_y$. Thus $S \geqslant l_x \ln l_y$ which in the thermodynamic limit $\{l_x, l_y\} \to \infty$ is much larger than the surface area $2(l_x + l_y)$. While it is easy to see that the area law is broken, it is nontrivial to find an upper bound to the entropy and the exact form of the correction term. This will be done in the following.

Using the spectral representation of V, the correlation matrices $V^{1/2}$ and $V^{-1/2}$ can be decomposed as

$$V^{1/2} = [(\Lambda - Q) \otimes \mathbf{1}_y + n_y Q \otimes \mathcal{P}_{n_y,n_y}]/\sqrt{n_y} \tag{5}$$

and

$$V^{-1/2} = \{(\Lambda - Q)^{-1} \otimes \mathbf{1}_y + [(\Lambda - Q + n_y Q)^{-1} - (\Lambda - Q)^{-1}] \otimes \mathcal{P}_{n_y,n_y}\}\sqrt{n_y}, \tag{6}$$

where $\mathbf{1}_y$ is the unity matrix of size $n_y \times n_y$ and $\mathcal{P}_{nm} = |P_{nm}\rangle\langle P_{nm}|$ is the projector onto the (in general nonnormalized) vector

$$|P_{nm}\rangle = \frac{1}{\sqrt{n}}(\underbrace{1,1,\ldots,1}_{m})^T.$$

Following Refs. [2–4,9], the von Neumann entropy or the entropy of entanglement of the two compact parts I and II can be calculated from a decomposition of $V^{1/2}$ into the two subsystems. To this end we express $V^{1/2}$ and $V^{-1/2}$ in a block form according to the two subsystems by proper reordering of rows and columns

$$V^{-1/2} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad V^{1/2} = \begin{bmatrix} D & E \\ E^T & F \end{bmatrix}. \tag{7}$$

Here A and D are $N_0 \times N_0$ matrices describing correlations within subsystem I, C and F are $(N - N_0) \times (N - N_0)$ matrices describing correlations within subsystem II, and the matrices B and E describe the correlations between them. The entropy of entanglement is then given by the eigenvalues $\mu_i \geqslant 1$ of the matrix product $A \cdot D$ [4]:

$$S = \sum_{i=1}^{N_0} f(\sqrt{\mu_i}), \tag{8}$$

$$f(x) = \frac{x+1}{2} \ln \frac{x+1}{2} - \frac{x-1}{2} \ln \frac{x-1}{2}. \tag{9}$$

Despite the simplicity of its form, expression (8) cannot be explicitly evaluated in general. Due to the special interaction matrix the eigenvalues can, however, be evaluated in the thermodynamic limit.

From the spectral decomposition of $V^{1/2}$, Eq. (5), one easily finds that the subsystem matrices read

$$A = [A_0 \otimes \mathbf{1}_{l_y} + (A_1 - A_0) \otimes \mathcal{P}_{n_y,l_y}]\sqrt{n_y},$$

$$D = [D_0 \otimes \mathbf{1}_{l_y} + n_y D_1 \otimes \mathcal{P}_{n_y,l_y}]/\sqrt{n_y}, \tag{10}$$

where A_0, A_1 and D_0, D_1 are $l_x \times l_x$ principal submatrices of $(\Lambda - Q)^{-1}, (\Lambda - Q + n_y Q)^{-1}$, and $(\Lambda - Q), Q$, respectively. For large n_y one has

$$A \cdot D \approx (A_0 \cdot D_0) \otimes \mathbf{1}_{l_y} + n_y (A_0 \cdot D_1) \otimes \mathcal{P}_{n_y,l_y}. \quad (11)$$

Here we have used that $\mathcal{P}^2_{n_y,l_y} = l_y/n_y \mathcal{P}_{n_y,l_y}$, which scales as $1/n_y$ for fixed l_y, and is thus negligible in the thermodynamic limit. Furthermore $\mathcal{P}_{n_y,l_y}$ has one nonzero eigenvalue l_y/n_y, which vanishes in the thermodynamic limit (l_y fixed and $n_y \to \infty$), and (l_y-1) zero eigenvalues. Thus the $l_x l_y$ eigenvalues of $A \cdot D$ can be decomposed into two sets. The first set consists of the l_x eigenvalues of $A_0 \cdot D_0$ each of which occurs (l_y-1) times:

$$\mu_1, \ldots, \mu_{l_y-1} = \alpha_1(A_0 \cdot D_0),$$

$$\mu_{l_y}, \ldots, \mu_{2(l_y-1)} = \alpha_2(A_0 \cdot D_0),$$

$$\vdots$$

$$\mu_{(l_x-1)(l_y-1)+1}, \ldots, \mu_{l_x(l_y-1)} = \alpha_{l_x}(A_0 \cdot D_0). \quad (12)$$

Here and in the following $\alpha_k(X)$ denotes the kth eigenvalues of the matrix X. The total number of these eigenvalues is $l_x(l_y-1)$. The second set consists of the l_x eigenvalues of $[A_0 \cdot D_0 + l_y(A_0 \cdot D_1)]$

$$\mu_k = \alpha_k[A_0 \cdot D_0 + l_y(A_0 \cdot D_1)]$$

$$\text{for } k = l_x(l_y-1)+1, \ldots, l_x l_y. \quad (13)$$

Expression (13) for the second set of eigenvalues can be simplified using Lidskii's theorem [10], which is stated as follows. Let X and Y be M-dimensional Hermitian matrices. Moreover, let $\alpha_k(X), \alpha_k(Y)$, and $\alpha_k(X-Y)$, $k=1,\ldots,M$, be the eigenvalues of X, Y, and $X-Y$, respectively, in ascending order $\{\alpha_1(X) \leqslant \alpha_2(X) \leqslant \cdots \leqslant \alpha_M(X)\}$. Then there exist numbers $w_{kj} \geqslant 0$ $(k,j=1,\ldots,M)$, such that $\Sigma_k w_{kj} = \Sigma_j w_{kj} = 1$ and

$$\alpha_k(X) = \alpha_k(Y) + \sum_{j=1}^{M} w_{kj} \alpha_j(X-Y). \quad (14)$$

Equation (14) implies that for sufficiently large l_y the eigenvalues of the matrix $A_0 \cdot D_0 + l_y(A_0 \cdot D_1)$ are

$$\alpha_k[A_0 \cdot D_0 + l_y(A_0 \cdot D_1)] \approx l_y \sum_{j=1}^{l_x} w_{kj} \alpha_j(A_0 \cdot D_1). \quad (15)$$

An *upper* bound to the entropy can be found by evaluating the sum over the eigenvalues (12) and (13) in Eq. (8) separately:

$$S = S_1 + S_2 = \sum_{j=1}^{l_x(l_y-1)} f(\sqrt{\mu_j}) + \sum_{j=l_x(l_y-1)+1}^{l_x l_y} f(\sqrt{\mu_j}). \quad (16)$$

Taking into account Eq. (12) one recognizes that S_1 is apart from a prefactor (l_y-1) formally equivalent to the von Neumann entropy of a linear oscillator chain of length l_x with interaction $\tilde{V} = (\Lambda - Q)^2$

$$S_1 = (l_y - 1) \sum_{k=1}^{l_x} f[\sqrt{\alpha_k(A_0 \cdot D_0)}]. \quad (17)$$

Since $\Lambda - Q$ was assumed to be strictly positive, the interaction $\tilde{V}$ has only nonzero eigenvalues and thus corresponds to a gapped oscillator chain. As shown in [4,6] the entropy of such a linear chain saturates in the thermodynamic limit, i.e., it becomes independent of the length l_x of the chain. Thus we have in the thermodynamic limit

$$S_1 \leqslant l_y c_1. \quad (18)$$

To obtain an upper bound to S_2 we use the inequality $f(x) < 1 - \ln 2 + \ln x$. This yields with Eq. (15)

$$S_2 < l_x(1 - \ln 2) + \frac{1}{2} \sum_{k=1}^{l_x} \ln\left(l_y \sum_{j=1}^{l_x} w_{kj} \alpha_j(A_0 \cdot D_1) \right). \quad (19)$$

To further evaluate the last term we make use of the convexity of the logarithm together with the arithmetic mean inequality

$$\begin{aligned} &\frac{1}{2} \sum_{k=1}^{l_x} \ln\left(l_y \sum_{j=1}^{l_x} w_{kj} \alpha_j(A_0 \cdot D_1) \right) \\ &\quad \leqslant \frac{l_x}{2} \ln\left(\frac{l_y}{l_x} \sum_{j=1}^{l_x} \sum_{k=1}^{l_x} w_{kj} \alpha_j(A_0 \cdot D_1) \right) \\ &\quad = \frac{l_x}{2} \ln\left(\frac{l_y}{l_x} \sum_{j=1}^{l_x} \alpha_j(A_0 \cdot D_1) \right), \end{aligned} \quad (20)$$

where we have used $\Sigma_k w_{kj} = 1$ in the last step.

We now have to evaluate the remaining logarithm. For this we make use of the fact that Λ and Q are regular (i.e., strictly positive) Toeplitz matrices. Because of this, their elements can be obtained from the non-negative spectral functions $\lambda(\theta)$ and $q(\theta)$ [11] $\Lambda_k = (1/2\pi) \int_0^{2\pi} \lambda(\theta) \exp(-ik\theta) d\theta$, $Q_k = (1/2\pi) \int_0^{2\pi} q(\theta) \exp(-ik\theta) d\theta$. Since we have assumed above that also $\Lambda - Q$ is strictly positive, the functions $\lambda(\theta)$, $q(\theta)$ are strictly positive and $\lambda(\theta) > q(\theta)$. In addition, we require also that $[\lambda(\theta) - q(\theta)]^{\pm 1}$ and $q(\theta)$ have bounded derivatives of second order. As a consequence, one finds (see [11] p. 221)

$$\frac{1}{l_x} \left(\sum_{j=1}^{l_x} \alpha_j(A_0 \cdot D_1) \right) \approx \frac{1}{2\pi} \int_0^{2\pi} \frac{q(\theta)}{\lambda(\theta) - q(\theta)} d\theta, \quad (21)$$

which is a constant independent of l_x. Thus the desired upper bound to the entropy for sufficiently large l_x, l_y is

$$S \leqslant c_1 l_y + c_2 l_x + \frac{l_x}{2} \ln l_y \quad (22)$$

where c_1, c_2 are some constants independent of the size of the subsystem.

A *lower* bound to the entropy can be found from the inequality $f(x) \geqslant \ln x$. This yields, with Eq. (15),

$$S \geqslant \frac{(l_y-1)}{2}\sum_{k=1}^{l_x} \ln[\alpha_k(A_0\cdot D_0)] + \frac{l_x}{2}\ln(l_y) + \frac{1}{2}\sum_{k=1}^{l_x} \ln\left(\sum_{j=1}^{l_x} w_{kj}\alpha_j(A_0\cdot D_1)\right). \quad (23)$$

Making use of Jensen's inequality for concave functions $\ln(\Sigma_j t_j\alpha_j)\geqslant\Sigma_j t_j \ln(\alpha_j)$ and $\Sigma_k w_{kj}=1$, we find

$$S \geqslant \frac{(l_y-1)}{2}\sum_{k=1}^{l_x} \ln[\alpha_k(A_0\cdot D_0)] + \frac{l_x}{2}\ln(l_y) + \frac{1}{2}\sum_{j=1}^{l_x} \ln[\alpha_j(A_0\cdot D_1)]. \quad (24)$$

To evaluate the sums over the logarithms we employ Szegö's theorem [11], for determinants of a Toeplitz matrices T. The theorem states that, for sufficiently large l_x

$$\ln[\det(T)] \approx q_0 l_x + \sum_{k=1}^{\infty} k|q_k|^2,$$

for a regular spectral function $q(\theta)$. Here q_k are the Fourier coefficients of $\ln q(\theta)$. Since, moreover,

$$\sum_j \ln[\alpha_j(A_0\cdot D_1)] = \ln\left(\prod_j \alpha_j(A_0\cdot D_1)\right) = \ln[\det(A_0)\det(D_1)], \quad (25)$$

we eventually find the lower bound

$$S \geqslant a_1 l_x + a_2 l_y + \frac{l_x}{2}\ln(l_y). \quad (26)$$

Here a_1 and a_2 are constants independent of the size of the subsystem, and we have ignored an unimportant constant term.

By combining the two estimates (22) and (26) one finds

$$c_1 l_x + c_2 l_y + \frac{l_x}{2}\ln(l_y) \geqslant S \geqslant a_1 l_x + a_2 l_y + \frac{l_x}{2}\ln(l_y).$$

Since both sides of this inequality have the same functional form, S approaches for large l_x, l_y the asymptotic value

$$S \approx \frac{l_x}{2}\ln(l_y), \quad l_x, l_y \gg 1. \quad (27)$$

This is the main result of our paper. It shows that the entropy-area law is violated for a set of harmonic chains, which for themselves have a gapped spectrum and are noncritical but become gapless by a collective interaction between the chains. Both upper and lower bounds to the entropy attain the same logarithmic correction term to the area law.

A physical system that can be approximated by the model studied here is an anisotropic ion crystal. In such a system the Coulomb interaction in the direction of the small lattice constant can in first approximation be considered as collective, while the one in an orthogonal direction is of finite range.

In conclusion, we derived an exact asymptotic expression for the entanglement entropy of a critical system of interacting oscillators in more than one dimension. We found that, as in one-dimensional systems [6], the entanglement area law is violated by a logarithmic correction proportional to the surface area in the critical direction. To our knowledge, the system of collectively interacting harmonic strings considered here, which is approximately realized, e.g., in an anisotropic ion crystal, is the first nontrivial example of a critical two-dimensional system for which the correction to the area law can explicitly be calculated.

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