Many-particle entanglement in the gaped antiferromagnetic Lipkin model

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Bipartite and global entanglement are analyzed for the ground state of a system of N spin-1/2 particles interacting via a collective spin-spin coupling described by the Lipkin-Meshkov-Glick Hamiltonian. Under certain conditions, which include the special case of supersymmetry, the ground state can be constructed analytically. In the case of antiferromagnetic coupling and for an even number of particles, the system has a finite energy gap and the ground state undergoes a smooth transition, as a function of the continuous anisotropy parameter γ, from a separable ($\gamma=\infty$) to a maximally entangled state ($\gamma=0$). From the analytic expression for the ground state, the bipartite entanglement between two subsets of spins as well as the global entanglement are calculated. Despite the absence of a quantum phase transition a discontinuous change of the scaling of the bipartite entanglement with the number of spins in the subsystem is found at the isotropy point $\gamma=0$: While at $\gamma=0$ the entanglement grows logarithmically with the system size with no upper bound, it saturates for $\gamma\neq 0$ at a level only depending on γ.

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I. INTRODUCTION

Since the early days of quantum theory it was realized that quantum systems can possess correlations that do not have a classical counterpart [1–3]. For a long time this phenomenon, called entanglement, has been of interest mostly in the context of foundations of quantum mechanics. With the advent of quantum-information science [4], it has been realized that entanglement is an essential resource for efficient computation and communication. This initiated a more systematic study of its properties. While entanglement in small systems is by now well understood, many-particle entanglement is still a wide open field. It is well known that quantum correlations and entanglement naturally occur in interacting many-particle systems, but we are only beginning to understand their role in these systems [5].

Recently the entanglement properties of quantum systems near the critical points of quantum phase transitions [6] have attracted much attention. Two-particle entanglement has been studied in terms of concurrence [7], e.g., in one-dimensional spin chains [8–12]. The concurrence contains, however, only limited information about the global distribution of entanglement, and other measures such as the bipartite entanglement between blocks of spins may be of larger interest. Bipartite entanglement was analyzed, e.g., in one-dimensional quantum spin chains in [12], where it was shown that it scales logarithmically with the system size in the critical regime, with a prefactor determined by the universality class, and saturates in the noncritical regime.

It is now commonly believed that a quantum phase transition is also reflected in the entanglement properties of a system. In the present paper we will show that the reverse is not true, however. We discuss a system that shows a discontinuous change of scaling of entanglement with the system size in the absence of a quantum phase transition.

In particular we study the bipartite entanglement in a system of spins with a collective coupling described by a generalization of the Lipkin-Meshkov-Glick (LMG) Hamiltonian [13]. Since this Hamiltonian is symmetric under the exchange of particles, the Hilbert space separates into subspaces whose dimensions grow only linearly with the number of particles, which makes them numerically accessible. In particular we consider the case of an *even* number of spins and an *antiferromagnetic* coupling. Furthermore we concentrate on the case where the Hamiltonian can be factorized in a product of two terms each being linear in the total spin operators. Under these conditions, which include the special case of supersymmetry (SUSY) [14,15], the ground state can be constructed explicitly [16]. This ground state undergoes a smooth transition from a separable to a maximally entangled state as a function of the parameter γ, which characterizes the asymmetry between the collective spin coupling in the x and y directions. The two-particle concurrence in this system has been analyzed in [17]. Recently also bipartite entanglement has been studied in [18] and [19], but for the *ferromagnetic* version of the LMG model, where there is a quantum phase transition. Although in the antiferromagnetic case considered here there is no quantum phase transition, we find a discontinuous behavior of the entanglement when γ is changed. While for $\gamma\neq 0$ the entanglement entropy is always finite, it grows logarithmically with the number of spins in the subsystem with no upper bound at the isotropy point $\gamma=0$. We show that we have a gapped system for arbitrary γ. Only very recently a similar behavior has been found in a quantum spin-1 chain with finite energy gap, but for localizable entanglement [20].

After discussing the LMG model and its ground state under the condition of supersymmetry in Sec. II, we analyze the bipartite entanglement of this state in terms of the von Neumann entropy [21] in Sec. III. We show that for an isotropic interaction ($\gamma=0$) and in the thermodynamic limit the entropy grows logarithmically with no upper bound. On the other hand, for any nonvanishing γ the entropy has an upper limit determined solely by γ. Furthermore it becomes a function of the ratio of the subsystem spin to the total spin rather than a function of the subsystem spin alone. In a finite system the transition between isotropic and anisotropic behavior occurs at $\gamma_{\mathrm{crit}}=J^{-1}$. To understand the saturation of the entanglement quantitatively, we give in Sec. IV an analytic estimate for the global entanglement by determining the geometric measure of entanglement.

II. COLLECTIVE SPIN COUPLING AND SUPERSYMMETRY

Let us consider an even number N of spin-1/2 particles interacting through a nonlinear coupling of the collective spin $\hat{J}_\mu=\Sigma_{j=1}^N\hat{\sigma}_\mu^j$, where $\hat{\sigma}_\mu$ denotes the μ's component of the single-particle spin. The interaction is assumed to be of second order in the total spin and is thus a generalization of the Lipkin-Meshkov-Glick Hamiltonian [13]

$$H=\alpha\hat{J}_z+\beta\hat{J}_x^2+\hat{J}_y^2-2\mu\hat{J}_y. \tag{1}$$

α and β are positive real numbers and thus the coupling is of the antiferromagnetic type. H commutes with the total spin $\hat{\mathbf{J}}^2$ and thus the total Hilbert space separates into subspaces determined by the spin quantum number J. We here restrict ourselves to the case of maximum spin, i.e., $J=N/2$. As has been shown in [16], Eq. (1) can be written as a product of two terms linear in the collective spin operators if $\beta=\alpha^2$:

$$H=(\alpha\hat{J}_x+i\hat{J}_y-i\mu)(\alpha\hat{J}_x-i\hat{J}_y+i\mu)-\mu^2. \tag{2}$$

$H+\mu^2$ is positive definite, and if $\mu=m$, with $m\in\{-J,-(J-1),\ldots,(J-1),J\},J=N/2$ being the total angular momentum, it possesses a nondegenerate ground state with $E=0$ obeying

$$(\alpha\hat{J}_x-i\hat{J}_y+im)|\Psi\rangle=0.$$

Since this equation is linear the ground state can be easily constructed, which yields

$$|\Psi\rangle=\mathcal{N}(\gamma,m)\exp(-\gamma\hat{J}_z)|m_y=m\rangle, \tag{3}$$

where we have introduced the real anisotropy parameter γ through $\tanh(\gamma)=\alpha\geqslant 0$. It is interesting to note that an anisotropy in the spin coupling is reflected here in the nonunitary term $\exp(-\gamma\hat{J}_z)$. In the fully isotropic limit $\gamma=0$, the ground state is the state $|m_y=m\rangle$ which is entangled for all $|m|<J$. In the maximally anisotropic case $\gamma=\infty$, the ground state is $|m_z=-J\rangle$, which is a product state. The loss of entanglement in this case is due to the nonunitary term $\exp(-\gamma\hat{J}_z)$. Changing γ, e.g., as function of time from ∞ to 0 causes a smooth transition from a factorized to an entangled many-body state.

Due to the symmetry of the coupling, all matrix elements of the Hamiltonian between states corresponding to different total spin J vanish exactly even for time-dependent parameter. Thus even though the ground state of Eq. (1) becomes degenerate for $\gamma=0$ with respect to the total spin J [17], the system cannot undergo a quantum phase transition upon changing γ. Furthermore the degeneracy in J at $\gamma=0$ can easily be lifted by adding a term $-\lambda\hat{\mathbf{J}}^2$ to Eq. (1), which has no effect on $|\Psi\rangle$.

In the following we will restrict ourselves to the most interesting special case $m=0$. As has been shown in [22] and [23] the collective state $|m=0\rangle$ has the largest global entanglement and should thus be considered as the state with maximum N-particle entanglement. A generalization to arbitrary m values is rather straightforward but less instructive. An additional feature of the $m=0$ case is the presence of a supersymmetry of the LMG Hamiltonian [16]. As a consequence in every spin sector J the spectrum of Eq. (1) has for all values of γ a nondegenerate ground state and all excited states are pairwise degenerate [15]. As shown in [16] the energy gap between the ground state and the pair of first excited states in every spin sector J does not close.

For $m=0$ the ground state (3) reads explicitly

$$|\Psi\rangle=\frac{e^{-\gamma\hat{J}_z}}{\sqrt{P_J(\cosh 2\gamma)}}|m_y=0\rangle \tag{4}$$

with P_J being Legendre polynomials.

III. BIPARTITE ENTANGLEMENT

In the following section we discuss the entanglement between two arbitrary partitions of the N particle system in the SUSY ground state of the LMG model. As mentioned in the Introduction it is not important here how the partitioning is done. Due to the symmetry of the Hamiltonian only the number of particles in each partition is of relevance.

A. Entropy of entanglement and distribution of Schmidt coefficients

A generally accepted quantitative measure for the entanglement between two subsystems 1 and 2, if the total system is in a pure state $|\Psi\rangle$, is the von Neumann entropy of either of the two subsystems (entropy of entanglement):

$$S(\Psi)=-\,\mathrm{tr}_1\{\rho_1\ln\rho_1\}=-\,\mathrm{tr}_2\{\rho_2\ln\rho_2\},$$

where

$$\rho_{1,2}=\mathrm{tr}_{2,1}\{|\Psi\rangle\langle\Psi|\}$$

are the reduced density matrices. $S(\Psi)$ is essentially a measure for the information loss due to division of the system and ignoring one of the subsystems. If there is entanglement between 1 and 2 in the original pure state $|\Psi\rangle$ of the total system, the entropy is nonzero. On the other hand if $|\Psi\rangle$ factorizes there is no information loss if we ignore one subsystem and the entropy vanishes.

The von Neumann entropy for pure states $S(\Psi)$ is identical to the minimum relative entropy of entanglement $E_2(\Psi)$ [24] with respect to all bipartite separable states $\sigma\in\mathcal{S}_2$:

$$\sigma=\sum_i p_i\sigma_1^i\otimes\sigma_2^i,\quad p_i\geqslant 0,\quad \sum_i p_i=1,$$

$$E_2(\Psi)=\min_{\sigma\in\mathcal{S}_2}S(\Psi\,\|\,\sigma), \tag{5}$$

$$S(\Psi \parallel \sigma) = \mathrm{tr}(\rho \log_2 \rho - \rho \log_2 \sigma), \tag{6}$$

and $\rho = |\Psi\rangle\langle\Psi|$.

Calculating the von Neumann entropy of a many-particle system is in general a very nontrivial task due to the exponential growth of the relevant Hilbert space. We will show now that the von Neumann entropy can be related to the variance of the distribution of the Schmidt coefficients, arranged in an appropriate order, in the limit of a large number of particles. For the symmetric spin states considered here this variance can easily be calculated, which will be done in the following subsection.

Let $|\Psi\rangle$ denote a pure state of a quantum system consisting of two parts labeled 1 and 2. In the case of finite-dimensional spaces, Schmidt's theorem [25] asserts that any state $|\Psi\rangle$ in the Hilbert space $H_1 \otimes H_2$ can be written in the form

$$|\Psi\rangle = \sum_{m=1}^{\chi} \lambda_m |\Phi_m^{(1)}\rangle \otimes |\Phi_m^{(2)}\rangle \tag{7}$$

where $\chi \leq \min\{d_1, d_2\}, d_1$ and d_2 being the dimensions of the corresponding Hilbert spaces. $\{|\Phi_m^{(1)}\rangle\}$ and $\{|\Phi_n^{(2)}\rangle\}$ are sets of orthonormal states for the subsystems 1 and 2, respectively, and λ_m are the positive Schmidt coefficients obeying the sum rule

$$\sum_{m=1}^{\chi} \lambda_m^2 = 1. \tag{8}$$

It is easy to see that the entropy of entanglement is related to the Schmidt coefficients via

$$S(\Psi) = -\sum_{m=1}^{\chi} \lambda_m^2 \log_2 \lambda_m^2. \tag{9}$$

The Schmidt rank, i.e., the number of Schmidt coefficients χ, provides a simple upper bound for the entropy of entanglement $S(\Psi) \leq \log_2 \chi$. If $\chi \sim \min\{d_1, d_2\}$, i.e., if it scales exponentially with the number of particles, $\log_2 \chi$ is a polynomial function of the number of particles in the smaller of the two subsystems. For the symmetric coupling considered here, the dimension of the relevant Hilbert space d increases only linearly in the number of particles, implying a logarithmic scaling of $\log_2 \chi$ with the system size. Thus also the von Neumann entropy S is expected to scale logarithmically; however, with a yet unknown coefficient. In order to calculate this coefficient it is obviously not sufficient to use $\log_2 \chi$ as an estimate for $S(\Psi)$.

It is possible, however, to find a better estimate for $S(\Psi)$ in terms of the variance of the distribution of appropriately ordered Schmidt coefficients. To show this we first note that, as shown in [26], the following inequality holds:

$$-\int |f(x)|^2 \log_2 |f(x)|^2 dx \leq \frac{1}{2}(1 + \log_2 \pi e) + \log_2 \Delta x, \tag{10}$$

where $\Delta x^2 = \int (x - \bar{x})^2 |f(x)|^2 dx$ is the variance of a probability distribution $f(x)$. This inequality becomes an equality if $f(x)$ is a Gaussian function.

For large values of χ the sum in Eq. (9) can be written as an integral with $\lambda_m^2 \to \lambda^2(m)$ representing a continuous, smooth probability distribution. The functional form of this distribution depends on the ordering of the Schmidt coefficients λ_m. For symmetric states this ordering can be chosen in such a way that $\lambda^2(m)$ becomes to a good approximation a Gaussian function [27]. In this case the entropy of entanglement is given by

$$S(\Psi) = \frac{1}{2}(1 + \log_2 \pi e) + \log_2 \Delta\lambda, \tag{11}$$

where $\Delta\lambda$ is the variance of the Schmidt coefficients.

B. Clebsch-Gordan decomposition and bipartite entanglement

We will now calculate the entropy of entanglement using Eq. (11) by finding a suitable bipartite decomposition of the ground state (4) corresponding to two subsets of spins. The scaling of $S(\Psi)$ with the system size will be studied in detail and in particular the prefactor of the logarithm determined. In the limit of a totally isotropic spin coupling $\gamma=0$ an explicit analytic expression for the variance of the Schmidt coefficients and the entanglement can be given. For nonzero values of γ numerical results will be presented.

As mentioned in the Introduction, we shall restrict ourselves to the ground state with the highest spin quantum number $J=N/2$. If we make such a choice and split the system of N spins into two sets with N_1 and N_2 spins, the magnitude of the total spin of the two subsets is fixed as $J_1=N_1/2$ and $J_2=N_2/2$. Let us illustrate this with the following example. Consider the permutation-invariant $|W\rangle$ state

$$|\psi\rangle = \frac{1}{\sqrt{3}}[|010\rangle + |001\rangle + |100\rangle] = \left| J = \frac{3}{2}, m = -\frac{1}{2} \right\rangle.$$

We split the system into two parts, one containing a single spin and the other one two spins. The bases for the one-particle subsystem and the two-particle one are, respectively, $\{|0\rangle\}\{|1\rangle\}$ and $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$, which leads to

$$|\psi\rangle = \sqrt{\frac{2}{3}}\left[\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right] \otimes |0\rangle + \frac{1}{\sqrt{3}}|00\rangle \otimes |1\rangle.$$

We can identify $|J=1, m=0\rangle = (1/\sqrt{2})[|01\rangle + |10\rangle], |J=1, m=-1\rangle = |00\rangle, |J=1/2, m=-1/2\rangle = |0\rangle$, and $J=1/2, m=+1/2\rangle$. Thus $|\psi\rangle$ reads

$$|\psi\rangle = \sqrt{\frac{2}{3}}|1,0\rangle \otimes |1/2,-1/2\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle \otimes |1/2,1/2\rangle.$$

This is actually the Clebsch-Gordan decomposition of the state $|\psi\rangle = |J=3/2, m=-1/2\rangle$ into two subsystems with angular momenta $J_1=1$ and $J_2=1/2$, with $J_1+J_2=J=3/2$.

In general, state (4) can be decomposed using the above method into two subsystems with angular momenta J_1 and J_2 that respect the condition $J_1+J_2=J$:

$$|\Psi\rangle = \mathcal{N}(\gamma)e^{-\gamma\hat{J}_z}|m_y=0\rangle = \mathcal{N}(\gamma)\sum_{m_1}\sum_{m_2} C^{J_1J_2J}_{m_1m_2m} d^J_{m,0}(-i\gamma)|J_1,m_1\rangle \otimes |J_2,m_2\rangle. \tag{12}$$

Here $C^{J_1J_2J}_{m_1m_2m}$ are the Clebsch-Gordan (Wigner) coefficients and $m=m_1+m_2$. $d^J_{m'm}(\beta)$ are the rotation matrices defined as [28]

$$d^J_{m'm}(\beta) = \langle J,m'|e^{-i\beta\hat{J}_y}|J,m\rangle = \sqrt{\frac{(J+m)!\,(J-m)!}{(J+m')!\,(J-m')!}}\left(\sin\frac{\beta}{2}\right)^{m-m'}\left(\cos\frac{\beta}{2}\right)^{m+m'} \times P^{(m-m',m+m')}_{J-m}(\cos\beta), \tag{13}$$

where $P^{(\alpha,\beta)}_n(x)$ are Jacobi polynomials. Since we are considering the special decomposition for $J=J_1+J_2$, the Wigner coefficients have the binomial distribution:

$$(C^{J_1J_2J}_{m_1m_2m})^2 = \frac{\binom{2J_1}{J_1+m_1}\binom{2J_2}{J_2+m_2}}{\binom{2J}{J+m}}. \tag{14}$$

This relation combined with Eq.(13) gives the following decomposition for the wave function $|\Psi\rangle$:

$$|\Psi\rangle = \sum_{m_1=-J_1}^{J_1}\sum_{m_2=-J_2}^{J_2} A_{m_1,m_2}(\gamma)|J_1,m_1\rangle \otimes |J_2,m_2\rangle, \tag{15}$$

with

$$A_{m_1,m_2}(\gamma) = \mathcal{N}(\gamma)(i)^{m_1+m_2} \times \sqrt{\frac{(J!)^2(2J_1)!\,(2J_2)!}{(2J)!\,(J_1+m_1)!\,(J_1-m_1)!\,(J_2+m_2)!\,(J_2-m_2)!}} \times\left(\coth\frac{\gamma}{2}\right)^{m_1+m_2} P_J^{(-m_1-m_2,m_1+m_2)}(\cosh\gamma). \tag{16}$$

Equation (15) is separable if all coefficients A_{m_1,m_2} factorize. This is the case in the limit $\gamma\to\infty$, where the coth term in Eq.(16) approaches unity and the Jacobi polynomials factorize in m_1 and m_2.

1. Isotropic spin coupling $\gamma=0$

Making use of the results of Sec. II A, the summation in Eq. (12) can be carried out explicitly for the limit of isotropic spin coupling. In this case one sees from Eq. (16) that only coefficients A_{m_1,m_2} with $m_1+m_2=0$ survive. Thus one has the following decomposition:

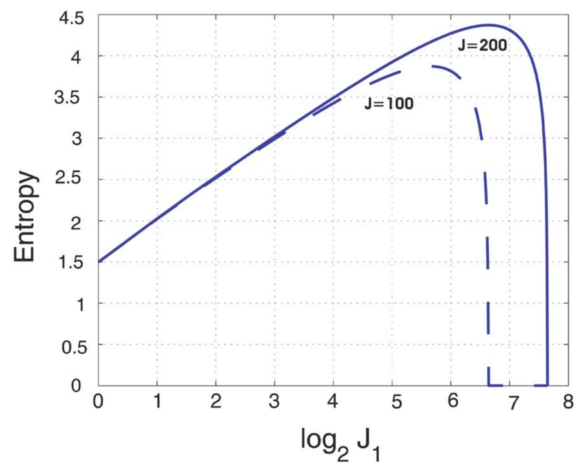

FIG. 1. (Color online) Entropy of entanglement for $\gamma=0$ as a function of the logarithm of the subsystem spin J_1 for different values of the total spin J. Due to the symmetry of S with respect to $J_1\leftrightarrow J-J_1$, the curves saturate at $J_1=J/2$.

$$|\Psi\rangle = \sum_{m=-J_1}^{J_1} C^{J_1J_2\,J_1+J_2}_{m-m0}|J_1,m\rangle \otimes |J_2,-m\rangle,$$

where $J_2\geqslant J_1$ was assumed without loss of generality. The Schmidt coefficients are, therefore, the Clebsch-Gordan coefficients. In the limit $J_2\gg J_1$, they have a Gaussian form [27]

$$(C^{J_1J_2\,J_1+J_2}_{m-m0})^2 \approx \frac{\exp(-m^2/J_1)}{\sqrt{J_1\pi}}$$

where $|m|\leqslant J_1\ll J_2$. The Gaussian form of the coefficients allows us to make use of the relation (11) to calculate the von Neumann entropy for $J_1\ll J$:

$$S \simeq \frac{1}{2}\log_2 J_1 + \frac{1}{2}(1+\log_2 \pi e), \quad \gamma=0. \tag{17}$$

In Fig. 1 we have plotted the von Neumann entropy for $\gamma=0$ as a function of the subsystem spin $J_1=N_1/2$ for different values of the total spin $J=N/2$. For $J_1\ll J$ a logarithmic scaling with prefactor 1/2 is evident. When J_1 approaches $J/2$ the entropy saturates since S is symmetric with respect to the replacement $J_1\leftrightarrow J-J_1$. It is important to note that for $J_1 \ll J$ the von Neumann entropy S does not depend on J.

2. Anisotropic spin coupling $\gamma\neq 0$

If the spin coupling is anisotropic, i.e., if $\gamma\neq 0$, the double sum in Eqs. (12) and (15) remains. Thus in order to discuss the influence of a finite γ it is necessary to explicitly evaluate the sum in Eq. (9). We have done this numerically for a total particle number up to 200 and subsystems up to 100 particles. The results are shown in Figs. 2 and 3. As can be seen from Fig. 2 in contrast to the isotropic case $\gamma=0$, the entropy is no longer independent of the total spin if

$$\gamma \geqslant \gamma_{\text{crit}} \equiv \frac{1}{J}.$$

In the thermodynamic limit the critical point is $\gamma=0$.

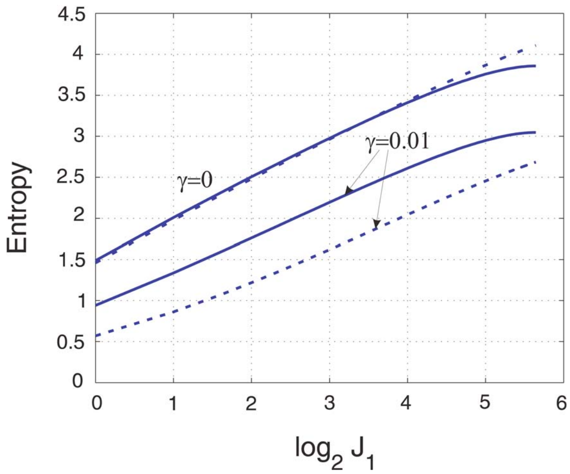

FIG. 2. (Color online) Entropy of entanglement for $J=100$ (full line) and 200 (dashed line) as a function of logarithm of subsystem spin J_1 for different values of γ. One recognizes that in contrast to the isotropic case the entropy now depends on the total spin J.

As can be seen from Fig. 3, for any $\gamma \geqslant J^{-1}$ the entropy becomes a function of the logarithm of the fraction of particles $J_1/J = N_1/N$.

Our numerical calculations suggest for $J_1 \ll J$ and $\gamma \gg J^{-1}$ a functional dependence of the form

$$S \sim f(\gamma)\log_2(J_1/J), \quad \gamma > \gamma_{\mathrm{crit}}.$$

The reduction of entanglement with increasing γ is expected. The state $\mathcal{N}(\gamma)e^{-\gamma \hat{J}_z}|m_y=0\rangle$ is maximally entangled for $\gamma=0$ and the prefactor $e^{-\gamma \hat{J}_z}$ corresponds to a local nonunitary operation which always decreases the amount of entanglement. For large values of γ the state becomes eventually separable.

The most peculiar feature of the von Neumann entropy is the change of the scaling behavior with J_1 from $S \sim \log_2 J_1$ for $\gamma < \gamma_{\mathrm{crit}}$ to $S \sim \log_2 J_1/J$ for $\gamma \geqslant \gamma_{\mathrm{crit}}$. The role of $\gamma_{\mathrm{crit}} = J^{-1}$ and the change of the scaling behavior is also reflected in the distribution of ordered Schmidt numbers. As can be seen from Fig. 4, the falloff of the Schmidt numbers λ_m becomes exponential when γJ exceeds unity.

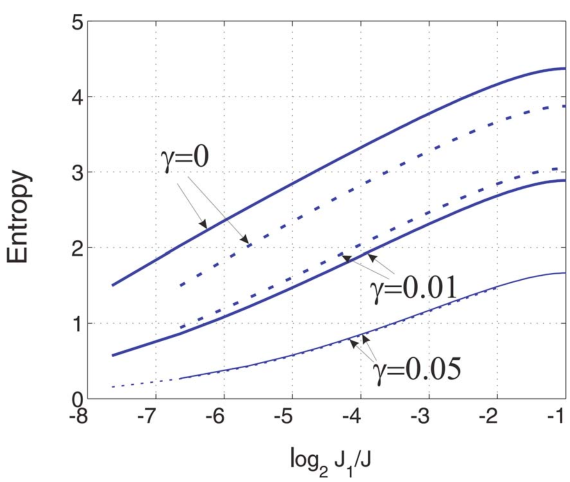

FIG. 3. (Color online) Entropy of entanglement for different values of γ as a function of $\log_2(J_1/J)$ for $J=100$ (solid line) and 200 (dashed line). For $\gamma J \geqslant 1$ the curves become virtually indistinguishable.

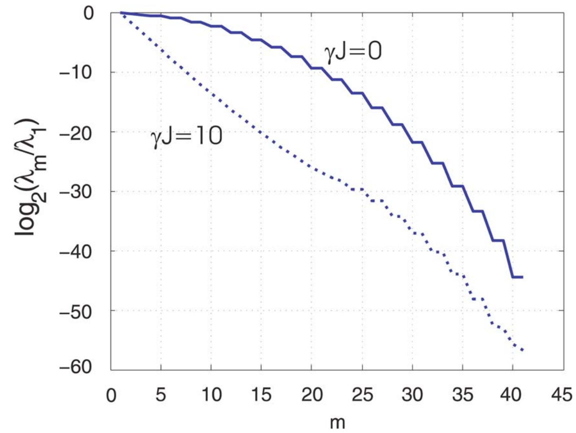

FIG. 4. (Color online) Ordered distribution of normalized Schmidt numbers for different values of γJ.

There is no obvious distinction of the point $\gamma = J^{-1}$ in the properties of the system. The system does not undergo a phase transition at this point. Due to the SUSY, the qualitative structure of the spectrum is the same for all values of γ and there is always an energy gap between the ground and first excited states. Thus the question remains whether there are any physical signatures for the change of the scaling behavior of the entanglement at $\gamma = \gamma_{\mathrm{crit}}$.

IV. GEOMETRIC ESTIMATE FOR GLOBAL ENTANGLEMENT

In the previous section we have discussed the bipartite entanglement of two partitions of the N spin-1/2 system. We have seen (see Fig. 3) that for $\gamma \geqslant \gamma_{\mathrm{crit}}$ the von Neumann entropy has a maximum value independent of J. In this section we will quantitatively analyze this maximum by determining the N-partite or global entanglement E_N of the SUSY ground state (4), which is an upper bound to the bipartite entanglement E_2. Although it is not possible to obtain an analytic expression for E_N, we can determine a very good estimate for it given by the geometric measure of entanglement.

A. Relative entropy and geometric measure of entanglement

A many-particle state is called N-partite separable if it can be written as a product of states of all N subsystems. Obviously a bipartite entangled state is always N-partite entangled, but not vice versa. A quantitative measure of many-particle or *global* entanglement of a state ρ is the minimum relative entropy that determines the minimum distance between ρ and the set $\mathcal{S}_N$ of N-partite product states σ [24]:

$$E_N = \min_{\sigma \in \mathcal{S}_N} S(\rho \,\|\, \sigma), \tag{18}$$

where

$$S(\rho \parallel \sigma) \equiv \mathrm{tr}(\rho \log_2 \rho - \rho \log_2 \sigma), \tag{19}$$

$\sigma \in \mathcal{S}_N$ being an N-partite separable state:

$$\sigma = \sum_{i=1}^{N} p_i \rho_1^i \otimes \rho_2^i \otimes \cdots \otimes \rho_n^i \tag{20}$$

with $p_i > 0$ and $\Sigma_i p_i = 1$. For the bipartite case E_N is equivalent to the entanglement of formation [24], which in the case of pure states is identical to the von Neumann entropy.

Since the set $\mathcal{S}_N$ is smaller than $\mathcal{S}_2$ for any partitioning, $\mathcal{S}_N \subset \mathcal{S}_2$, it follows immediately that

$$E_N(\Psi) \geqslant E_2(\Psi),$$

i.e., the global entanglement represents an upper bound to the bipartite entanglement.

In order to compute E_N for any state ρ, one has to find its closest product state σ. This is in general a quite difficult task and can be done only in very special cases. There is, however, a lower bound to E_N which gives a good estimate for the behavior of the global entanglement. This lower bound is the geometric entanglement $E_G(\Psi)$ [22,23]

$$E_G(\Psi) \equiv -2 \log_2 \Lambda_{\max}(\Psi), \tag{21}$$

where

$$\Lambda_{\max}(\Psi) = \max_{\phi} |\langle \phi | \Psi \rangle| \tag{22}$$

is the maximum overlap of $|\Psi\rangle$ with an N-partite separable state $|\phi\rangle$. $E_G(\Psi)$ is not an entanglement monotone and thus, in the strict sense, not a valid measure of entanglement. It does give, however, a close lower bound to E_N which for some states such as the Dicke states is a tight bound, i.e., $E_G = E_N$ [22]. The geometric entanglement can easily be calculated for states that are permutation invariant, which is the case for the SUSY ground state (4).

B. Geometric measure of entanglement for the SUSY state

To calculate the geometric entanglement E_G or equivalently the maximum overlap $\Lambda_{\max}$ of the SUSY ground state (4) with N-partite separable states it is sufficient to construct the most general N-partite separable state which is invariant under permutation of spins [22]. This state is given by rotations of the state $|m_z = -J\rangle$:

$$|\phi\rangle = e^{-i\alpha \hat{J}_z} e^{-i\beta \hat{J}_y} e^{-i\xi \hat{J}_z} |m_z = -J\rangle.$$

Calculating the overlap of $|\phi\rangle$ with Eq. (4) and maximizing it with respect to the real parameters α, β and ξ leads to

$$|\phi\rangle = |m_z = -J\rangle.$$

The corresponding entanglement eigenvalue reads

$$\Lambda_{\max}(\gamma) = \frac{\sqrt{(2J)!}}{2^J J!} \frac{e^{\gamma J}}{\sqrt{P_J(\cosh 2\gamma)}}. \tag{23}$$

1. Isotropic spin coupling $\gamma = 0$

In the isotropic case Eq. (23) reduces to [22]

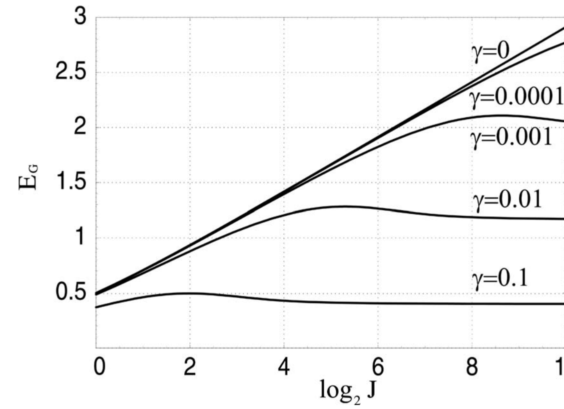

FIG. 5. Geometric measure of entanglement as a function of $\log_2 J$ for different values of the anisotropy parameter γ. One recognizes a saturation at $J \geqslant \gamma_{\mathrm{crit}}^{-1}$.

$$\Lambda_{\max}(\gamma = 0) = \frac{\sqrt{(2J)!}}{2^J J!} \tag{24}$$

and thus the geometric entanglement is given by $E_G(\Psi) = \frac{1}{2}\log_2 J$. Since the SUSY state for $\gamma = 0$ is the Dicke state $|J, m_y = 0\rangle$ the geometric entanglement is a tight lower bound to the relative entropy and thus

$$E_N(\Psi) = E_G(\Psi) = \frac{1}{2}\log_2 J. \tag{25}$$

2. Anisotropic spin coupling $\gamma \neq 0$

It is obvious that the largest entanglement is obtained for $\gamma = 0$, where the maximum overlap $\Lambda_{\max}$ with separable states is the smallest. On the other hand, for $\gamma \to \infty$, the state becomes identical to the separable state $|m_z = -J\rangle$. The same conclusion can of course be obtained from Eq. (23), employing the asymptotic expansion of the Legendre polynomials.

In Fig. 5 we have plotted the geometric entanglement as a function of J for different values of γ. For sufficiently small values of J, one recognizes a logarithmic growth which saturates when J exceeds the value γ^{-1}. One can easily obtain an analytic expression for the saturation value of E_G. Making use of the asymptotics of the Legendre polynomials for large J and $\gamma \neq 0$,

$$P_J(\cosh 2\gamma) \underset{\text{large } J}{\to} \frac{1}{\sqrt{1 - e^{-4\gamma}}} e^{2\gamma J} \frac{(2J-1)!!}{2^J J!},$$

one arrives at the simple expression

$$\Lambda_{\max}(\gamma) = (1 - e^{-4\gamma})^{1/4},$$

leading to

$$E_G(\gamma, J) \underset{\text{large } J}{\to} -\frac{1}{2}\log_2(1 - e^{-4\gamma}).$$

Comparing the numerical values for E_2 obtained in the previous section, one finds that $E_G < E_2$. This shows that in the case of nonisotropic coupling $\gamma \neq 0$ the geometric entanglement is not a tight lower bound to the global entanglement, i.e., here $E_G < E_N$. Thus E_G can only be used as a qualitative measure for the global entanglement E_N.

V. CONCLUSIONS

In the present paper we have studied the bipartite entanglement between blocks of spins in the antiferromagnetic Lipkin-Meshkov-Glick model under conditions of supersymmetry. The supersymmetry of the model allows for an explicit construction of the ground state which undergoes a smooth transition from a separable to a maximally entangled state when changing the anisotropy of the collective spin coupling. Making use of the Clebsch-Gordan decomposition of angular momenta, the von Neumann entropy, which quantifies the bipartite entanglement, can be calculated analytically in the isotropic case or numerically in the case of anisotropic coupling. Although the structure of the spectrum stays always the same, with one nondegenerate ground state and pair-wise degenerate excited states, and no level crossing or merging occurs, the entanglement shows a discontinuous behavior at the isotropy point. When the anisotropy parameter γ vanishes exactly, the von Neumann entropy grows logarithmically with the number of particles in the subsystem. For any nonvanishing value of γ (in the thermodynamic limit) the entropy saturates at a finite value determined by γ. The maximum bipartite entanglement can be estimated by the geometric measure of global entanglement, which has been determined analytically. Furthermore in this case the entropy becomes a function of the ratio of particle number in the subsystem to the total particle number rather than a function of the subsystem size alone. For finite systems the transition between the two cases happens at a small but finite value of γ corresponding to the inverse of the total number of spins.

The discontinuous scaling behavior of entanglement, as observed here, is usually attributed to the disappearance of an energy gap, i.e., a quantum phase transition. Only very recently it was realized in [20] that also gapped quantum spin chains can show a behavior reminiscent of a quantum phase transition, where the characteristic length of spin-spin entanglement diverges, while the correlation length remains finite. This shows that the analysis of entanglement can reveal very interesting properties of interacting many-particle systems, which cannot be seen in simple correlations.

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