Symmetry-protected creation of superposition states and entanglement using circulant Hamiltonians

R. G. Unanyan,[1,2] B. W. Shore,[1,*] M. Fleischhauer,[1] and N. V. Vitanov[3,4]
[1]*Technical University of Kaiserslautern, D-67653 Kaiserslautern, Germany*
[2]*Insitut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany*
[3]*Department of Physics, Sofia University, James Bourchier 5 blvd., 1164 Sofia, Bulgaria*
[4]*Institute of Solid State Physics, Bulgarian Academy of Sciences, Tsarigradsko chaussée 72, 1784 Sofia, Bulgaria*
(Received 6 October 2006; published 6 February 2007)

We describe the use of a special interaction symmetry for the robust generation of the totally symmetric superposition state or entangled state of an N-state system. The required symmetry of the Hamiltonian is that of a circulant matrix. Such a matrix has the important property that its eigenstates are independent of the matrix elements as long as the circulant symmetry is maintained. One of the eigenvectors is the target superposition. By inducing a slow evolution of the Hamiltonian into the circulant form, adiabatic following will generate the desired superposition out of a convenient initial state such as a product state. The creation process is robust: it is insensitive to details of the interaction as long as the final Hamiltonian has the required symmetry. We illustrate the procedure with a simple example: a ring of quantum wells that permit interwell tunneling, into which a single atom is placed. By carrying out adiabatic evolution the state vector approaches an equal distribution of probability amplitudes in each well.

DOI: 10.1103/PhysRevA.75.022305 PACS number(s): 03.67.Mn, 03.65.Xp, 03.67.Lx

I. INTRODUCTION

The creation of coherent superpositions of discrete degenerate quantum states has become increasingly important for applications in quantum information processing [1] and quantum metrology [2]. Particularly interesting examples occur in N-state quantum systems when the state vector Ψ has the simple form

$$\Psi = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \psi_n. \tag{1}$$

That is, each basis state ψ_n appears in the superposition with equal amplitude. Such a superposition forms the initial state of various quantum algorithms, such as that of Shor or Grover [1]. It is therefore desirable to produce such a state with high fidelity.

For many applications the system comprises N distinguishable particles, or subsystems. We can express the joint basis states ψ_n as products of separate quantum states φ, each associated with a particular particle. For example, a set of N qubits are representable as $\varphi_0^{(n)}$ and $\varphi_1^{(n)}$ for $n=1,\ldots,N$. An important class of such situations occurs when we have a total excitation number of unity in which case we deal with product states such as

$$\psi_n = \varphi_0^{(1)}\varphi_0^{(2)}\cdots\varphi_0^{(n-1)}\varphi_1^{(n)}\varphi_0^{(n+1)}\cdots\varphi_0^{(N)}. \tag{2}$$

When such product states appear as the basis states for the superposition of Eq. (1) the resulting state vector is *maximally entangled*. It is the so-called W state [3].

The generation of superposition states composed of a large number of basis states, such as Eq. (1) with Eq. (2), constitutes a major challenge for practical quantum information science and quantum engineering, particularly if many particles are involved, as occurs when entangled states are considered. Many-particle entanglement is very fragile: fluctuations of system parameters or noise tend to rapidly collapse the wave function into a classical statistical mixture. Thus there is a need for techniques which are insensitive to parameter fluctuations and noise.

An intrinsic obstacle to identifying such a technique is that a Hamiltonian which is capable of creating *arbitrary* superposition states must, by construction, depend sensitively on continuous parameters. Thus it is prone to decoherence caused by fluctuations of these parameters.

By contrast, methods that are capable of generating only *specific* superposition states do not have this principle problem. For example, when there is a sufficiently large gap between the energy of a nondegenerate ground state and that of the lowest-energy excited state, then during adiabatic changes the system will remain in the ground state even though its construction from bare states alters. Choosing the ground state is, of course, only necessary if spontaneous processes act to produce transition; in general it suffices to have a large energy gap between the target state and any other state.

Using this property it is possible, with a slowly varying time-dependent Hamiltonian, to alter the bare-state construction of the ground state to produce a specific superposition state or many-particle entanglement. The procedure can be made resistant to decoherence if the energy gap is sufficiently large [4]. In such evolution the initial and final asymptotic forms of the ground state should coincide, respectively, with the desired initial and target states; for robustness, the asymptotic forms of these states should depend only on the symmetry of the Hamiltonian, and not on the specific values of its elements.

We here describe a scheme that meets these requirements. To this end we employ a Hamiltonian with a special symmetry, namely that of a circulant matrix [5]. We make use of an important property of circulant matrices: although their ei-

*Permanent address: 618 Escondido Circle, Livermore, CA 94550.

genvalues depend on the matrix elements of the Hamiltonian, the eigenstates do not. These are entirely fixed by the circulant symmetry.

II. CIRCULANT HAMILTONIANS

Let us consider a Hilbert space spanned by N eigenstates $\psi_1,\ldots,\psi_N$ of some Hamiltonian $\hat{H}_0$. These basis states, which will be referred to as *bare* states, could correspond, for example, to different energy levels of a single system, such as an atom or molecule, or they could be product states of a noninteracting many-particle Hamiltonian. We assume that the system can easily be prepared in the ground state of $\hat{H}_0$, denoted by the Hilbert space vector Ψ_0. Our goal is to create out of this initial state a totally symmetric superposition, as specified in Eq. (1). For this purpose we consider a time-dependent Hamiltonian of the form

$$\hat{H}(t) = f(t)\hat{H}_0 + g(t)\hat{H}_1. \tag{3}$$

Here $f(t)$ and $g(t)$ are real-valued functions of time t, chosen such that $|f(t)| \gg |g(t)|$ for $t \to -\infty$ and $|f(t)| \ll |g(t)|$ for $t \to +\infty$. We take the Hamiltonian $\hat{H}_1$ to be represented by a real matrix H_1 having the symmetry properties of a circulant matrix [5].

An N-dimensional (real) matrix A is called circulant if it has the form

$$\mathsf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_0 \end{bmatrix}, \tag{4}$$

where each row is a shift of the row above it. This banded structure is a special example of a Toeplitz matrix [5]. The structure can also be characterized by noting that the (k,j) entry of A is given by

$$(\mathsf{A})_{kj} = a_{(j-k) \bmod N}. \tag{5}$$

It is quite easy to verify that any N-dimensional circulant matrix has the eigenvectors

$$\Phi_n = \frac{1}{\sqrt{N}}[1, r_n, r_n^2, \ldots, r_n^{N-1}]^T, \tag{6}$$

where $r_n = \exp(2\pi n i/N)$ is the nth root of the scalar equation $r^N = 1$ and the superscript T denotes matrix transpose. Of particular interest is the $n=0$ eigenvector,

$$\Phi_0 = \frac{1}{\sqrt{N}}[1,1,\ldots,1]^T = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} \psi_n, \tag{7}$$

which is an equal real superposition of the unit basis vectors ψ_n. This is exactly the symmetry desired for the superposition construction of Eq. (1); it is the *target state* of our process, $\Psi(t) \to \Phi_0$ for $t \to \infty$.

It is important to note that the eigenvectors of a circulant matrix do not depend on the actual values of the matrix elements a_k. This means the eigenstates are resistant to fluctuations of the parameters (or noise) as long as the circulant form is maintained, the noise does not break this symmetry, and there are no nonadiabatic transitions between the eigenstates.

Unlike the eigenvectors, the eigenvalues do depend on the individual elements of the circulant matrix: they are phased sums of the elements a_k,

$$\lambda_n = \sum_{k=0}^{N-1} a_k \exp\left(-i\frac{2\pi kn}{N}\right). \tag{8}$$

The eigenvalue λ_0 corresponding to the eigenvector Φ_0 is the sum of all the N elements of the circulant matrix,

$$\lambda_0 = \sum_{k=0}^{N-1} a_k. \tag{9}$$

Other eigenvalues involve sums with complex-valued terms. For our purposes, that of constructing a Hamiltonian with circulant symmetry, we require additional constraints upon the matrix: specifically, it must be Hermitian. Then all complex-valued terms in the eigenvalues (8) sum to real numbers. Thus if all the elements of the circulant matrix are real and positive (or negative), then the eigenvalue λ_0 is the largest (smallest).

There is always a finite gap $\Delta\lambda_0$ between this eigenvalue and all other eigenvalues. In the special case when all a_k are equal, i.e., $a_k \equiv a$, then all eigenvalues except λ_0 vanish: $\lambda_{k\neq 0} = 0$ $(k = 1,2,\ldots,N-1)$, and the gap is

$$\Delta\lambda_0 = Na. \tag{10}$$

The assumption of equal matrix elements of the Hamiltonian is not realistic for large N. A more physical assumption is a polynomial dependence upon index, say $a_k \approx ak^{-\gamma}$. In this case it can be shown that the gap approaches a constant value as $N \to \infty$ when $\gamma < 2$; the gap diminishes as $1/N^2$ if $\gamma \geqslant 2$ or if the coupling decreases exponentially with k.

We will write the matrix representation of the Hamiltonian $\hat{H}_1$ in Eq. (3) as

$$\mathsf{H}_1 = -\begin{bmatrix} 0 & V_1 & V_2 & V_3 & \cdots & V_2 & V_1 \\ V_1 & 0 & V_1 & V_2 & \cdots & V_3 & V_2 \\ V_2 & V_1 & 0 & V_1 & \cdots & V_4 & V_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_1 & V_2 & V_3 & V_4 & \cdots & V_1 & 0 \end{bmatrix}. \tag{11}$$

Without loss of generality we have set the diagonal elements equal to zero; such elements will appear only in the Hamiltonian $\hat{H}_0$. We take the interactions V_n to be real valued. For even-integer N there are $N/2$ distinct interactions, while for odd-integer N there are $(N-1)/2$ distinct interactions. For later convenience we assume that the interactions V_n are all positive [note the overall minus sign in Eq. (11)]. In this case the lowest-energy (ground) state of $\hat{H}_1$ is the target state Φ_0.

This follows immediately from the relations

$$\lambda_n - \lambda_0 = \begin{cases} 4\sum_{k=1}^{M} V_k \sin^2 \dfrac{\pi k n}{N} & (N = 2M+1), \quad (12a) \\ 4\sum_{k=1}^{M-1} V_k \sin^2 \dfrac{\pi k n}{N} + [1-(-1)^n]V_M & (N = 2M), \quad (12b) \end{cases}$$

which are easily obtained from Eqs. (8), (9), and (11), and which imply $\lambda_n > \lambda_0$ for any $n \neq 0$.

We choose the time dependences $g(t)$ and $f(t)$ such that $|f(t)| \gg |g(t)|$ for $t \to -\infty$ and $|f(t)| \ll |g(t)|$ for $t \to +\infty$. With this choice the ground state of $\hat{H}(t)$ corresponds, for $t \to -\infty$, to the initial state Ψ_0 and for $t \to +\infty$ to the symmetric superposition state Φ_0. Let $\Delta\lambda(t)$ be the gap between the ground state and the next-nearest eigenvalue of $\hat{H}(t)$. We then require that $g(t)$ and $f(t)$ change sufficiently slowly, such that the energy gap obeys the inequality ($\hbar = 1$)

$$\Delta\lambda(t)|_{min} \gg 1/T, \tag{13}$$

where T is the interaction duration. Under these conditions the state vector $\Psi(t)$ will adiabatically rotate from the initial state Ψ_0 to the target superposition Φ_0.

III. REALIZATION: A RING OF QUANTUM WELLS

As a possible realization of a circulant Hamiltonian we consider a ring of N quantum wells, coupled by tunneling to each other. To be specific we consider a ring of $N=6$ wells, as shown in Fig. 1. Initially we place a single atom into one of these wells, as illustrated on the left. Then we adiabatically shrink the well structure, as shown on the right. Tunneling interactions occur between pairs of wells. These are of three sorts, labeled V_1, V_2, and V_3, and distinguished by the separation between wells; V_1 is associated with the shortest and V_3 with the largest separation. We assume that all V's are positive, and that $V_1 > V_2 > V_3$. When all wells are identical the interaction Hamiltonian matrix H_1 has the form of a circulant,

$$\mathsf{H}_1 = -\begin{bmatrix} 0 & V_1 & V_2 & V_3 & V_2 & V_1 \\ V_1 & 0 & V_1 & V_2 & V_3 & V_2 \\ V_2 & V_1 & 0 & V_1 & V_2 & V_3 \\ V_3 & V_2 & V_1 & 0 & V_1 & V_2 \\ V_2 & V_3 & V_2 & V_1 & 0 & V_1 \\ V_1 & V_2 & V_3 & V_2 & V_1 & 0 \end{bmatrix}. \tag{14}$$

The desired superposition state (1) is the eigenstate of $\mathsf{H}_1(t)$ with the lowest energy.

The initial Hamiltonian matrix H_0 describes a variable well depth for well number 1, and uniform values for the other wells. We take the latter to be zero, thereby defining the zero point of the energy scale. Thus H_0 has only a single element, in the upper left corner,

$$\mathsf{H}_0 = -\begin{bmatrix} \Delta & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & 0 \end{bmatrix}. \tag{15}$$

We remark that, in practice, the depth of one well will also affect the tunneling rates from this well. This alteration is not relevant to the present discussion: we require only that, asymptotically, the interactions have the circulant form and that the energy of the initial state connect unambiguously with that of the target state.

Initially, by construction, the wells are far apart, as indicated in the left frame of Fig. 1; all the tunneling interactions are therefore weak. We lower the potential of one well, with respect to all others, and place an atom into it, as shown on the left of the figure. We next force the ring of wells together, adiabatically, until they reach the conditions, shown in the right frame, where tunneling interactions dominate.

We describe the desired change in the Hamiltonian matrix by introducing for Eq. (3) a time-dependent function $g(t)$ which changes from 0 to 1. During the same time interval we change the depth of the first well until it becomes identical with the other wells [i.e., until $f(t)=0$]. As the wells draw closer together the interactions are all much stronger and quantum tunneling steers the system adiabatically into the target superposition state; the population becomes evenly distributed among all the wells.

We demonstrate the proposed procedure by numerically integrating the time-dependent Schrödinger equation for tanh shapes of the time dependences,

$$g(t) = \frac{1}{2}[1 + \tanh(t/T)], \tag{16a}$$

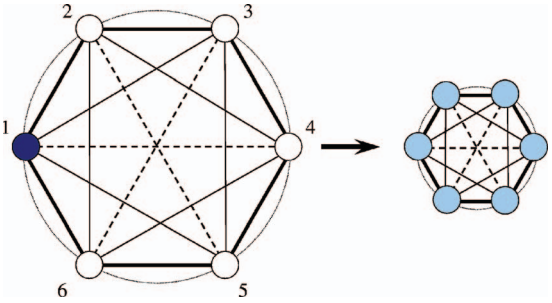

FIG. 1. (Color online) Implementation of circulant Hamiltonian with ring of quantum wells (small circles) evenly spaced around a ring (large thin circle). Tunneling interactions appear as thick lines (strongest interactions, V_1), thin lines (weaker interactions, V_2) and dashed lines (weakest interactions, V_3). (a) Initially population is in one well, shown as dark. (b) After adiabatic passage to a smaller ring where tunneling becomes dominant, the population is evenly distributed.

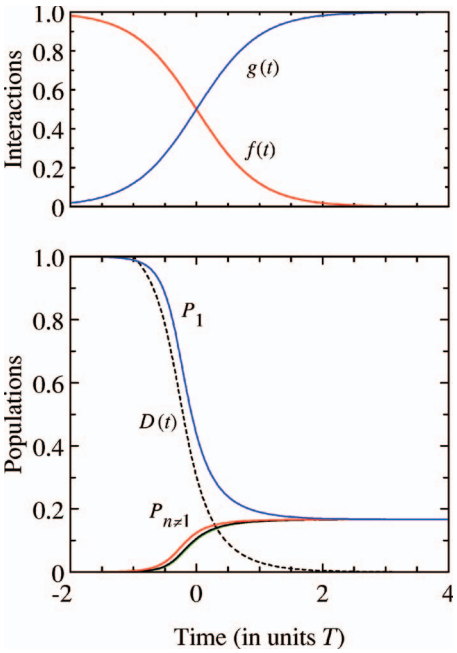

FIG. 2. (Color online) Top: Time variations $f(t)$ and $g(t)$. Bottom: Population evolution. After adiabatic passage to a smaller ring, the population is evenly distributed. The fields are given by Eqs. (16a) and (16b) with $V_1T=10$, $V_2T=10/\sqrt{3}$, $V_3T=5$, and $\Delta T=50$. The dashed line is the distance $D(t)=|\Psi(t)-\Phi_0|$ between the state vector $\Psi(t)$ and the target superposition state Φ_0.

$$f(t)=\frac{1}{2}[1-\tanh(t/T)]. \tag{16b}$$

Figure 2 shows the calculated evolution of the populations for values of the parameters that ensure nearly adiabatic evolution. We show also the distance $D(t)=|\Psi(t)-\Phi_0|$ between the target state Φ_0 and the actual state vector $\Psi(t)$ [with the unimportant global phase in $\Psi(t)$ factored out], a measure of the error in state construction. This vanishes as time increases, showing that not only the populations, but also the phases of the created superposition match the target superposition Φ_0.

We have presented and analyzed a model in which the circulant symmetry arises during the course of adiabatic evolution. Though this is conceptually simple, physical realization poses fabrication difficulties. A more promising approach is to hold the geometry fixed but alter the individual interactions—the well depths. Such a realization can, in principle, be obtained with a two-dimensional optical lattice in which the potential acting on each atom depends upon the internal excitation state of the atom, as has been demonstrated [6]. Adiabatic alteration of the internal state of a specific atom (say, by using the STIRAP technique [7]) will alter the depth of a single well while preserving the overall spatial symmetry.

IV. CONCLUSION

When the matrix representation of an N-state Hamiltonian has the symmetry of a circulant matrix, its eigenvectors are independent of its elements. We propose to use this symmetry to construct, via adiabatic passage, a specific superposition state, Eq. (1), in a way that is least affected by parameter fluctuations. The formation of such a state depends only on the symmetry of the Hamiltonian, and is therefore robust to variations of the matrix elements that maintain this symmetry.

Furthermore, there is always a finite energy gap between the adiabatic energy of the fully symmetric superposition state (1) and that of the nearest neighboring state. This gap allows us to maintain adiabatic evolution.

When the interaction originates with long-range forces, such that the coupling falls off with distance r more slowly than $1/r^2$, the gap approaches a constant as the number of states increases. This scaling should allow application of this method to the generation of multiparticle entanglement.

We have illustrated the formation of such a superposition state in a system of linked quantum wells, into which a single atom is placed. Our simulations illustrate the successful formation of this superposition state.

ACKNOWLEDGMENTS

This work was supported by the EU Marie Curie ToK project CAMEL and the EU Marie Curie RTN project EMALI. The work of R.U. was also supported by the EU through the Integrated Project FET/QIC SCALA. B.W.S. acknowledges support from funds available through the Max Planck Forschungspreis 2003.

[1] C. P. Williams and S. H. Clearwater, *Explorations in Quantum Computing* (Springer, Berlin, 1997); M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000); D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation* (Springer, Berlin, 2001).

[2] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio, and J. I. Cirac, Phys. Rev. Lett. **79**, 3865 (1997); D. Liebfried, M. D. Barrett, T. Schaetz, J. Britton, J. Chiaverini, W. M. Itano, J. D. Jost, C. Langer, and D. J. Wineland, Science **304**, 1476 (2004).

[3] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[4] R. G. Unanyan and M. Fleischhauer, Phys. Rev. Lett. **90**, 133601 (2003).

[5] F. R. Gantmacher, *Matrix Theory* (Springer, Berlin, 1986).

[6] P. Treutlein, T. Steinmetz, Y. Colombe, B. Lev, P. Hommelhoff, J. Reichel, M. Greiner, O. Mandel, A. Widera, T. Rom, I. Bloch, and T. W. Haensch, Fortschr. Phys. **54**, 702 (2006).

[7] N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, Annu. Rev. Phys. Chem. **52**, 763 (2001).