Quantized transport induced by topology transfer between coupled one-dimensional lattice systems

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We show that a topological pump in a one-dimensional insulator can induce a strictly quantized transport in an auxiliary chain of noninteracting fermions weakly coupled to the first. The transported charge is determined by an integer topological invariant of the fictitious Hamiltonian of the insulator, given by the covariance matrix of single-particle correlations. If the original system consists of noninteracting fermions, this number is identical to the Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) invariant of the original system and thus the coupling induces a transfer of topology to the auxiliary chain. When extended to particles with interactions, for which the TKNN number does not exist, the transported charge in the auxiliary chain defines a topological invariant for the interacting system. In certain cases this invariant agrees with the many-body generalization of the TKNN number introduced by Niu, Thouless, and Wu. We illustrate the topology transfer to the auxiliary system for the Rice-Mele model of noninteracting fermions at half filling and the extended superlattice Bose-Hubbard model at quarter filling. In the latter case the induced charge pump is fractional.

I. INTRODUCTION

Topological states of matter [1–3] have fascinated physicists for a long time as they can give rise to interesting phenomena such as protected edge states and edge currents, quantized bulk transport in insulating states, and exotic elementary excitations [4–8]. Free-fermion systems with topological band structure are very well understood by now and a full classification can be given in terms of generalized symmetries of the single-particle Hamiltonian [9–11]. An important subclass of these systems is that of Chern insulators, such as the Harper-Hofstadter model, where time-reversal symmetry is broken. A hallmark feature of them in the case of two spatial dimensions is the quantized Hall conductivity. A related phenomenon in one-dimensional (1D) systems is the quantized bulk transport upon adiabatic cyclic variations of system parameters [12]. Both are strictly quantized in the thermodynamic limit and at zero temperature and are related to a topological invariant. In the last decade there has been a growing interest in the field of interacting topological insulators and a number of topological states have been discovered that exist only because of interactions. Prime examples are fractional quantum Hall states [7,8] and their generalizations or the Haldane state in antiferromagnetic spin-1 chains [13]. While being incompressible like their noninteracting counterparts, interacting topological insulators are fundamentally different and are characterized by degeneracies and fractional topological charges. One of the challenges in the field of interacting topological insulators is to find suitable and measurable invariants to distinguish between topological phases.

In the present paper we argue that the covariance matrix of single-particle correlations can be used to define directly measurable topological invariants of one-dimensional lattice models, with and without interactions and also at finite

temperatures. In the absence of interactions all equilibrium properties, including topological ones, are encoded in single-particle correlations. Interestingly, as shown in Ref. [14], the latter extends also to finite-temperature states. We here show that topological properties encoded in the covariance matrix of an (interacting) one-dimensional lattice system can be transferred to a second auxiliary chain of noninteracting fermions giving rise to a quantized transport in the latter upon cyclic adiabatic variations of parameters. In mean-field approximation the dynamics of the auxiliary fermions is governed by a single-particle Hamiltonian matrix, which is identical to the covariance matrix of the original model, which is therefore also termed a fictitious Hamiltonian.

If the 1D system is composed of noninteracting fermions the transported charge in the auxiliary chain is just given by the Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) invariant of the original fermion Hamiltonian [5]. If the original system is interacting, the number of auxiliary particles transported in a full cycle is still quantized and defines a topological invariant. We show that this invariant is fully determined by the Zak phase of the single-particle Bloch states of the fictitious Hamiltonian matrix. In many cases it is identical to the many-body generalization of the TKNN invariant by Niu, Thouless, and Wu (NTW) [15,16], but a general relation cannot be derived.

The idea to relate topological properties of interacting systems to single-particle quantities is not new. Building on early work by Volovik [17], Wang *et al.* [18–20], and Gurarie and coworkers [21,22] suggested simplified topological invariants of interacting systems in terms of single-particle Green's functions. As is true for the fictitious Hamiltonian discussed in the present paper, the Green's function at zero frequency is simply related to the single-particle Hamiltonian in the case of noninteracting fermions. Also all generalized symmetries

of the Hamiltonian are inherited by the Green's function in this case.

Our paper is organized as follows. In Sec. II we discuss the topological equivalence between the single-particle Hamiltonian of free fermions and the covariance matrix in a gapped many-body ground state or in a finite-temperature state. In Sec. III we show that the topological properties of an auxiliary chain of fermions weakly coupled to the original one are controlled by an effective single-particle Hamiltonian, the fictitious Hamiltonian above. Thus topological effects, such as quantized bulk transport, can be induced in the auxiliary chain. We illustrate our findings for a simple 1D topological model, the Rice-Mele model (RMM) at half filling. The coupling scheme proposed in Sec. III is diagonal in momentum space and difficult to implement. Therefore we discuss in Sec. IV an approximate implementation using quasilocal couplings only. In Sec. V we extend the discussion to insulating ground states of interacting systems in one dimension, including the case of degeneracy. As a specific example we consider the superlattice Bose-Hubbard model with strong nearest-neighbor (NN) and next-nearest-neighbor (NNN) interactions of bosons. This model has two degenerate, Mott-insulating ground states at quarter filling associated with a fractional topological charge. Finally, making use of the simple relation between the covariance matrices of a noninteracting fermion system at finite and zero temperature, we give an outlook to measurements of finite-temperature topological invariants [14,23,24].

II. FREE FERMIONS: TOPOLOGICAL EQUIVALENCE OF THE HAMILTONIAN AND THE EQUILIBRIUM COVARIANCE MATRIX

A. Model and the equilibrium covariance matrix

Let us first consider gapped ground states of noninteracting fermions on a lattice with particle number conservation. We consider a one-dimensional lattice with lattice constant $a = 1$ and L unit cells and set $\hbar = 1$ throughout this paper. The operators $\hat{c}_{\mu,j}$, $\hat{c}_{\mu,j}^{\dagger}$ describe the annihilation and creation of a fermion in the jth unit cell and with the index $\mu \in \{1, \ldots, p\}$ denoting a possible internal degree of freedom within a unit cell. Assuming translational invariance for simplicity, the Hamiltonian can be written in second quantization in the form

$$H_s = \sum_k \sum_{\mu,\nu=1}^{p} \hat{c}_{\mu}^{\dagger}(k)\, \mathsf{h}_{\mu\nu}(k)\, \hat{c}_{\nu}(k). \tag{1}$$

Some remarks about disordered systems will be made later. We consider a grand canonical setting with chemical potential μ which fixes the total particle number in the system. We assume that H_s has multiple bands and consider an insulator, i.e., assume that μ is chosen within a band gap of H_s. Topological properties of lattice models are characterized by integer-valued invariants, which in many cases are based on geometric phases, such as the Zak phase [1], which is the phase picked up by a Bloch state $|u_n(k)\rangle$ in a specific band n upon parallel transport through the Brillouin zone. In the thermodynamic limit, where the lattice momentum becomes

continuous it reads

$$\phi_{\mathrm{Zak}}^{(n)} = i \int_{\mathrm{BZ}} dk\, \langle u_n(k)|\partial_k u_n(k)\rangle. \tag{2}$$

The Zak phase is in general not unique as it depends on the gauge choice of the Bloch states $|u_n(k)\rangle$. One-dimensional lattice models can possess nontrivial topological properties only when protected by symmetries. A well-known example is the Su-Schrieffer-Heeger model [25], where chiral symmetry enforces topology. The latter is characterized by a quantized value of the Zak phase of zero or π [1], which allows one to distinguish between two topologically different phases. These phases can only be connected by going through a phase transition or breaking the symmetry. As suggested by Rice and Mele [26] adding an appropriate symmetry-breaking term allows one to smoothly connect the phases avoiding the critical point. Adiabatic cyclic variations of the Hamiltonian parameters enclosing the critical point lead to a nontrivial winding of the Zak phase, which defines a topological invariant as integral over the two-dimensional parameter space (k, t) on a torus:

$$\nu_s = \frac{i}{2\pi} \int_0^\tau dt \int_{\mathrm{BZ}} dk\, \langle \partial_t u_n(k)|\partial_k u_n(k)\rangle. \tag{3}$$

In two-dimensional systems with lattice momenta k_x and k_y, one of the two momenta can take over the role of t.

We now argue that the topological properties of the systems, determined by the single-particle Hamiltonian $\mathsf{h}(k)$, are also encoded in the covariance matrix of normal-ordered, single-particle correlations in a gapped many-body ground state:

$$\mathsf{m}_{\mu\nu}(k) = \langle \hat{c}_{\mu}^{\dagger}(k)\hat{c}_{\nu}(k)\rangle. \tag{4}$$

The ground state of a system of free fermions on a lattice is a Gaussian state

$$\rho = \frac{1}{Z} \exp\left\{ -\sum_k \hat{\mathbf{c}}^{\dagger}(k)\, \mathsf{g}(k)\, \hat{\mathbf{c}}(k) \right\} \tag{5}$$

which is fully determined by a $p \times p$ Hermitian matrix $\mathsf{g}(k)$, and we have used the abbreviation $\hat{\mathbf{c}}(k) = (\hat{c}_1(k), \ldots, \hat{c}_p(k))$. The covariance matrix of such a state is directly related to g by [23]

$$\mathsf{m}_{\mu\nu}(k) = \frac{1}{2}\left[1 - \tanh\left(\frac{\mathsf{g}(k)}{2} \right) \right]_{\mu,\nu}. \tag{6}$$

One notices that Eq. (5) has the form of a Gibbs state and indeed an equilibrium state of H_s at finite $\beta = 1/(k_B T)$ is also a Gaussian state with

$$\mathsf{g}(k) = \beta(\mathsf{h}(k) - \mu). \tag{7}$$

The ground state is obtained in the limit $\beta \to \infty$. Most importantly all (single-particle) eigenstates $|\epsilon_n(k)\rangle$ of $\mathsf{h}(k)$ are also eigenstates of $\mathsf{g}(k)$ and thus of the covariance matrix $\mathsf{m}(k)$.

B. Fictitious Hamiltonian

Let us now consider a free-fermion lattice system with fictitious Hamiltonian [23]

$$H_{\mathrm{fict}} = \eta \sum_k \sum_{\mu,\nu=1}^{p} \hat{a}_{\mu}^{\dagger}(k)\, \mathsf{m}_{\mu\nu}(k)\, \hat{a}_{\nu}(k) \tag{8}$$

where $\hat{a}_\mu(k)$ and $\hat{a}_\mu^\dagger(k)$ are fermion annihilation and creation operators in momentum space and $\mathsf{m}(k)$ is the single-particle covariance matrix, Eq. (6).

If $\mathsf{m}(k)$ corresponds to a gapped ground state of a fermion lattice model, the fictitious Hamiltonian $\mathsf{h}_{\text{fict}}(k) = \eta\mathsf{m}(k)$ is also gapped. In an insulating nondegenerate ground state of the original system, the fictitious Hamiltonian $\mathsf{h}_{\text{fict}}(k)$ has a flat spectrum

$$\epsilon_n^{\text{fict}}(k) = \begin{cases} 0 & \text{if } \epsilon_n(k) > \mu \\ \eta & \text{if } \epsilon_n(k) < \mu, \end{cases} \qquad (9)$$

with only two energy "bands." Note that for positive values of η the spectrum of the fictitious single-particle Hamiltonian is reversed with respect to that of $\mathsf{h}(k)$. Depending on the sign of η, the gapped many-body ground state of the fictitious Hamiltonian thus contains all single-particle eigenstates with energies $\epsilon_n(k)$ either above (for $\eta > 0$) or below (for $\eta < 0$) the chemical potential μ. In particular if $\eta > 0$ and the original Hamiltonian has d bands with energy $\epsilon_n(k) < \mu$, $\mathsf{h}_{\text{fict}}(k)$ has a p-d-fold degenerate ground state for every lattice momentum k with energy $\epsilon_n^{\text{fict}}(k) = 0$ and a d-fold degenerate excited state with energy $\epsilon_n^{\text{fict}}(k) = \eta$. If $\eta < 0$ the situation is reversed.

We now argue that the fictitious system inherits the topological properties of the original one. Let us consider adiabatic parameter variations of the system Hamiltonian $H_s(t)$ in a closed loop in time from $t = 0$ to $t = \tau$, such that $H_s(t) = H_s(t + \tau)$. Then the initial and final states of the original system are the same apart from a phase and also the ground state of the fictitious system will return to itself. Since the single-particle eigenstates of the fictitious Hamiltonian are identical to those of the system Hamiltonian, the Wilson loop [27] of the d-fold ($\eta < 0$) degenerate ground-state manifold of H_{fict} reads

$$\nu_{\text{fict}} = \frac{1}{2\pi} \int_0^\tau dt \, \frac{\partial}{\partial t} \operatorname{Im} \ln \det \mathsf{W}$$
$$= \frac{1}{2\pi} \int_0^\tau dt \, \frac{\partial}{\partial t} \operatorname{Im} \operatorname{Tr} \ln \mathsf{W}, \qquad (10)$$

with the $d \times d$ matrix ($n, m \in \{1, \dots, d\}$)

$$\mathsf{W}_{nm} = \exp\left\{ i \int_{\text{BZ}} dk \, \langle u_n(k) | \partial_k u_m(k) \rangle \right\}, \qquad (11)$$

which is identical to the winding of the total Zak phase of all occupied bands for $\eta < 0$:

$$\nu_s = \frac{i}{2\pi} \int_0^\tau dt \sum_{n;\text{occup.}} \int_{\text{BZ}} dk \, \langle \partial_t u_n(k) | \partial_k u_n(k) \rangle. \qquad (12)$$

Here the $|u_n(k)\rangle$ are the Bloch states of the original Hamiltonian (1), with $\langle \mathbf{r} | \epsilon_n(k) \rangle = e^{i\mathbf{k}\cdot\mathbf{r}} \langle \mathbf{r} | u_n(k) \rangle$, which are also eigenstates of the fictitious Hamiltonian (8).

For $\eta > 0$, ν_{fict} corresponds to the winding number of all unoccupied bands $\tilde{\nu}_s = -\nu_s$ of the original system. That the two Hamiltonians H and H_{fict} have the same topological properties in their corresponding insulating ground states is no surprise as one recognizes from Eqs. (6) and (7) that $\mathsf{h}(k)$ can be smoothly deformed into $\mathsf{m}(k)$ without closing the many-body gap and thus both are topologically equivalent.
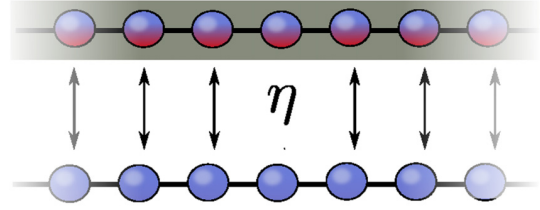

FIG. 1. Sketch of the topology transfer scheme: The original chain of free or interacting fermions (top) is weakly coupled to an auxiliary system of noninteracting fermions (bottom). The coupling is diagonal in momentum space and conserves the particle numbers in both chains.

III. TOPOLOGY TRANSFER FROM A FREE-FERMION SYSTEM TO AN AUXILIARY SYSTEM

We now want to show that the fictitious Hamiltonian of a 1D system can be physically realized by a weak coupling to an auxiliary chain of otherwise noninteracting fermions. In this way topological properties are transferred from one system to a second, auxiliary one. This "topology transfer" manifests itself, e.g., in a quantized charge transport in the auxiliary system upon periodic adiabatic modulations of the original Hamiltonian.

In this section we consider as the "system" noninteracting fermions in a gapped many-body ground state, which is nondegenerate. The generalization to interacting fermions, which also includes the possibility of degeneracies, will be discussed in a subsequent section.

A. Model

To be specific we consider two one-dimensional chains of fermions weakly coupled to each other as indicated in Fig. 1. The system, represented by the top chain, is described by a free-fermion Hamiltonian H_s with annihilation and creation operators $\hat{c}_\mu(k)$ and $\hat{c}_\mu^\dagger(k)$, where k is the lattice momentum and μ labels the degrees of freedom within a unit cell. It is weakly coupled to an "auxiliary" system of otherwise noninteracting fermions with respective annihilation and creation operators $\hat{a}_\mu(k)$ and $\hat{a}_\mu^\dagger(k)$ according to

$$H = H_s + H_\eta, \qquad (13)$$

$$H_\eta = \eta \sum_k \sum_{\mu,\nu=1}^p \hat{c}_\mu^\dagger(k)\hat{c}_\nu(k)\hat{a}_\mu^\dagger(k)\hat{a}_\nu(k). \qquad (14)$$

Here we have assumed a unit cell of p sites. The number of fermions in both chains is individually conserved and we assume that it is chosen such that the combined system has an insulating many-body ground state $|\Phi_0\rangle$. If $|\eta|$ is small compared to the gap of H_s, then the original system is only little affected by the coupling and is approximately described by its ground state $|\Phi_0^s\rangle$, which is also insulating.

For the following discussion it is useful to rewrite the total Hamiltonian in the form

$$H = H_0 + H_1 \qquad (15)$$

where

$$H_0 = H_s + \eta \sum_k \sum_{\mu,\nu=1}^p \langle \hat{c}_\mu^\dagger(k)\hat{c}_\nu(k)\rangle \hat{a}_\mu^\dagger(k)\hat{a}_\nu(k) \qquad (16)$$

contains the system Hamiltonian and the mean-field interaction Hamiltonian, where $\mathsf{m}_{\mu\nu} = \langle \hat{c}_\mu^\dagger(k)\hat{c}_\nu(k)\rangle$ is evaluated in the ground state $|\Phi_0^s\rangle$ of H_s. Thus Eq. (16) represents the fictitious Hamiltonian (8) experienced by the auxiliary fermions. The second term in (15) formally describes the coupling of the auxiliary system to fluctuations in the original system

$$H_1 = \eta \sum_k \sum_{\mu,\nu=1}^p (\hat{c}_\mu^\dagger(k)\hat{c}_\nu(k) - \langle \hat{c}_\mu^\dagger(k)\hat{c}_\nu(k)\rangle)\hat{a}_\mu^\dagger(k)\hat{a}_\nu(k)$$
$$(17)$$

and is responsible for the buildup of entanglement between the two fermion chains.

B. Topology transfer and induced quantized particle transport

We now argue that an adiabatic cyclic modulation of $H_s \rightarrow H_s(t)$ in time will lead to a topological charge pump in the auxiliary system. A modulation $H_s \rightarrow H_s(t) = H_s(t + \tau)$ in time with period τ implies a corresponding cyclic modulation of the covariance matrix $\mathsf{m}(t)$ and the fictitious Hamiltonian. As we will show this gives rise to an adiabatic transport of Q_a particles in the auxiliary system.

If the auxiliary system is prepared in the ground state, Q_a is equal to the total winding number of occupied bands of the original system, ν_s, for an attractive coupling $\eta < 0$, or the total winding number of unoccupied bands, $\tilde{\nu}_s = -\nu_s$, for the case of a repulsive coupling $\eta > 0$:

$$Q_a = \begin{cases} +\nu_s, & \text{for } \eta < 0 \\ -\nu_s, & \text{for } \eta > 0 \end{cases}. \qquad (18)$$

If, on the other hand, the auxiliary system is prepared in the excited state of the fictitious Hamiltonian, the transported charge is opposite, i.e.,

$$Q_a = \begin{cases} -\nu_s, & \text{for } \eta < 0 \\ +\nu_s, & \text{for } \eta > 0 \end{cases}. \qquad (19)$$

Thus the transport in the auxiliary system is indeed of topological nature and not just due to an interaction-induced drag of auxiliary particles with particles in the original system.

To calculate the number of transported particles Q_a, assuming initial preparation in the ground state, for simplicity, we follow the procedure of Niu, Thouless, and Wu [15] and integrate the expectation value of the total momentum operator of the auxiliary fermions $\hat{P}_a = \sum_{i=1}^N \hat{p}_i$ in a nondegenerate adiabatic eigenstate $|\Phi(t)\rangle$ with $|\Phi(t=0)\rangle = |\Phi_0\rangle$ of the total system. Since there is no transport in instantaneous insulating eigenstates, one has to consider the lowest-order nonadiabatic correction:

$$|\Phi(t)\rangle = |\Phi_0(t)\rangle + i \sum_{n\neq0} \frac{|\Phi_n(t)\rangle\langle\Phi_n(t)|\partial_t \Phi_0(t)\rangle}{E_n(t) - E_0(t)}. \qquad (20)$$

$|\Phi_n(t)\rangle$ are the excited eigenstates with energy $E_n(t)$. This gives in lowest order of nonadiabatic corrections

$$Q_a = \frac{1}{L} \int_0^\tau dt\, \langle\Phi(t)|\hat{P}_a|\Phi(t)\rangle$$
$$= \frac{1}{L} \int_0^\tau dt \sum_{n\neq0} \frac{\langle\Phi_0(t)|\hat{P}_a|\Phi_n(t)\rangle\langle\Phi_n(t)|\partial_t \Phi_0(t)\rangle}{E_n(t) - E_0(t)} + \text{c.c.}$$
$$(21)$$

In order to calculate the matrix elements of the momentum operator it is useful to perform a canonical transformation of the Hamiltonian $H(\alpha) = e^{-i\alpha\hat{X}_a} H e^{i\alpha\hat{X}_a}$, with $\hat{X}_a$ being the position operator of the particles of the auxiliary system. Making use of $\hat{P}_a = -i[\hat{X}_a, H] = \partial_\alpha H(\alpha)|_{\alpha=0}$ one finds

$$\langle\Phi_0|\hat{P}_a|\Phi_n\rangle = \langle\Phi_0|\frac{\partial H(\alpha)}{\partial\alpha}|\Phi_n\rangle\bigg|_{\alpha=0}$$
$$= -\langle\partial_\alpha\Phi_0(\alpha)|\Phi_n(\alpha)\rangle(E_n(\alpha) - E_0(\alpha))|_{\alpha=0}. \quad (22)$$

This gives for the transported charge

$$Q_a = -\frac{i}{L} \int_0^\tau dt\, (\langle\partial_\alpha\Phi_0|\partial_t\Phi_0\rangle - \langle\partial_t\Phi_0|\partial_\alpha\Phi_0\rangle)_{\alpha=0}. \quad (23)$$

Following Ref. [15] Q_a is identical to its average over all values of α between $\{-\pi/L, \pi/L\}$ in the thermodynamic limit $L \rightarrow \infty$. Note that $\alpha = \pm\pi/L$ correspond to the same situation. This gives an integral of a Berry curvature over a two-dimensional torus

$$Q_a = -\frac{i}{2\pi} \int_0^\tau dt \int_{-\pi/L}^{\pi/L} d\alpha (\langle\partial_\alpha\Phi_0|\partial_t\Phi_0\rangle - \langle\partial_t\Phi_0|\partial_\alpha\Phi_0\rangle)$$
$$(24)$$

which must be an integer and is a topological invariant ν_a, corresponding to a winding number.

We now show that this winding number is identical to the topological invariant characterizing all occupied bands (respectively, all unoccupied bands) of the original system. To this end we assume a weak coupling η where H_1 can be considered as a perturbation to H_0. For simplicity we assume in the rest of this subsection $\eta < 0$. All arguments can however straightforwardly be applied to $\eta > 0$.

1. Zeroth-order perturbation in H_1

In lowest order of H_1 the evolution of the system chain is unaffected by the auxiliary one and the dynamics of the latter is determined by the mean-field (fictitious) Hamiltonian:

$$H_{\text{fict}}(t) = \eta \sum_k \sum_{\mu,\nu=1}^p \hat{a}_\mu^\dagger(k)\, \mathsf{m}_{\mu\nu}(k,t)\, \hat{a}_\nu(k). \qquad (25)$$

Since the Hamiltonian of the system chain $\mathsf{h}_{\mu\nu}$ is modulated in time with period τ, the covariance matrix is also τ periodic, $\mathsf{m}_{\mu\nu}(k, t + \tau) = \mathsf{m}_{\mu\nu}(k, t)$. In lowest-order perturbation there is no buildup of entanglement between the two subsystems and all eigenstates of the combined system factorize. In particular one has for the ground state of the total system

$$\big|\Phi_0^{(0)}\big\rangle = \big|\Phi_0^s\big\rangle\big|\Phi_0^a\big\rangle \qquad (26)$$

where $|\Phi_0^a\rangle$ is the ground state of (25). The transported charge Q_a is thus given by Eq. (24), with $|\Phi_0\rangle$ replaced by the ground state $|\Phi_0^a\rangle$ of H_{fict}. Since for noninteracting fermions the covariance matrix has the same eigenfunctions as the single-particle Hamiltonian $\mathsf{h}_{\mu\nu}(k, t)$ of the original system, Q_a is given in lowest-order perturbation by the winding number of the total Zak phase of the system Hamiltonian, i.e.,

$$Q_a^{(0)} = \nu_{\text{fict}} = \begin{cases} +\nu_s, & \text{for } \eta < 0 \\ -\nu_s, & \text{for } \eta > 0 \end{cases}. \tag{27}$$

The excited states with energy $E_{n,m}^{(0)}$ can be labeled with two indices n and m corresponding to the system and auxiliary chain, respectively:

$$\big|\Phi_{n,m}^{(0)}\big\rangle = \big|\Phi_n^s\big\rangle\big|\Phi_m^a\big\rangle. \tag{28}$$

2. First-order perturbation in H_1

In first order of H_1 the instantaneous ground state of the combined system reads

$$\big|\Phi_0^{(1)}\big\rangle = \sqrt{p}\big|\Phi_0^{(0)}\big\rangle + \sqrt{1-p}|\tilde{\Phi}\rangle \tag{29}$$

where $\sqrt{p}$ is the overlap between the exact ground state of the system and the unperturbed one. The normalized correction to the state vector reads

$$|\tilde{\Phi}\rangle = \sqrt{\frac{p}{1-p}} \sum_{n \neq 0} \sum_m \frac{\big|\Phi_{n,m}^{(0)}\big\rangle\big\langle\Phi_{n,m}^{(0)}\big|H_1\big|\Phi_0^{(0)}\big\rangle}{E_{n,m}^{(0)} - E_0^{(0)}}. \tag{30}$$

Since $H_1 \sim (\hat{c}_\mu^\dagger \hat{c}_\nu - \langle\Phi_0^s|\hat{c}_\mu^\dagger \hat{c}_\nu|\Phi_0^s\rangle)$, the only states contributing to $|\tilde{\Phi}\rangle$ are those where the system chain is excited $|\Phi_{n>0}^s\rangle|\Phi_m^a\rangle$. Thus the denominator is always larger than the energy gap of the system, i.e., $E_{n,m}^{(0)} - E_0^{(0)} \geqslant \Delta_{\text{gap}}$, and the probability $1-p$ for the exact ground state to contain components orthogonal to the unperturbed one scales as

$$(1-p) \sim \sum_{n \neq 0} \sum_m \frac{\big\langle\phi_0^{(0)}\big|H_1\big|\Phi_{n,m}^{(0)}\big\rangle\big\langle\Phi_{n,m}^{(0)}\big|H_1\big|\Phi_0^{(0)}\big\rangle}{\big(E_{n,m}^{(0)} - E_0^{(0)}\big)^2}$$

$$\sim \mathcal{O}\left(\frac{\eta^2}{\Delta_{\text{gap}}^2}\right). \tag{31}$$

Plugging $|\Phi_0^{(1)}\rangle$ into expression (24) for the transported charge yields

$$Q_a^{(1)} = -\frac{i}{2\pi} \int dt \int d\alpha \left\{ \langle\partial_\alpha \Phi_0^{(1)}|\partial_t \Phi_0^{(1)}\rangle - \text{c.c.} \right\} \tag{32}$$

$$= -\frac{i}{2\pi} \int dt \int d\alpha \left\{ p\big(\langle\partial_\alpha \Phi_0^{(0)}|\partial_t \Phi_0^{(0)}\rangle - \text{c.c.}\big) \right.$$

$$+ (1-p)\big(\langle\partial_\alpha \tilde{\Phi}|\partial_t \tilde{\Phi}\rangle - \text{c.c.}\big)$$

$$+ \sqrt{p(1-p)}\big(\langle\partial_\alpha \Phi_0^{(0)}|\partial_t \tilde{\Phi}\rangle + \langle\partial_\alpha \tilde{\Phi}|\partial_t \Phi_0^{(0)}\rangle$$

$$\left. - \langle\partial_t \tilde{\Phi}|\partial_\alpha \Phi_0^{(0)}\rangle - \langle\partial_t \Phi_0^{(0)}|\partial_\alpha \tilde{\Phi}\rangle\big)\right\}. \tag{33}$$

As can be seen from Eq. (32), also $Q_a^{(1)}$ is an integral of a Berry curvature of a gapped many-body state over a two-dimensional torus and thus an integer. Apart from the prefactors p and $1-p$ the same holds for the integrals in the second and third line of the above expression. Furthermore,

as we will show in the Appendix, the last integral in (33) vanishes exactly.

Thus we can write

$$Q_a^{(1)} = Q_a^{(0)} + (1-p)\tilde{Q}_a, \tag{34}$$

with $\tilde{Q}_a$ being an integer. As long as η is sufficiently small compared to the gap of the unperturbed system, Δ_{gap}, there is no phase transition in the combined system and thus the winding number of the total system as well as the transported charge $Q_a^{(1)}$ must be the same as in the limit $(1-p) \to 0$. The only integer in Eq. (34) compatible with this is $\tilde{Q}_a \equiv 0$. Thus also in first-order perturbation one has

$$Q_a^{(1)} = \nu_{\text{fict}} = \begin{cases} +\nu_s, & \text{for } \eta < 0 \\ -\nu_s, & \text{for } \eta > 0 \end{cases}, \tag{35}$$

i.e., a cyclic adiabatic variation of parameters of the original system Hamiltonian induces a strictly quantized topological charge transport in the auxiliary system.

C. Example

To illustrate the topology transfer let us consider the simplest topologically nontrivial model in one dimension, the RMM [26]. This model, which is sketched in Fig. 2(a), describes lattice fermions in the tight-binding limit with alternating tunnel couplings t_1 and t_2 between neighboring sites and a staggered on-site potential $\pm\Delta$. The unit cell thus contains two sites, labeled A and B. The Hamiltonian can be written in momentum space as

$$H_{\text{RM}} = \sum_k \begin{pmatrix} \hat{c}_A^\dagger(k) \\ \hat{c}_B^\dagger(k) \end{pmatrix}^{\text{T}} \begin{pmatrix} \Delta & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & -\Delta \end{pmatrix} \begin{pmatrix} \hat{c}_A(k) \\ \hat{c}_B(k) \end{pmatrix}. \tag{36}$$

Here $\hat{c}_{A,B}(k)$ denote the fermion annihilation operators of the A or B sublattice in momentum space. The RMM has two bands

$$\varepsilon_\pm(k) = \pm\varepsilon(k) = \pm\sqrt{\Delta^2 + t_1^2 + t_2^2 + 2t_1 t_2 \cos(k)} \tag{37}$$

and the gap closes only for $\Delta = t_1 - t_2 = 0$. Except at this point in parameter space, the RMM has thus an insulating ground state at half filling. Adiabatically changing $\Delta = \Delta(t)$ as well as $t_{1,2} = t_{1,2}(t)$ in a closed loop encircling the degeneracy point leads to a nontrivial winding of the Zak phase of the lower (upper) band, $\nu_s = +1$ ($\tilde{\nu}_s = -1$), associated with a quantized topological transport of one particle. This simple model has been implemented in cold-atom experiments with fermions [28] or strongly interacting, fermionized bosons [29]. The effects of going away from the strong interaction limit of hard-core bosons was discussed in Ref. [30].

We now couple the RMM to an auxiliary chain of otherwise noninteracting fermions at half filling, according to Eq. (14). If we choose, say, a positive value of η, we expect a quantized transport of particles in the auxiliary chain in the opposite direction as in the Rice-Mele ground state.

Figures 2(b) and 2(c) show numerical simulations of the particle transport in the auxiliary chain as a function of time for different values of the coupling $\eta > 0$ ($\eta/\Delta_{\text{gap}}^{\text{RM}} = 0.1, 1, 10$) for the parameter path $\Delta(t) = -6\sin(2\pi\frac{t}{\tau})$, $t_1(t) = 2[1 + \cos(2\pi\frac{t}{\tau})]$, $t_2(t) = 2[1 - \cos(2\pi\frac{t}{\tau})]$, where τ

was chosen sufficiently large to be adiabatic. At $t = 0$ the Rice-Mele system decouples into unit cells, which are symmetric double wells, and we assume that it is prepared in its ground state $|\Phi_0^s\rangle = \prod(|01\rangle + |10\rangle)$. The auxiliary system, on the other hand, is prepared either in the ground state of the fictitious Hamiltonian [Fig. 2(b)] or in its excited state [Fig. 2(c)]. One recognizes quantized transport in opposite directions depending on the preparation of the auxiliary

fermions, despite the same interaction. The latter shows that the induced transport is topological and not just an interaction-induced drag.

On first glance it is surprising that the transport in the auxiliary chain remains strictly quantized to one also beyond the perturbative limit of small η. This can be understood as follows: The Hamiltonian of the total system, consisting of the RMM and the auxiliary chain, can be written in the form

$$H = \sum_k \begin{pmatrix} \hat{c}_A^\dagger(k) \\ \hat{c}_B^\dagger(k) \end{pmatrix}^T \begin{pmatrix} \Delta + \eta \hat{a}_A^\dagger(k)\hat{a}_A(k) & (-t_1 - t_2 e^{-ik}) + \eta \hat{a}_A^\dagger(k)\hat{a}_B(k) \\ (-t_1 - t_2 e^{ik}) + \eta \hat{a}_B^\dagger(k)\hat{a}_A(k) & -\Delta + \eta \hat{a}_B^\dagger(k)\hat{a}_B \end{pmatrix} \begin{pmatrix} \hat{c}_A(k) \\ \hat{c}_B(k) \end{pmatrix}, \tag{38}$$

where $\hat{a}_{A,B}(k)$ denote the fermion annihilation operators of the A or B sublattice in the auxiliary system. The total Hamiltonian conserves the particle number in each subsystem and for every momentum. Thus we can express (38) at double half filling in the particle-number basis $|n_A^c n_B^c n_A^a n_B^a\rangle_k$ as follows:

$$H = \sum_k \begin{pmatrix} |1001\rangle \\ |0101\rangle \\ |1010\rangle \\ |0110\rangle \end{pmatrix}^T \begin{pmatrix} \Delta & -(t_1 + t_2 e^{-ik}) & 0 & 0 \\ -(t_1 + t_2 e^{ik}) & -\Delta + \eta & \eta & 0 \\ 0 & \eta & \Delta + \eta & -(t_1 + t_2 e^{-ik}) \\ 0 & 0 & -(t_1 + t_2 e^{ik}) & -\Delta \end{pmatrix} \begin{pmatrix} \langle 1001| \\ \langle 0101| \\ \langle 1010| \\ \langle 0110| \end{pmatrix}. \tag{39}$$

The 4×4 Hamiltonian has four eigenvalues for every lattice momentum, which can easily be calculated:

$$\varepsilon_\pm(k) = \pm\varepsilon(k) = \pm\sqrt{\Delta^2 + t_1^2 + t_2^2 + 2t_1 t_2 \cos(k)}, \tag{40}$$

$$\varepsilon_\pm(k,\eta) = \eta \pm \sqrt{\varepsilon(k)^2 + \eta^2}. \tag{41}$$

The corresponding bands are separated by gaps, which close for any $\eta \neq 0$ only for $\Delta = t_1 - t_2 = 0$. There is no band crossing because, for $\eta > 0$, $\varepsilon_+(k,\eta) > \varepsilon_+(k) > 0 > \varepsilon_-(k,\eta) > \varepsilon_-(k)$. In the absence of a gap closing upon changing η, there is no topological phase transition and thus the transported charge in both systems remains quantized to the same value for arbitrary values of η, i.e., in particular also beyond the perturbative regime. The many-body gap Δ_{gap} of the full system at double half filling ranges between $|\eta|$ and half the gap of the RMM, $\Delta_{\text{gap}}^{\text{RM}}$, with

$$\Delta_{\text{gap}} \simeq \begin{cases} |\eta| & \text{for } |\eta| \ll 1 \\ \frac{\Delta_{\text{gap}}^{\text{RM}}}{2} & \text{for } |\eta| \gg 1 \end{cases}. \tag{42}$$

Another surprising feature of the simulations in Figs. 2(b) and 2(c) is that all curves lie on top of each other. This is due to the choice of η being positive. In this case the lowest eigenstate of (39), $|\phi_-\rangle$, does not depend on η:

$$|\phi_\pm\rangle \sim -\frac{(\Delta \pm \varepsilon(k))^2}{(t_1 + t_2 e^{ik})^2}|1001\rangle + \frac{\Delta \pm \varepsilon(k)}{t_1 + t_2 e^{ik}}|0101\rangle$$
$$-\frac{\Delta \pm \varepsilon(k)}{t_1 + t_2 e^{ik}}|1010\rangle + |0110\rangle, \tag{43}$$

$$|\phi_\pm^\eta\rangle \sim \frac{t_1 + t_2 e^{-ik}}{t_1 + t_2 e^{ik}}|1001\rangle + \frac{\Delta - \varepsilon_\pm(k,\eta)}{t_1 + t_2 e^{ik}}|0101\rangle$$
$$-\frac{\Delta + \varepsilon_\pm(k,\eta)}{t_1 + t_2 e^{ik}}|1010\rangle + |0110\rangle. \tag{44}$$

Thus not only is the net charge transported in one full cycle of the Thouless pump exactly quantized, but the transport is completely independent of η.

D. Comment on the effect of disorder

Until this point we have assumed lattice translational invariance, which excludes the presence of disorder. Topological properties are however expected to be robust against weak perturbations and thus we have to discuss if the topology transfer survives in the presence of disorder. To this end we can modify the discussion in Sec. III B and add a disorder potential acting on the original system:

$$H_1 \rightarrow H_1 + H_{\text{dis}}. \tag{45}$$

We can repeat the perturbation arguments given in Sec. III B and see that as long as H_{dis} does not lead to a gap closing the transported charge in the auxiliary system remains strictly quantized to the value set by the fictitious Hamiltonian.

IV. IMPLEMENTATION OF TOPOLOGY TRANSFER

The coupling Hamiltonian (14), which induces the topology transfer between system and auxiliary chain, is diagonal in momentum space and as such difficult to realize. In the following we will show that it can be approximately implemented by couplings that are short ranged in real space and thus can be realized, e.g., in experiments with ultracold gases. We will restrict the following discussion to a system with translational invariance of the ground state by two lattice sites. Then transforming to a real-space description via

$$\hat{c}_\mu(k) = \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{\frac{2\pi i j k}{L}} \hat{c}_{j,\mu} \tag{46}$$

where the index $j \in \{1, \ldots, L\}$ denotes the unit cell and $\mu \in \{A, B\}$ denotes the intracell degree of freedom, the fictitious
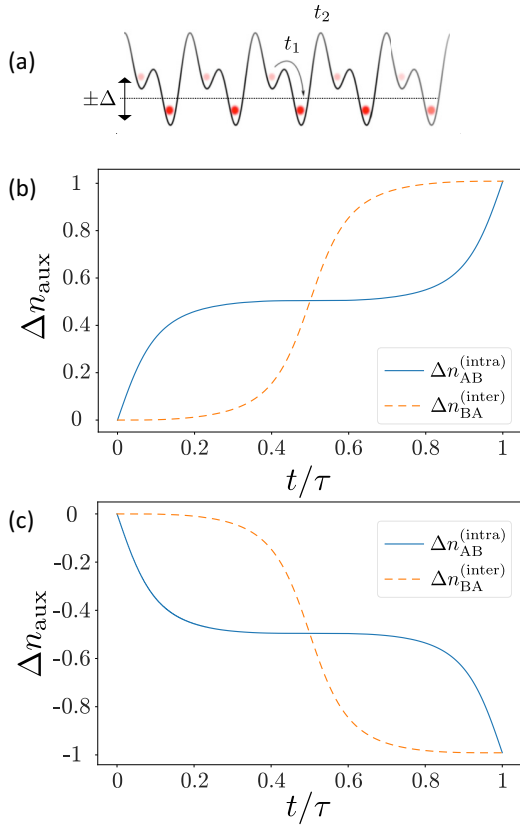
FIG. 2. (a) Rice-Mele model (RMM) with alternating hopping t_1, t_2 and staggered potential $\pm\Delta$. (b) Particle transport in the auxiliary chain between adjacent unit cells ($\Delta n_{BA}^{(inter)}$) and within the unit cell ($\Delta n_{AB}^{(intra)}$) as a function of time upon adiabatic change of parameters of RMM for the auxiliary system prepared in the ground state $|\phi_-\rangle$ of the fictitious Hamiltonian. (c) The same for the auxiliary system being prepared in the excited state $|\phi_+\rangle$. In (b) the curves, which are plotted for $\eta/\Delta_{gap}^{RM} = 0.1, 1, 10$, lie exactly on top of each other. Here Δ_{gap}^{RM} is the minimum gap along the parameter path of the RMM: $\Delta(t) = -6\sin(2\pi\frac{t}{\tau})$, $t_1(t) = 2[1 + \cos(2\pi\frac{t}{\tau})]$, $t_2(t) = 2[1 - \cos(2\pi\frac{t}{\tau})]$.

Hamiltonian (8) can be expressed as

$$H_{fict} = \eta \sum_{m,n} \sum_{\mu,\nu} (\langle \hat{c}_{m,\mu}^\dagger \hat{c}_{n,\nu}\rangle \hat{a}_{m,\mu}^\dagger \hat{a}_{n,\nu} + \text{H.a.}). \qquad (47)$$

Here we have used the translational invariance of ground-state correlations. In an insulating state of the original system, off-diagonal first-order coherences decay exponentially with distance and to good approximation it is sufficient to consider nearest-neighbor correlations. If we denote the left site of a unit cell with the index $\mu = A$ and the right site by $\mu = B$, the only relevant correlations are thus

$$\langle \hat{c}_{m,\mu}^\dagger \hat{c}_{m,\mu}\rangle, \quad \langle \hat{c}_{m,A}^\dagger \hat{c}_{m,B}\rangle, \quad \langle \hat{c}_{m,B}^\dagger \hat{c}_{m+1,A}\rangle, \quad \langle \hat{c}_{m,A}^\dagger \hat{c}_{m-1,B}\rangle.$$

With this we find

$$H_{fict} \approx 2\eta \sum_l \langle \hat{c}_l^\dagger \hat{c}_l\rangle \hat{a}_l^\dagger \hat{a}_l + 2\eta \sum_l (\langle \hat{c}_{l+1}^\dagger \hat{c}_l\rangle \hat{a}_{l+1}^\dagger \hat{a}_l + \text{H.a.}), \qquad (48)$$

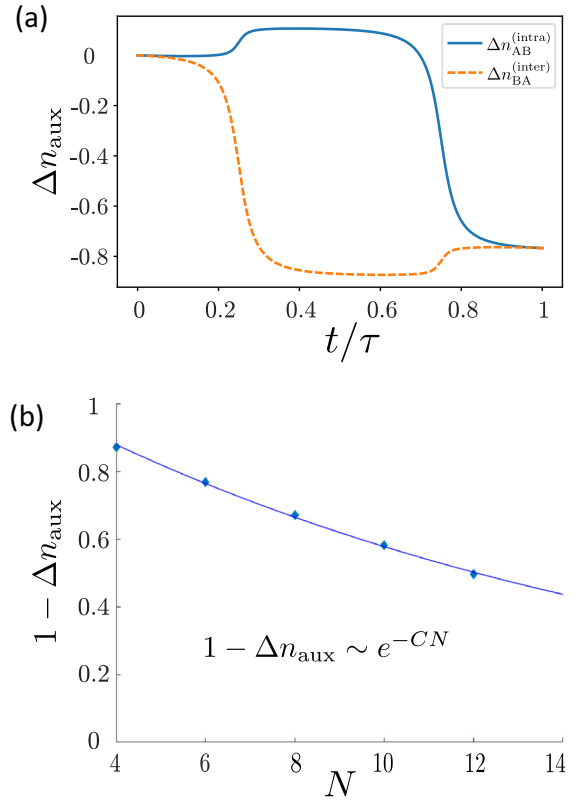FIG. 3. (a) Particle transport in the auxiliary system for system size of $L = 2$ unit cells, and $\eta/\Delta_{gap}^{RM} = 0.25$ between sites within a unit cell (blue) and across unit cells (orange), using the approximate real-space coupling scheme (49). All other parameters are as in Fig. 2. (b) Deviation from quantized transport as a function of number of sites $N = 2L$ for strong coupling to the auxiliary system $\eta/\Delta_{gap}^{RM} = 1$. Even for this large coupling, clearly outside the perturbative regime, one recognizes exponential decay with increasing system size N and a quantized value of the transported charge is approached in the thermodynamic limit.

where we have switched to a a simpler notation for the spatial indices $(m, A) \to l = 2m$ and $(m, B) \to l = 2m + 1$. Equation (48) is the mean-field approximation corresponding to a coupling Hamiltonian

$$\tilde{H}_\eta = 2\eta \sum_l \hat{c}_l^\dagger \hat{c}_l \hat{a}_l^\dagger \hat{a}_l + 2\eta \sum_l (\hat{c}_{l+1}^\dagger \hat{c}_l \hat{a}_{l+1}^\dagger \hat{a}_l + \text{H.a.}), \qquad (49)$$

which describes a *local* density-density coupling and correlated *nearest-neighbor* hopping.

In Fig. 3(a) we have shown numerical results for the adiabatic transport in the auxiliary system coupled to a RMM chain by the approximate interaction (49) obtained from exact diagonalization for a system of $L = 2$ unit cells (of 2×2 sites). Shown is the time dependence in one cycle of a Thouless pump for repulsive interaction ($\eta > 0$) and $\eta/\Delta_{gap}^{RM} = 0.25$. One recognizes that the transport deviates from the expected value of unity. The deviation increases with increasing coupling strength η but decreases exponentially with system size. This is shown to hold true even for a very strong

coupling $\eta/\Delta_{\text{gap}}^{\text{RM}} = 1$, i.e., outside of the perturbative regime, in Fig. 3(b).

V. TOPOLOGY TRANSFER FROM INTERACTING SYSTEMS TO AUXILIARY FERMIONS

We have seen that a chain of auxiliary fermions coupled to a one-dimensional lattice according to Eq. (14) inherits the topological properties encoded in the fictitious Hamiltonian. In particular the transport in the auxiliary chain upon cyclic adiabatic variations of parameters directly detects the topological invariant of the fictitious Hamiltonian.

If the original system is in a gapped ground state of *noninteracting* fermions, ν_{fict} is identical to the TKNN invariant. It is near at hand to ask what happens in the presence of interactions. We will show in the following that the *transfer of topology* also applies if the original system is an *interacting* system with an insulating many-body ground state.

In the absence of interactions, gapped ground states occur only at integer fillings of fermions per unit cell, which implies integer-valued topological charges. This changes with interactions. Here gapped ground states can exist which have fractional fillings and the Lieb-Schultz-Mattis theorem [31] tells us that they are degenerate [32]. We will thus have to discuss the topology transfer in the case of interactions with and without degeneracies separately.

A. Nondegenerate ground state

In Sec. III we have shown that if the total system is prepared in a nondegenerate and gapped ground state $|\Phi_0\rangle = |\Phi_0^s\rangle|\Phi_0^a\rangle + \mathcal{O}(\eta)$ the charge transported in the auxiliary chain is integer quantized and the integer is given by the winding number ν_{fict} of the Zak phase of the fictitious Hamiltonian. Since the discussion in Sec. III B made no reference to the system being a noninteracting one, all results apply equally to interacting particles. Thus

$$Q_a = \nu_{\text{fict}}. \tag{50}$$

Equation (50) gives us the license to interpret the winding number ν_{fict} as topological invariant also for interacting one-dimensional systems with a nondegenerate, gapped ground state [33].

It is near at hand to ask if and what relation exists between ν_{fict} and, e.g., the winding number ν_s of the many-body Zak phase $\phi_{\text{Zak}}^{\text{MB}}$ introduced by NTW [15]:

$$\phi_{\text{Zak}}^{\text{MB}} = i \int_{-\pi/L}^{\pi/L} d\theta \, \langle \Phi_0^s(\theta) | \partial_\theta \Phi_0^s(\theta) \rangle. \tag{51}$$

Here $|\Phi_0^s(\theta)\rangle$ is the nondegenerate many-body ground state of the twisted system Hamiltonian $H_s(\theta) = e^{-i\theta\hat{X}} H_s e^{i\theta\hat{X}}$, with $\hat{X}$ being the total position operator of all particles, and periodic boundary conditions are assumed. While for the examples we studied, such as the superlattice Bose-Hubbard model in a Mott insulating state with one boson per unit cell, it appeared that they are the same, i.e., $\nu_{\text{fict}} = \nu_s$, a strict proof is not obvious. Furthermore a word of caution should be given: It is not guaranteed that the fictitious Hamiltonian remains gapped whenever the many-body ground state of the original system

does. The latter holds true for noninteracting systems but may fail in the presence of interactions.

B. Ground-state degeneracies and fractional transport

Let us now discuss the case of a d-fold degenerate ground state $|\Phi_{0,\lambda}^s\rangle$ of the interacting system, with $\lambda = (1, 2, \ldots, d)$. Furthermore let us restrict ourselves to cases where the degeneracy is accompanied by a spontaneous breaking of lattice-translational invariance. Then it is possible to find a basis $\{|\Phi_{0,\lambda}^s\rangle\}$ such that the application of the lattice shift operator $\hat{T}$ by one unit cell transforms between the basis states:

$$|\Phi_{0,\lambda+1}\rangle = \hat{T} |\Phi_{0,\lambda}\rangle. \tag{52}$$

d-fold application of $\hat{T}$ returns any eigenstate back to itself (up to a phase). Real-space correlations $\langle \hat{c}_j^\dagger \hat{c}_l \rangle$ in any of the degenerate ground states are then only invariant under d successive lattice translations. It is therefore useful to introduce a new, enlarged unit cell. If the single-particle Hamiltonian of the system has a unit cell of p lattice sites, the enlarged unit cell consists of pd sites, and the new Brillouin zone is correspondingly reduced. The fictitious Hamiltonian expressed in this reduced Brillouin zone has thus in general pd bands:

$$H_{\text{fict}}(t) = \eta \sum_k \sum_{\mu,\nu=1}^{pd} \hat{a}_\mu^\dagger(k) \, \mathsf{m}_{\mu\nu}^\lambda(k, t) \, \hat{a}_\nu(k) \tag{53}$$

where $\mathsf{m}_{\mu\nu}^\lambda(k, t) = \langle \Phi_{0,\lambda}(t) | \hat{c}_\mu^\dagger(k) \hat{c}_\nu(k) | \Phi_{0,\lambda}(t) \rangle$ is the covariance matrix of single-particle correlations in the λth ground state. In general the bands of H_{fict} will be separated by gaps and the fictitious fermion system will have a *nondegenerate* gapped ground state when the number of auxiliary fermions is chosen appropriately.

The coupling Hamiltonian (14) that realizes the fictitious Hamiltonian for the auxiliary chain can also be rewritten in the reduced Brillouin zone:

$$H_\eta = \eta \sum_k \sum_{\mu,\nu=1}^{pd} \hat{c}_\mu^\dagger(k) \hat{c}_\nu(k) \hat{a}_\mu^\dagger(k) \hat{a}_\nu(k). \tag{54}$$

Due to the degeneracy only a d-fold cycle of the parameters of the original system returns the fictitious Hamiltonian back to itself. Thus we expect that there is a quantized transport in the auxiliary system only after d pump cycles. This property is in full agreement with the corresponding property of the many-body Zak phase, Eq. (51). As shown by Niu, Thouless, and Wu [15], the topological invariant of an interacting system with a degenerate many-body ground state is an integral of the Berry curvature over an enlarged torus, extending the time integration to τd:

$$\nu_s = \frac{1}{2\pi} \int_0^{\tau d} dt \int_{-\pi/L}^{\pi/L} d\theta \, \text{Im} \langle \partial_t \Phi_{0,\lambda}^s | \partial_\theta \Phi_{0,\lambda}^s \rangle. \tag{55}$$

Let us now consider, as a specific example with degenerate ground states, the one-dimensional extended superlattice Bose-Hubbard model (ExtSLBHM) [34,35]. The single-particle part of the Hamiltonian is identical to the RMM, Eq. (36), only for bosons. In addition there are interactions
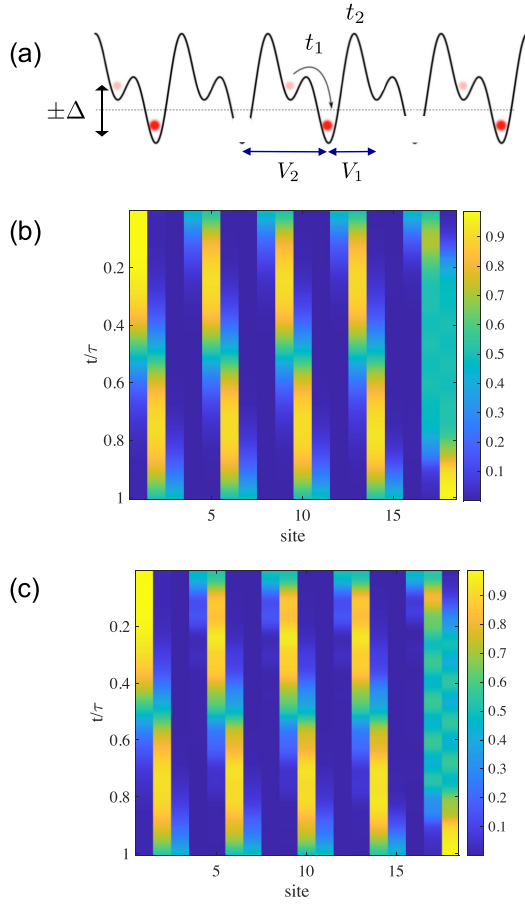
FIG. 4. (a) Extended superlattice Bose-Hubbard model with NN (V_1) and NNN (V_2) interactions. Particle transport at quarter filling in the boson system (b) and the auxiliary fermion system (c) for $\eta = 0.1$, $U = 4000$, $V_1 = 2V_2 = 200$, $t_{1,2}(t) = 50[1 \pm \cos(2\pi t/\tau)]$, $\Delta(t) = 100\sin(2\pi t/\tau)$, and $\tau = 200$.

between particles at the same lattice site with strength U NN and NNN couplings V_1 and V_2, respectively [see Fig. 4(a)]:

$$
\begin{aligned}
H = & -t_1 \sum_{j,\text{even}} \hat{a}_j^\dagger \hat{a}_{j+1} - t_2 \sum_{j,\text{odd}} \hat{a}_j^\dagger \hat{a}_{j+1} + \text{c.c.} \\
& + \Delta \sum_j (-1)^j \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2} \sum_j \hat{n}_j(\hat{n}_j - 1) \\
& + V_1 \sum_j \hat{n}_j \hat{n}_{j+1} + V_2 \sum_j \hat{n}_j \hat{n}_{j+2}
\end{aligned}
\tag{56}
$$

where $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ denotes the particle number at lattice site j. For strong interactions this model has Mott-insulating phases at fractional fillings, e.g., at average filling $\rho_s = 1/4$ per site. The corresponding ground state is doubly degenerate and topologically nontrivial. Adiabatically varying the staggered potential $\Delta(t)$ and the hoppings $t_1(t) - t_2(t)$ in a loop enclosing the origin realizes a topological Thouless pump. Since the ground state is doubly degenerate, a single cycle transfers one ground state into the other one. Here two cycles are needed for an integer quantized particle transport, which reflects the fractional topological charge of this model [35,36]. If we prepare
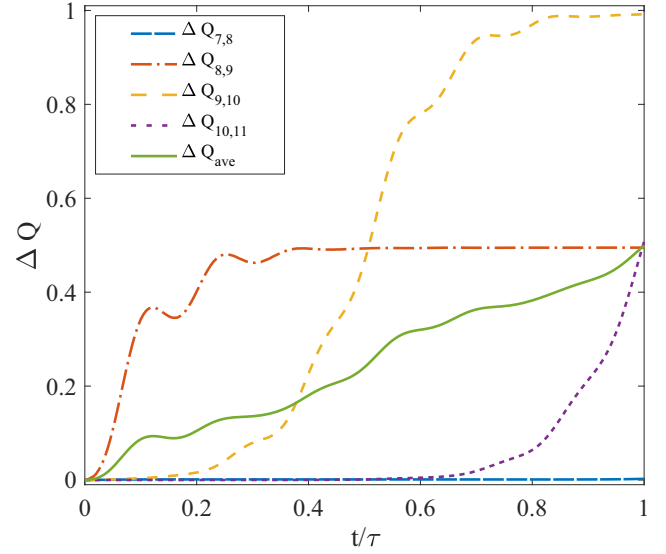

FIG. 5. Particle transport in the auxiliary system across different bonds between adjacent lattice sites as a function of time during a single cycle of the Thouless pump for parameters of Fig. 4. While the net transport through different bonds varies due to the density-wave structure of the ground state, the average net transport ΔQ_{ave} is exactly 0.5, reflecting the fractional topological charge of the ExSLBHM.

the system in one of the two ground states $|\Phi_{0,\pm}\rangle$, with spontaneously broken translational invariance, the corresponding fictitious Hamiltonian $\mathsf{m}_{\mu,\nu}^\pm$ has a single-particle gap, and an insulating many-body ground state exists at average filling of auxiliary fermions of $\rho_{\text{aux}} = 1/4$ per lattice site.

In order to calculate the particle transport in such a state from the full Hamiltonian, we use time-evolving block decimation (TEBD) simulations [37–39], which are based on a representation of the many-body wave function in terms of matrix-product states (MPSs) [40]. Since MPS simulations are much more difficult for periodic boundary conditions, we here choose a finite system of 2×18 sites with open boundary conditions. Furthermore we used the approximate real-space coupling Hamiltonian (49). The results of our simulations for strong interactions are shown in Figs. 4(b) and 4(c), where the particle densities in the ExtSLBHM [Fig. 4(b)] and the auxiliary fermion chain [Fig. 4(c)] are shown as a function of time during a single cycle of the Thouless pump. One recognizes that exactly 1/2 of the particle was transported in both chains, reflecting the fractional topological charge of the ExtSLBHM. We here have chosen $\eta < 0$ corresponding to an attractive interaction between the particles in the two chains. Note that due to the use of open boundary conditions we cannot simulate the full period of two pump cycles since then a particle would be driven into excited states. Due to the open boundary conditions the ExSLBHM has an occupied edge state at the left end at $t = 0$, where $\Delta = 0$, and an occupied edge state at the right after a single cycle, i.e., at $t = \tau$. For the chosen parameter, which corresponds to the atomic limit, the parallel transport in both chains becomes clearly visible.

The time evolution of the pump can be seen in more detail in Fig. 5, where we have shown the particle transport in the

auxiliary chain across different bonds as a function of time. One recognizes that the net transport averaged over one enlarged unit cell is exactly 0.5 as expected.

VI. OUTLOOK TO FINITE-TEMPERATURE SYSTEMS

A. Fictitious Hamiltonian of a free-fermion system at $T > 0$

It is interesting to note that the discussion in Sec. II B for free fermions carries over to a finite-temperature state of the original system. As can be seen from Eqs. (6) and (7) the fictitious Hamiltonian remains gapped in thermal states as long as $T < \infty$, i.e., $\beta > 0$. For the eigenvalues $\tilde{\epsilon}_n^{\text{fict}}(k)$ in a thermal equilibrium state of (1) holds:

$$\tilde{\epsilon}_n^{\text{fict}}(k) = \frac{\eta}{2}\left[1 - \tanh\left(\frac{\beta(\epsilon_n(k) - \mu)}{2}\right)\right], \qquad (57)$$

i.e., the spectrum is no longer flat for $T \neq 0$. If the chemical potential μ is in the middle of a band gap of Hamiltonian (1) of size Δ, the fictitious Hamiltonian is also gapped:

$$\Delta_{\text{gap}}^{\text{fict}} = |\eta| \tanh\left(\frac{\beta\Delta}{4}\right). \qquad (58)$$

Increasing the temperature leads to a reduction of the gap size, which approaches zero only at infinite temperature ($\beta = 0$). Most importantly the Zak phase of the fictitious Hamiltonian has the same topological winding as that of the *ground state* of the original Hamiltonian. From this it was concluded in Ref. [14] that noninteracting fermions at any finite temperature are topologically equivalent to the ground state.

Let us consider again the Rice-Mele model, Eq. (36). One easily calculates the finite-temperature covariance matrix, which has the following nonvanishing terms:

$$\langle \hat{c}_A^\dagger(k)\hat{c}_A(k)\rangle = \frac{1}{2} - \frac{\Delta}{2\varepsilon_k}\tanh\left(\frac{\beta\varepsilon_k}{2}\right),$$

$$\langle \hat{c}_B^\dagger(k)\hat{c}_B(k)\rangle = \frac{1}{2} + \frac{\Delta}{2\varepsilon_k}\tanh\left(\frac{\beta\varepsilon_k}{2}\right),$$

$$\langle \hat{c}_A^\dagger(k)\hat{c}_B(k)\rangle = \frac{(t_1 + t_2 e^{ik})}{2\varepsilon_k}\tanh\left(\frac{\beta\varepsilon_k}{2}\right),$$

$$\langle \hat{c}_B^\dagger(k)\hat{c}_A(k)\rangle = \frac{(t_1 + t_2 e^{-ik})}{2\varepsilon_k}\tanh\left(\frac{\beta\varepsilon_k}{2}\right).$$

This then leads to a fictitious Hamiltonian which, apart from an uninteresting overall energy shift, is again of the structure of the RMM with parameters directly related to those of the original model:

$$t_{1,2} \to \tilde{t}_{1,2}(k) = \frac{\eta}{2\varepsilon_k}\tanh\left(\frac{\beta\varepsilon_k}{2}\right)t_{1,2},$$

$$\Delta \to \tilde{\Delta}(k) = \frac{\eta}{2\varepsilon_k}\tanh\left(\frac{\beta\varepsilon_k}{2}\right)\Delta.$$

Its spectral gap is shown for different temperatures in Fig. 6. One recognizes that except for the singular point of the original RMM ($\Delta = t_1 - t_2 = 0$) the fictitious Hamiltonian remains gapped for all values of T.
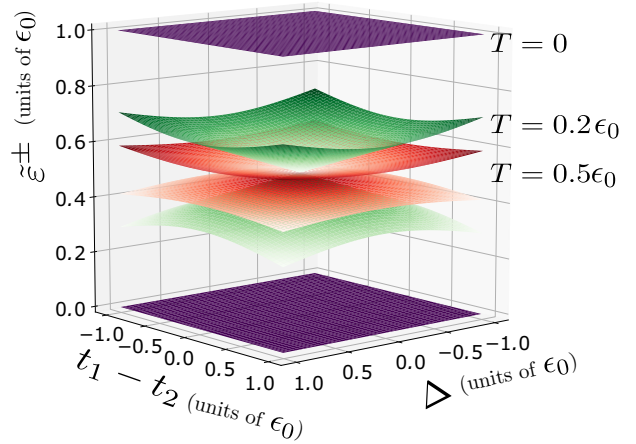

FIG. 6. Minimum energy of the upper band $\tilde{\varepsilon}^+ = \min[\tilde{\epsilon}_0^+(k)]$ and maximum energy of the lower band $\tilde{\varepsilon}^- = \max[\tilde{\epsilon}_0^-(k)]$ of the fictitious Hamiltonian of a Rice-Mele model $H = \epsilon_0 H_{\text{RM}}$ at different temperatures T, where H_{RM} is from Eq. (36) and ϵ_0 sets the overall energy scale. Apart from the singular point of the original RMM ($\Delta = t_1 - t_2 = 0$), the fictitious Hamiltonian remains gapped for all temperatures. (Note that for $T = 0$ the bands are flat but there is a point singularity at the origin, which cannot be resolved in the graph.)

B. Topological invariant of the fictitious Hamiltonian: Many-body polarization

We have argued in the previous sections that the topological properties of the *ground state* of noninteracting lattice fermions are mapped to those of the covariance matrix and the fictitious Hamiltonian, respectively, and both are characterized by the same topological invariant. This invariant can be detected via a quantized transport in an auxiliary system using the topology-transfer scheme discussed in this paper. In the case of an equilibrium state of the fermion system with *nonzero temperature*, or even in a *nonequilibrium steady state*, the fictitious Hamiltonian is still well defined and one can ask for its topological properties encoded in the Zak phase (Wilson loop) of its lowest band. As was shown in Ref. [14] the latter can in fact be used to generalize the concept of topology to finite-temperature and nonequilibrium steady states of noninteracting or Gaussian fermionic systems. It is interesting to note that the topological invariant of the fictitious Hamiltonian for any pure or mixed Gaussian state is identical to the many-body polarization of the system, introduced by Resta (see Refs. [41,42]):

$$\phi_{\text{Zak}}^{\text{fict}} = 2\pi P \equiv \text{Im} \ln \text{Tr}\{\rho \hat{U}\}. \qquad (59)$$

Here $\hat{U} = e^{2\pi i \hat{X}}$ is the unitary momentum shift operator, which shifts the lattice momentum of every particle by one unit. $\hat{X} = \frac{1}{L}\sum_{j=1}^L \sum_{s=1}^p (j + r_s)\hat{n}_{js}$ is the center of mass of all particles in the lattice consisting of L unit cells, with the lattice constant set to 1. $\hat{n}_{js}$ denotes the particle number in the sth site ($s \in \{1, 2, \ldots p\}$) of the jth unit cell and periodic boundary conditions are considered. $0 \leqslant r_s \leqslant 1$ describes the position of the site within the unit cell.

VII. SUMMARY AND CONCLUSION

In the present paper we have shown that topological properties encoded in the covariance matrix of single-particle correlations of a one-dimensional lattice system can be transferred to a second, auxiliary chain of noninteracting fermions, weakly coupled to the first. The coupling is constructed in such a way that the auxiliary particles experience an effective mean-field Hamiltonian, called the fictitious Hamiltonian, which is given by the covariance matrix of the first system. As a consequence an adiabatic cyclic variation of parameters of the original system induces a transport of auxiliary fermions in an insulating ground state. The charge pump is quantized and the transport is determined by the Zak-phase winding number of the fictitious Hamiltonian, which is an integer-valued topological invariant. For noninteracting fermions this number is just the TKNN invariant corresponding to either all occupied or all unoccupied bands. We illustrated the topology transfer for a simple topologically nontrivial system of noninteracting fermions, the Rice-Mele model, coupled to an auxiliary fermion chain, for which exact solutions for the state evolution can be derived. The coupling between the two chains, required for the topology transfer, is diagonal in momentum space and thus difficult to implement. We showed that it can however be well approximated by an interaction that contains only local density-density couplings and correlated nearest-neighbor hoppings.

In the presence of interactions in the original system, the transport induced in the auxiliary chain is still quantized and given by the winding number of the fictitious Hamiltonian, which therefore defines a topological invariant for the interacting system. While without interactions insulating states require integer fillings of fermions per unit cell, here gapped ground states can exist with fractional fillings and degeneracies. In such a case multiple loops in parameter space are needed for the eigenfunctions of the fictitious Hamiltonian to return to themselves, indicating fractional topological charges. We illustrated this for the example of the ExtSLBH model with nearest- and next-nearest-neighbor interactions coupled to a chain of noninteracting fermions. The ExtSLBHM possesses a doubly degenerate Mott-insulating state at quarter filling of bosons. It has a fractional topological charge of 1/2 since only a twofold cyclic variation of the single-particle terms in the Hamiltonian leads to a winding of the many-body Zak phase by 2π. Using numerical TEBD simulations we showed that a coupling of the ExtSLBHM to a chain of noninteracting fermions induces the same fractional transport in the auxiliary system. This suggests that the Zak-phase winding of the fictitious Hamiltonian is identical to the many-body topological invariant of NTW. While this is true in many cases, we could not derive a general relation to this invariant and such a relation may not exist in general. Nevertheless if the winding number of the fictitious Hamiltonian is nontrivial, it provides an observable invariant which allows one to classify topological properties of an interacting system.

The matrix describing the fictitious Hamiltonian of noninteracting fermions in thermal equilibrium is topologically equivalent to the corresponding matrix of the ground state. Thus, as sketched in the last section of the paper, the discussed transfer scheme may also provide a tool to directly observe topological invariants of finite-temperature states, such as the ensemble geometric phase in noninteracting [14,24] or interacting systems [43].

APPENDIX

Here we give a proof that the mixed-term contribution in Eq. (33) vanishes. We consider

$$\varepsilon = \frac{i}{2\pi} \int_0^\tau dt \int_{-\pi/L}^{\pi/L} d\alpha \big(\langle \partial_\alpha \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle - \langle \partial_t \tilde{\Phi} | \partial_\alpha \Phi_0^{(0)} \rangle$$
$$+ \langle \partial_\alpha \tilde{\Phi} | \partial_t \Phi_0^{(0)} \rangle - \langle \partial_t \Phi_0^{(0)} | \partial_\alpha \tilde{\Phi} \rangle \big).$$

Pulling out the derivatives in the bra vectors gives

$$\varepsilon = \frac{i}{2\pi} \int_0^\tau dt \int_{-\pi/L}^{\pi/L} d\alpha \big(\partial_\alpha \langle \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle - \partial_t \langle \tilde{\Phi} | \partial_\alpha \Phi_0^{(0)} \rangle$$
$$+ \partial_\alpha \langle \tilde{\Phi} | \partial_t \Phi_0^{(0)} \rangle - \partial_t \langle \Phi_0^{(0)} | \partial_\alpha \tilde{\Phi} \rangle \big)$$
$$= \frac{i}{2\pi} \int_0^\tau dt \big(\langle \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle + \langle \tilde{\Phi} | \partial_t \Phi_0^{(0)} \rangle \big)_{\pi/L}^{\pi/L}$$
$$- \frac{i}{2\pi} \int_{-\pi/L}^{\pi/L} d\alpha \big(\langle \tilde{\Phi} | \partial_\alpha \Phi_0^{(0)} \rangle + \langle \Phi_0^{(0)} | \partial_\alpha \tilde{\Phi} \rangle \big)_0^\tau.$$

Making use of the orthogonality of the state vectors $\langle \Phi_0^{(0)} | \tilde{\Phi} \rangle = 0$ one thus finds

$$\varepsilon = \frac{i}{2\pi} \int_0^\tau dt \big(\langle \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle - \text{c.c.} \big)_{-\pi/L}^{\pi/L}$$
$$- \frac{i}{2\pi} \int_{-\pi/L}^{\pi/L} d\alpha \big(\langle \tilde{\Phi} | \partial_\alpha \Phi_0^{(0)} \rangle - \text{c.c.} \big)_0^\tau.$$

Since upon a full cycle in either t or α the perturbed ground state $|\Phi_0^{(1)}\rangle$ has to return to itself apart from a phase factor, one has

$$\big| \Phi_0^{(0)}(\alpha, \tau) \big\rangle = e^{i\vartheta(\alpha)} \big| \Phi_0^{(0)}(\alpha, 0) \big\rangle,$$
$$\big| \tilde{\Phi}(\alpha, \tau) \big\rangle = e^{i\vartheta(\alpha)} \big| \tilde{\Phi}(\alpha, 0) \big\rangle,$$
$$\big| \Phi_0^{(0)}(\pi/L, t) \big\rangle = e^{i\lambda(t)} \big| \Phi_0^{(0)}(-\pi/L, t) \big\rangle,$$
$$\big| \tilde{\Phi}(\pi/L, t) \big\rangle = e^{i\lambda(t)} \big| \tilde{\Phi}(-\pi/L, t) \big\rangle,$$

which yields

$$\langle \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle \big|_{\frac{\pi}{L}} = i \frac{\partial \lambda(t)}{\partial t} \langle \Phi_0^{(0)} | \tilde{\Phi} \rangle \big|_{\frac{\pi}{L}} + \langle \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle \big|_{-\frac{\pi}{L}}$$
$$= \langle \Phi_0^{(0)} | \partial_t \tilde{\Phi} \rangle \big|_{-\frac{\pi}{L}}, \qquad \text{etc.,}$$

where in the second line we have used again the orthogonality $\langle \Phi_0^{(0)} | \tilde{\Phi} \rangle = 0$. This finally gives

$$\varepsilon = 0. \tag{A1}$$

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