Quantum Semiclass. Opt. 7 (1995) 357–371. Printed in the UK

The N-atom laser below saturation: a density matrix approach without large-N scaling

Michael Fleischhauer and Stephan Sandhäger

Sektion Physik, Ludwig-Maximilians-Universität, Theresienstrasse 37, D-80333 München, Germany

Abstract. We present a method of deriving a reduced density operator equation for a quantized field interacting with an arbitrary number of atoms that does not neglect atom–atom correlations. As a particular example we analyse a three-level laser operating well below saturation and solve the density matrix equations in steady state. For weak pumping the system is equivalent to the Scully–Lamb model and the corresponding equations are recovered. Using the saturation photon number as a (large) scaling parameter one can derive a Fokker–Planck equation for the Glauber-P-distribution. The stationary solution of the reduced density matrix equation is compared with the solution of the Fokker–Planck equation and—for the case of a single-atom laser—with a numerical solution of the total density matrix equation.

1. Introduction

There are two kinds of standard approaches to the quantum theory of the laser. The first one is based on phase-space representations of the density operator and assumes the existence of a large scaling parameter, which allows the derivation of a Fokker–Planck type dynamical equation. The most common scaling parameter is the number of atoms N as in the approaches of Haken *et al* [1] and Lax and Louisell [2]. The second type of approach is effective single-atom models, which neglect atom–atom correlations, examples being the approaches of Scully and Lamb [3], Mandel [4] and Lugiato [5].

For most laser systems the large-N limit is naturally fulfilled and the single-atom micromaser [6] can accurately be described with the Scully–Lamb master equation. On the other hand, in recent years much progress has been made in the design of high-Q microcavities, which provide a means to experimentally realize lasers with a few atoms [7]. It is therefore of interest to ask for the properties of few-atom lasers, in which case the large-N scaling cannot be applied.

An approach which does not make use of large N-scaling has been given by Smith and Gardiner [8]. They have shown that many of the results of Haken's laser theory do not depend on N being large, and other parameters may be introduced to scale away higher-order derivatives in the equations of motion for the quasi-probabilities.

In this paper we derive a reduced density operator equation for a single-mode three-level laser under adiabatic conditions, for an arbitrary number of atoms. We thereby apply a technique which was introduced by one of us (MF) in [9] which eliminates the atomic degrees of freedom without throwing away atom–atom correlations. In order to truncate the infinite number of terms in the resulting equation for the reduced density matrix, we use the atomic saturation as an expansion parameter. This means we restrict our analysis to laser operation well below saturation.

1355-5111/95/030357+15$19.50 © 1995 IOP Publishing Ltd

If we assume weak pumping our closed three-level system can be compared with the open two-level system of Scully and Lamb [3] and we recover their master equation. The latter demonstrates explicitly that the single-atom approximation in this model is exact, as long as the adiabatic assumption holds. This is a consequence of the relation between N-atom lasers and single-atom lasers in the adiabatic limit derived in [9].

The paper is organized as follows: In section 2 we outline the general method of deriving a reduced density matrix equation for the field in adiabatic and perturbation approximations. In section 3 we introduce a specific laser model, which is a three-level system pumped by an incoherent light field. We calculate the photon number distribution and discuss the relation to results from standard approaches. Furthermore, a comparison to numerical solutions of the total density matrix equations for the case of a single-atom laser is made. A summary is given in section 4.

2. Effective time-evolution and the density matrix equation

We here consider the interaction of a single quantized mode of the radiation field, described by the annihilation and creation operators a and $a^\dagger$, with transitions $|a\rangle_j - |b\rangle_j$ $(j = 1, \ldots, N)$, of N independent atoms. The corresponding interaction Hamiltonian reads

$$V = -\hbar g \sum_{j=1}^{N} A \Sigma_j \tag{1}$$

where $A = a + a^\dagger$ and $\Sigma_j = \sigma_j + \sigma_j^\dagger$; $\sigma_j = |b\rangle_{jj}\langle a|$ being the atomic flip operator. For the sake of notational convenience we did not use the rotating-wave approximation in (1), we will do so, however, at a later point.

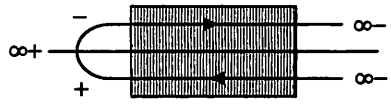

Figure 1. Schwinger–Keldysh time contour. The arrows indicate the time ordering.

We now introduce an interaction picture in which the density matrix or the state vector evolves with V and the operators evolve according to their free Hamiltonians plus the interaction with reservoirs, which we assume to be different for each individual atom. The total density operator W_0 at some initial time t_0 is assumed to factorize into subsystem operators, $W_0 = \rho_0 \otimes \prod_j \rho_j$, where ρ_0 is the field vacuum (or thermal state) and ρ_j is the initial density operator of the jth atom. Following standard techniques of non-equilibrium quantum statistics [10, 11] we can express the expectation value of any operator X_H in the Heisenberg picture (indicated by the subscript 'H') at time t in terms of interaction picture quantities. This is done with the help of a time-evolution operator S_C defined on a Schwinger–Keldysh time contour C [11] depicted in figure 1

$$\langle X_H(t)\rangle = \mathrm{Tr}\left\{W_0 T_C[S_C X(t)]\right\} \tag{2}$$

where

$$S_C = T_C \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_C \mathrm{d}\bar{\tau}\, V(\bar{\tau})\right\}. \tag{3}$$

Here the subscript C denotes integration along the time contour and the overbar indicates a time argument on the same. T_C is the time-ordering operator on C, which corresponds

to causal time ordering, T_+, on the upper branch of C and to anticausal time-ordering, T_-, on the lower branch. In addition it orders all operators with time arguments from the upper branch to the right of all operators with time arguments from the lower branch.

If $X(t)$ is a field variable, we may perform the partial trace of S_C with respect to the atomic degrees of freedom including the degrees of freedom of the atomic reservoirs

$$\begin{aligned}\langle X_H(t)\rangle &= \mathrm{Tr}_{\mathrm{field}}\left\{\rho_0 T_C \langle S_C\rangle_{\mathrm{atom}} X(t)\right\} \\ &= Tr_{\mathrm{field}}\left\{\rho_0 T_C S_C^{\mathrm{eff}} X(t)\right\}. \end{aligned} \tag{4}$$

As shown in [9], the effective (non-unitary) time evolution operator S_C^{eff} reads

$$S_C^{\mathrm{eff}} = T_C \exp\left\{N \sum_{m=1}^{\infty} \frac{(\mathrm{i}g)^m}{m!} \int_C \mathrm{d}\bar{\tau}_1 \cdots \int_C \mathrm{d}\bar{\tau}_m \langle\langle \Sigma(\bar{\tau}_1), \ldots, \Sigma(\bar{\tau}_m)\rangle\rangle A(\bar{\tau}_1)\cdots A(\bar{\tau}_m)\right\} \tag{5}$$

where the $\langle\langle m\rangle\rangle$'s are the so-called single-atom cumulants. They are functions of T_C-ordered correlation functions of m operators of a *single* atom *in the interaction picture*, and therefore do not contain the coupling to the laser mode. In order to calculate these correlation functions, we have to solve the equations of motion for the atomic operators only in the presence of the reservoir interaction. In the usual Born- and Markov-approximations these are linear quantum Langevin equations and the correlation functions can in most cases easily be derived with the help of the quantum regression theorem [12]. As can be seen from (5), the elimination of the atomic degrees of freedom yielded an effective time evolution, which is equivalent to a dissipative nonlinear optical process with all-order nonlinearities.

At this point we apply the rotating-wave approximation, that is we take into account only combinations of atomic lowering operators σ with creation operators $a^\dagger$ of the field mode and correspondingly atomic raising operators $\sigma^\dagger$ with annihilation operators a. Since the reservoirs do not generate atomic coherences, only correlation functions with equal numbers of lowering and raising operators are non-zero. In this case we may write S_C^{eff} as

$$\begin{aligned} S_C^{\mathrm{eff}} = T_C \exp\Bigg\{N \sum_{j=1}^{\infty} \frac{(\mathrm{i}g)^{2j}}{(2j)!} \int_C \mathrm{d}1 \cdots \int_C \mathrm{d}j' \binom{2j}{j} \\ \times S_j(1,\ldots,j;1',\ldots,j')\, a(1)\cdots a(j) a^\dagger(1')\cdots a^\dagger(j')\Bigg\} \end{aligned} \tag{6}$$

where $1 \triangleq \bar{\tau}_1$ etc. The lowest-order cumulants S_j have the following explicit form

$$S_1(1;1') = \langle T_C[\sigma^\dagger(1)\sigma(1')]\rangle \tag{7}$$

$$\begin{aligned} S_2(1,2;2',1') = \langle T_C[\sigma^\dagger(1)\sigma^\dagger(2)\sigma(2')\sigma(1')]\rangle - \langle T_C[\sigma^\dagger(1)\sigma(1')]\rangle\langle T_C[\sigma^\dagger(2)\sigma(2')]\rangle \\ - \langle T_C[\sigma^\dagger(1)\sigma(2')]\rangle\langle T_C[\sigma^\dagger(2)\sigma(1')]\rangle. \end{aligned} \tag{8}$$

They are symmetric in the time arguments that belong to the same kind of atomic operators. S_C^{eff} contains the atomic degrees of freedom only via these interaction-picture correlation functions. Since the atoms interact with each other only via the laser mode, there are no explicit atom–atom correlations in S_C^{eff} and the number of atoms enters as a common prefactor. This has the important consequence that the master equations of the single-atom laser and the N-atom laser (in the adiabatic limit) differ only by a factor N in front of the interaction Liouvillian [9]. The price one has to pay for the elimination of the atomic degrees of freedom in S_C^{eff} at such a general level is the infinite sum in the exponent. It is clear that any further analysis is only possible if this infinite sum can be cast into a finite form or may be truncated. The latter case requires that a perturbation expansion in the

atom–field coupling is permissible. If we restrict our discussion to a laser operating well below saturation, such that for all relevant photon numbers n

$$\frac{g^2 n}{\Gamma^2} \ll 1 \qquad \text{or} \qquad n \ll n_s \equiv \frac{\Gamma^2}{g^2} \tag{9}$$

where n_s is the saturation photon number, a perturbation expansion is possible and we may disregard all terms with $j > 2$. Thus we have

$$S_C^{\text{eff}} \simeq T_C \exp\Big\{-g^2 N \int\!\!\int_C \mathrm{d}1\,\mathrm{d}2\, S_1(1;2) a(1) a^\dagger(2) + \frac{g^4 N}{4} \int\!\!\int\!\!\int\!\!\int_C \mathrm{d}1 \cdots \mathrm{d}4\, S_2(1,2;3,4) a(1) a(2) a^\dagger(3) a^\dagger(4)\Big\}. \tag{10}$$

We now discuss the derivation of a master equation for the reduced field density operator ρ from S_C^{eff}. For this, we note that the dynamical evolution of the matrix element $\rho_{nm} = \langle n|\rho|m\rangle$ in the Schrödinger picture, where $|n\rangle$ and $|m\rangle$ are Fock states, is determined by the time evolution of the operator $P_{mn}^H = |m\rangle\langle n|$ in the Heisenberg picture

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho_{nm}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle P_{mn}^H(t)\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big\langle T_C P_{mn}(t) S_C^{\text{eff}}\big\rangle \\ &= \Big\langle T_C\Big(\frac{\mathrm{d}}{\mathrm{d}t}P_{mn}(t)\Big) S_C^{\text{eff}}\Big\rangle + \Big\langle T_C P_{mn}(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t} S_C^{\text{eff}}\Big)\Big\rangle. \end{aligned} \tag{11}$$

The first term in the second line describes the free evolution of the field and the cavity damping. The second term accounts for the interaction with the atomic system and will be analysed in the following. Using the decomposition of a contour integral into ordinary time integrals

$$\int_C \mathrm{d}\bar{\tau}\, f(\bar{\tau}) = \Big[\int_{t_0}^{t} \mathrm{d}\tau^{(+)} - \int_{t_0}^{t} \mathrm{d}\tau^{(-)}\Big] f(\bar{\tau}) \tag{12}$$

where the superscripts $(\pm)$ denote the branch of the time contour, we can determine the time derivative of S_C^{eff}. We find from (11) that

$$\begin{aligned} \dot{\rho}_{nm}(t) = {} & -\mathrm{i}\omega(n-m)\rho_{nm}(t) + \dot{\rho}_{nm}\big|_{\text{diss}} \\ & - g^2 N \int_{t_0}^{t} \mathrm{d}\tau_1 \Big\{ S_1^{++}(t;\tau_1)\big\langle P_{mn}^H(t) a_H(t) a_H^\dagger(\tau_1)\big\rangle \\ & - S_1^{-+}(t;\tau_1)\big\langle a_H(t) P_{mn}^H(t) a_H^\dagger(\tau_1)\big\rangle + 6 \text{ terms}\Big\} \\ & + \frac{g^4 N}{2} \int_{t_0}^{t} \mathrm{d}\tau_1 \int_{t_0}^{t} \mathrm{d}\tau_2 \int_{t_0}^{t} \mathrm{d}\tau_3 \Big\{ S_2^{++++}(t,\tau_1;\tau_2,\tau_3) \\ & \times \big\langle P_{mn}^H(t) a_H(t) T_+\big[a_H^\dagger(\tau_1) a_H(\tau_2) a_H^\dagger(\tau_3)\big]\big\rangle \\ & - S_2^{-+++}(t,\tau_1;\tau_2,\tau_3)\big\langle a_H(t) P_{mn}^H(t) T_+\big[a_H^\dagger(\tau_1) a_H(\tau_2) a_H^\dagger(\tau_3)\big]\big\rangle + 30 \text{ terms}\Big\} \end{aligned} \tag{13}$$

where the superscripts '$\pm$' on the cumulants denote again the branch of the time contour, i.e. $S_1^{-+}(t_1;t_2) \equiv S_1(\bar{t}_1^{(-)};\bar{t}_2^{(+)})$ etc. Here ω is the resonance frequency of the laser cavity

and $\dot{\rho}_{nm}|_{\text{diss}}$ denotes the contribution from the coupling to the cavity reservoir. The r.h.s. of (13) can be written in terms of matrix elements of the field density operator if all cumulants are essentially zero for $\tau_\mu \neq t$. This is the case if the atomic correlations decay on a time scale much smaller than the one of the field evolution, i.e. in the *adiabatic limit.* In this case we have

$$\begin{aligned} &S_1^{\mu\nu}(t_1; t_2) \sim s_1^{\mu\nu}(t_1, t_2)\,\delta(t_1 - t_2) \\ &S_2^{\mu\nu\eta\gamma}(t_1, t_2; t_3, t_4) \sim s_2^{\mu\nu\eta\gamma}(t_1, t_2, t_3, t_4)\,\delta(t_1 - t_2)\,\delta(t_1 - t_3)\,\delta(t_1 - t_4). \end{aligned} \tag{14}$$

s_1 and s_2 are proportional to step-functions $\Theta(t_\mu - t_\nu)$, which keep track of the order of the time arguments, and therefore determine the ordering of the field operators when applying (14) to (13). With (14) we can carry out the time integrations on the r.h.s. of (13). A typical term resulting from this would read, for example

$$\dot{\rho}_{nm} = +\frac{g^4 N}{2} s_2^{+-++}\langle a_H(t) P_{mn}^H(t) a_H^\dagger(t) a_H(t) a_H^\dagger(t)\rangle + \cdots. \tag{15}$$

Making use of the cyclic property of the trace, the term on the r.h.s. can, furthermore, be expressed in terms of a density matrix element in the Schrödinger picture

$$\dot{\rho}_{nm} = \cdots + \frac{g^4 N}{2} s_2^{+-++}\langle n|a^\dagger a a^\dagger \rho(t) a|m\rangle + \cdots. \tag{16}$$

Since (16) holds for any n and m, we can write it in operator form

$$\dot{\rho} = \cdots + \frac{g^4 N}{2} s_2^{+-++} a^\dagger a a^\dagger \rho(t) a + \cdots \tag{17}$$

which gives the desired master equation for the field density operator ρ.

The only assumptions made so far are that a perturbation expansion in the atom–field coupling is permissible and that the field evolution is much slower than the atomic dynamics (adiabatic limit). We have not yet introduced a specific laser model and the theory is therefore applicable to any kind of laser system. We could also have introduced a coherent driving field in order to describe optical bistability. In the next section we will pick a specific laser model, calculate the cumulants and derive an explicit form of the master equation.

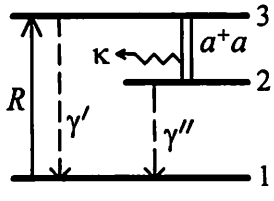

Figure 2. Three-level system. The laser mode couples levels 3 and 2. Population in 3 is provided by the pump out of level 1 with rate R and relaxation occurs between 3 and 1 with rate γ' and 2 and 1 with rate γ''.

3. The three-level laser

In this section we take as a specific model for the laser medium the three-level system shown in figure 2. The cavity mode is coupled to the 2–3 transition. Population in level 3 is provided by an incoherent pump with rate R out of level 1 and relaxation occurs between levels 3 and 1 with a rate γ' and between levels 2 and 1 with a rate γ''. In order to simplify the calculations we here neglect spontaneous decay between levels 3 and 2, but note that this is not an essential requirement.

3.1. Semiclassical dynamics

We first analyse the semiclassical dynamics of the three-level laser in the rate equation approximation. The corresponding density matrix equations of the atomic system read in a rotating frame [3]

$$\begin{aligned}\dot{\rho}_{33} &= -\gamma'\rho_{33} + R\rho_{11} + \mathrm{i}(\Omega\rho_{32} - \text{c.c.})\\ \dot{\rho}_{22} &= -\gamma''\rho_{22} - \mathrm{i}(\Omega\rho_{32} - \text{c.c.})\\ \dot{\rho}_{32} &= -\Gamma\rho_{32} + \mathrm{i}\Omega(\rho_{33} - \rho_{22})\end{aligned} \tag{18}$$

where $\Omega = g\alpha$ is the Rabi frequency of the laser field with complex amplitude α and $\Gamma = (\gamma' + \gamma'')/2$. The equation of motion for the 'number of photons', $n = \alpha^*\alpha$, is

$$\dot{n} = -2\kappa n - (\mathrm{i}\Omega^* N\rho_{32} - \text{c.c.}). \tag{19}$$

In the rate equation approximation we set the time derivatives in (18) equal to zero and substitute the result into (19). This yields in a second-order perturbation expansion in g^2

$$\dot{n} \simeq -2\kappa n + \frac{2g^2NR}{\gamma\Gamma}n - \frac{8g^4NR}{\gamma\Gamma^2}\frac{(R+\Gamma)}{\gamma\gamma''}n^2 \tag{20}$$

where $\gamma = \gamma' + R$. The steady-state solution of this equation is

$$n_{\mathrm{sc}} = \frac{A-\kappa}{B} \tag{21}$$

where $A = g^2NR/(\gamma\Gamma)$ is the linear gain coefficient and $B = 4g^2(R+\Gamma)A/(\Gamma\gamma\gamma'')$ is the saturation coefficient.

3.2. Heisenberg–Langevin equations and cumulants

We now turn to the calculation of the cumulants of the atomic system in order to derive a quantum master equation for the field mode as outlined in section 2. In the absence of the atom–field coupling, we can write down the Heisenberg–Langevin equation for the atomic operators using the standard Born- and Markov-approximations for the reservoir interactions [13]. Denoting

$$\sigma = |2\rangle\langle 3| \tag{22}$$

$$\sigma_\mu = |\mu\rangle\langle\mu| \tag{23}$$

we find

$$\begin{aligned}\dot{\sigma}_3 &= R\sigma_1 - \gamma'\sigma_3 + F_3\\ \dot{\sigma}_2 &= -\gamma''\sigma_2 + F_2\\ \dot{\sigma}_1 &= -R\sigma_1 + \gamma'\sigma_3 + \gamma''\sigma_2 + F_1\\ \dot{\sigma} &= -(\Gamma + \mathrm{i}\omega_0)\sigma + F_\sigma.\end{aligned} \tag{24}$$

ω_0 is the resonance frequency of the 2–3 transition, and the F_μ are noise operators with zero mean value and δ-type correlations [13]. For simplicity, we assume in the following resonance between the cavity mode and the atomic transition, i.e. $\omega_0 = \omega$. The steady-state

values of the mean populations and the mean polarization are

$$\langle\sigma_2\rangle = \langle\sigma\rangle = 0 \tag{25}$$

$$\langle\sigma_3\rangle = R/\gamma \tag{26}$$

$$\langle\sigma_1\rangle = \gamma'/\gamma. \tag{27}$$

The correlation functions appearing in the cumulants, equations (7) and (8), can be obtained with the help of the regression theorem [12] either by a Laplace transformation or by a direct integration of (24). Such an integration of the Heisenberg–Langevin equations yields

$$\sigma(t) = \sigma(t')\mathrm{e}^{-(\Gamma+\mathrm{i}\omega)(t-t')} \tag{28}$$

$$\sigma_2(t) = \sigma_2(t')\mathrm{e}^{-\gamma''(t-t')} \tag{29}$$

$$\sigma_3(t) = \sigma_3(t')\mathrm{e}^{-\gamma(t-t')} + \frac{R}{\gamma}\left(1-\mathrm{e}^{-\gamma(t-t')}\right) - \frac{R}{\gamma-\gamma''}\sigma_2(t')\left(\mathrm{e}^{-\gamma''(t-t')} - \mathrm{e}^{-\gamma(t-t')}\right). \tag{30}$$

We here have dropped the noise terms, since they do not contribute in the calculation of correlation functions if we start expressing the operator with the largest time argument in terms of operators with the second largest time argument and so on (regression theorem). Note, that due to the T_C-time ordering of the cumulants, operators with successive time arguments always stand next to each other. In this way we find the first-order cumulants

$$\begin{aligned}
S_1^{++}(1;2) &= \Theta(1-2)\langle\sigma_3\rangle\mathrm{e}^{\mathrm{i}\omega(1-2)}\mathrm{e}^{-\Gamma(1-2)} \\
S_1^{-+}(1;2) &= \langle\sigma_3\rangle\mathrm{e}^{\mathrm{i}\omega(1-2)}\mathrm{e}^{-\Gamma|1-2|} \\
S_1^{+-}(1;2) &= 0 \\
S_1^{--}(1;2) &= \Theta(2-1)\langle\sigma_3\rangle\mathrm{e}^{\mathrm{i}\omega(1-2)}\mathrm{e}^{-\Gamma(2-1)}.
\end{aligned} \tag{31}$$

The derivation of the second-order cumulants is also not difficult but somewhat lengthy. For illustration purposes we here only give

$$\begin{aligned}
S_2^{-+-+}(1,2;3,4) = \langle\sigma_3\rangle\mathrm{e}^{i\omega(1+2-3-4)} \\
\times\Big\{&\Theta(3,2,1,4)\left[\mathrm{e}^{-\Gamma(3-2)}\mathrm{e}^{-\gamma''(2-1)}\mathrm{e}^{-\Gamma(1-4)} - \langle\sigma_3\rangle\mathrm{e}^{-\Gamma(3-2)}\mathrm{e}^{-2\Gamma(2-1)}\mathrm{e}^{-\Gamma(1-4)}\right] \\
&+\Theta(3,2,4,1)\left[\mathrm{e}^{-\Gamma(3-2)}\mathrm{e}^{-\gamma''(2-4)}\mathrm{e}^{-\Gamma(4-1)} - \langle\sigma_3\rangle\mathrm{e}^{-\Gamma(3-2)}\mathrm{e}^{-2\Gamma(2-4)}\mathrm{e}^{-\Gamma(4-1)}\right] \\
&+\Theta(3,1,2,4)\left[\mathrm{e}^{-\Gamma(3-1)}\mathrm{e}^{-\gamma(1-2)}\mathrm{e}^{-\Gamma(2-4)}\langle\sigma_1\rangle\right] \\
&+\text{terms with } (1\leftrightarrow 4) \text{ and } (2\leftrightarrow 3)\Big\}.
\end{aligned} \tag{32}$$

Here $\Theta(1,2,3,4) \equiv \Theta(1-2)\Theta(2-3)\Theta(3-4)$. One recognizes from (31) and (32) that the cumulants behave essentially like δ-functions, if the atomic decay rates are large compared to the inverse of the characteristic time of the field evolution, which in many cases is the cavity decay rate κ.

3.3. Master equation for the field density operator

Substituting the adiabatic versions of the cumulants (31) and (32) into (13) we can derive a master equation for the field density operator. After some algebra we find

$$\dot{\rho}(t) = -i\omega[a^\dagger a, \rho] - \kappa[a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger] - \frac{g^2 N}{\Gamma}\langle\sigma_3\rangle[aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a]$$
$$+ \frac{g^4 N}{\Gamma^2}\langle\sigma_3\rangle\left\{\frac{\langle\sigma_1\rangle}{\gamma}[aa^\dagger aa^\dagger\rho + \rho aa^\dagger aa^\dagger - 2aa^\dagger\rho aa^\dagger]\right.$$
$$- \left(\frac{2}{\gamma''} + 2\frac{\langle\sigma_1\rangle}{\gamma} - \frac{\langle\sigma_3\rangle}{\Gamma}\right)[a^\dagger aa^\dagger\rho a + a^\dagger\rho aa^\dagger a - 2aa^\dagger\rho aa^\dagger]$$
$$+ \langle\sigma_3\rangle\left(2\left(\frac{1}{\gamma} + \frac{1}{\gamma''}\right) + \frac{3}{\Gamma}\right)[aa^\dagger a^\dagger\rho a + a^\dagger\rho aaa^\dagger - 2a^\dagger a^\dagger\rho aa]$$
$$\left. - \frac{\langle\sigma_3\rangle}{\Gamma}[aaa^\dagger a^\dagger\rho + \rho aaa^\dagger a^\dagger - 2a^\dagger a^\dagger\rho aa]\right\} \tag{33}$$

which is the main result of our paper. The first term in (33) describes the free evolution of the field, the second and third terms the cavity damping and the linear gain. The curly brackets contain the lowest-order saturation terms, which—because of the wrong signs—are not all of Lindblad form [14]. As a consequence, the positivity of the equation is not generally guaranteed, which is an artifact of any perturbative density operator equation for gain systems†. Problems arise however only for Fock-state matrix elements with very large photon numbers n, such that $g^2 n/\Gamma^2$ is no longer small, violating the perturbation assumption (9). If we derive an equation for the mean number of photons from (33) and take the semiclassical limit by setting $\langle n^2\rangle \approx \langle n\rangle^2$, we obtain the semiclassical steady-state expression (21).

As is well known for incoherently pumped lasers, there is no preferred phase and (33) is diagonal in the Fock representation. In the steady state we find a recurrence relation between ρ_{nn}, $\rho_{n\pm1\,n\pm1}$ and $\rho_{n-2\,n-2}$. The $\rho_{n+2\,n+2}$-term is missing in this relation since we have neglected the spontaneous decay between levels 3 and 2. Due to the absence of this term, we can easily calculate ρ_{nn}/ρ_{00} with $(n = 1, 2, \ldots)$ by successive application of the recurrence relation. Since the perturbation assumption is violated for very large n, we have to set $\rho_{nn} \equiv 0$ for n larger than some critical value n_{cr} which is of the order of the saturation photon number. ρ_{00} is eventually fixed by the normalization condition $\sum_{n=0}^{n_{cr}} \rho_{nn} = 1$. In figure 3 we have plotted the photon number distribution ρ_{nn} for different relative linear gain $a \equiv A/\kappa = g^2 N\langle\sigma_3\rangle/\Gamma\kappa$. The graphs show the typical behaviour for a laser. Below threshold ($a = 0.92$), the distribution is very similar to that of a thermal state and above threshold ($a = 1.08$) we observe a nearly Gaussian distribution, which will eventually becomes Poissonian if the gain is further increased. In figure 4 the mean number of photons $\langle n\rangle$ and the normalized photon-number fluctuations $\langle\Delta n^2\rangle/\langle n\rangle$ are plotted as a function of the relative linear gain. In order to stay within the perturbative regime, a should only be slightly larger than one. A typical feature of a laser at threshold is the maximum of the photon-number fluctuations for $a \approx 1$. It should be noted that figures 3 and 4 display the laser properties for arbitrary numbers of atoms, since the steady-state properties only depend on κ/N. Consequently, we may compare the results with those from a numerical evaluation of the total (field + atoms) density matrix equation for the case of a single atom.

† A similar problem arises, for example, in the perturbation approximation of the Scully–Lamb laser equation (see [3]).

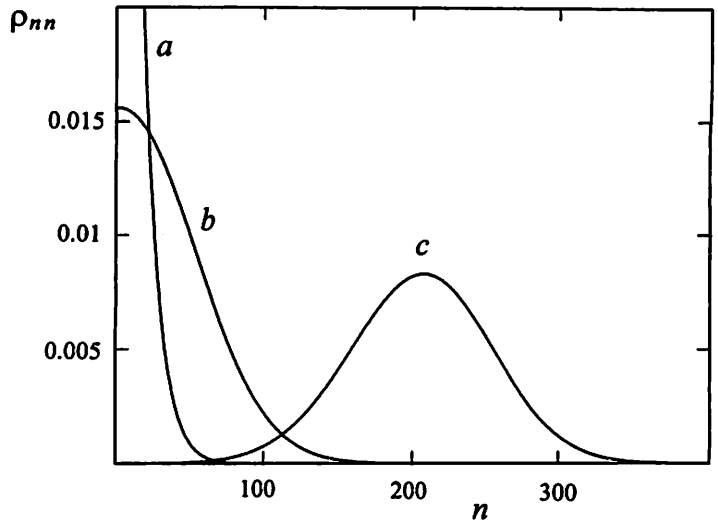

Figure 3. Photon number distribution for relative linear gain $a = 0.92$ (curve (a)), $a = 1$ (curve (b)) and $a = 1.08$ (curve (c)). The parameters are $g^2 = 5.4 \times 10^{-4}$, $\gamma' = 1$, $\gamma'' = 3$ and $\kappa/N = 2.5 \times 10^{-4}$.

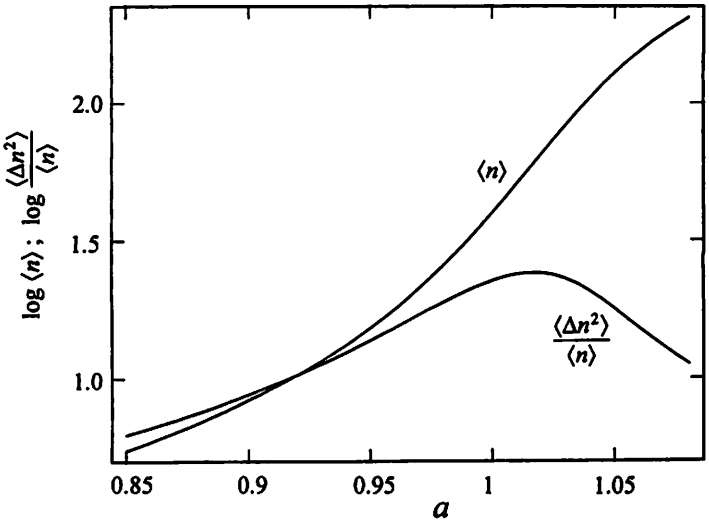

Figure 4. Mean number of photons and photon-number fluctuations as a function of linear gain for the parameter values of figure 3.

For this we use the damping basis method introduced by Briegel and Englert [15]. The results are shown in figures 5 and 6; we recognize good agreement. The small deviation in the photon-number fluctuations above threshold is due to the perturbative nature of the density matrix approach as opposed to the all-order numerical analysis.

3.4. Relation to the Scully–Lamb master equation

The closed three-level system analysed in the present paper can be identified with the open two-level system of the Scully–Lamb model [3], if we assume weak pumping into the upper

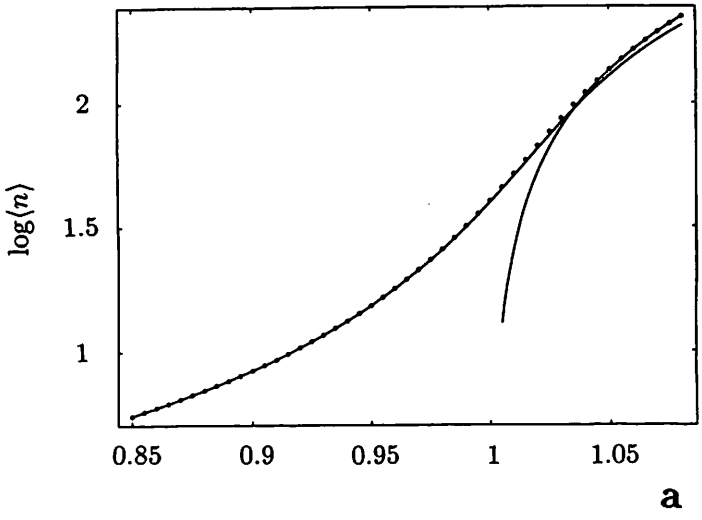

Figure 5. Comparison of mean number of photons from the reduced density matrix equation (full curve) and numerical solution of the single-atom equation (dotted curve). The semiclassical solution is also shown.

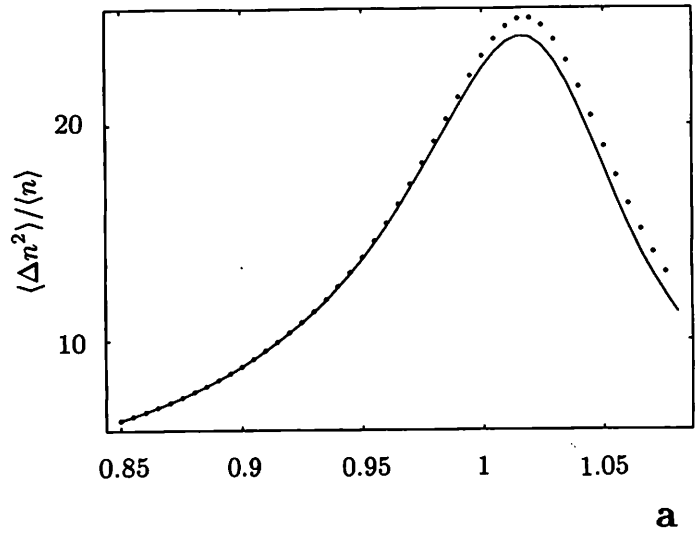

Figure 6. Comparison of photon-number fluctuations from the reduced density matrix equation (full curve) and numerical solution of the single-atom equation (dotted curve).

level 3 and set $\gamma' = \gamma'' = \gamma$. In this case level 1 acts as an atomic reservoir and the pump into level 3 corresponds to the injection of excited two-level atoms in the Scully–Lamb model with a rate $r = RN$. If we neglect the terms in the curly brackets in (33), which are proportional to $\langle\sigma_3\rangle$, set $\langle\sigma_1\rangle \to 1$ and make the above mentioned identifications, we find the density matrix equation

$$\dot{\rho}(t) = -i\omega[a^\dagger a, \rho] - \kappa[a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger] - \frac{g^2 r}{\gamma^2}[aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a]$$

$$+\frac{g^4 r}{\gamma^4}[aa^\dagger aa^\dagger\rho + \rho aa^\dagger aa^\dagger + 6aa^\dagger\rho aa^\dagger - 4a^\dagger aa^\dagger\rho a - 4a^\dagger\rho aa^\dagger a] \qquad (34)$$

which is precisely the fourth-order approximation of the Scully–Lamb master equation [3].

3.5. Fokker–Planck equation

The master equation for the density operator (33) can be transformed into a Fokker–Planck-type equation for the P-distribution [13] using the transformation rules

$$\begin{aligned} a\rho &\to \alpha P \\ a^\dagger\rho &\to \left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P \\ \rho a &\to \left(\alpha - \frac{\partial}{\partial\alpha^*}\right)P \\ \rho a^\dagger &\to \alpha^* P. \end{aligned} \tag{35}$$

One can see from these transformation rules that the saturation terms in (33) give rise to third- and fourth-order derivatives in the equation for P

$$\begin{aligned} \dot{P} = \Bigg\{ & -4\frac{\partial^4}{\partial\alpha^2\partial\alpha^{*2}}\left[b\langle\sigma_3\rangle\left(\frac{1}{\gamma''}+\frac{1}{\Gamma}+\frac{1}{\gamma}\right)\right] \\ & + \left(\frac{\partial^3}{\partial\alpha^2\partial\alpha^*}\alpha + \frac{\partial^3}{\partial\alpha\partial\alpha^{*2}}\alpha^*\right)b\left[2\left(\frac{1}{\gamma''}+\frac{1}{\gamma}\right)+\langle\sigma_3\rangle\left(\frac{3}{\gamma''}+\frac{1}{\Gamma}+\frac{1}{\gamma}\right)\right] \\ & - \left(\frac{\partial^2}{\partial\alpha^2}\alpha^2 + \frac{\partial^2}{\partial\alpha^{*2}}\alpha^{*2}\right)b\left[2\frac{1}{\gamma''}+\langle\sigma_3\rangle\left(\frac{1}{\gamma}+\frac{1}{\Gamma}+\frac{1}{\gamma''}\right)\right] \\ & + 2\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\left\{A-(\alpha^*\alpha+1)b\left[2\left(\frac{1}{\gamma''}+\frac{1}{\gamma}\right)+\langle\sigma_3\rangle\left(\frac{3}{\gamma''}+\frac{1}{\Gamma}+\frac{1}{\gamma}\right)\right]\right. \\ & \left. - \frac{\langle\sigma_1\rangle}{\gamma}b\alpha^*\alpha\right\} + \left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) \\ & \times\left\{\kappa - A + 2b\left[(\alpha^*\alpha+2)\left(\frac{1+\langle\sigma_3\rangle}{\gamma''}+\frac{1}{\gamma}\right)+\frac{\langle\sigma_1\rangle}{\gamma}+\frac{2\langle\sigma_3\rangle}{\Gamma}\right]\right\}\Bigg\}P \end{aligned} \tag{36}$$

where A is again the linear gain coefficient and $b = g^2A/\Gamma$. The truncation of the density operator equation in section 2 required that $n \ll n_s$ for all relevant photon numbers. We therefore may use the saturation photon number as a large parameter to scale away the higher-order derivatives in (36). We note that we do not, however, make any assumption about the number of atoms N and the Fokker–Planck equation can give accurate results also for the case of a single-atom laser as long as the saturation photon number is sufficiently large. It should be noted that a Fokker–Planck approach to a two-level laser, which uses the saturation photon number as a scaling parameter, was first given by Lugiato and Casagrande [16].

Because of the phase symmetry we, furthermore, have $P(\alpha^*, \alpha) = P(n)$. In steady state we eventually find the differential equation

$$[a - cd_2 n]\frac{\mathrm{d}}{\mathrm{d}n}P(n) = -[1 - a + cd_1 n]P(n) \tag{37}$$

where

$$d_1 = 2\left(\frac{1+\langle\sigma_3\rangle}{\gamma''}+\frac{1}{\gamma}\right) \tag{38}$$

$$d_2 = 4\frac{1}{\gamma''} + \frac{3}{\gamma} + 2\langle\sigma_3\rangle\left(\frac{3}{\gamma''} + \frac{1}{\gamma} + \frac{1}{\Gamma}\right) \tag{39}$$

$$c = \frac{b}{\kappa} = \frac{g^4 N}{\Gamma^2 \kappa}. \tag{40}$$

For the reasons discussed above, the solution of this equation is valid only for $n < n_{cr}$, where $n_{cr} = a/cd_2 \sim n_s$. We therefore have to set $P(n) \equiv 0$ for $n \geqslant n_{cr}$ and find

$$P(n) = \mathcal{N} e^{\frac{d_1}{d_2} n}\left[1 - \frac{n}{n_{cr}}\right]^{\mu} \Theta(n_{cr} - n) \tag{41}$$

with

$$\mu = n_{cr}\left(\frac{d_1}{d_2} - \frac{g^2 N - \Gamma\kappa}{g^2 N}\right). \tag{42}$$

In (41) $\mathcal{N}$ is a normalization constant. Figure 7 shows the mean number of photons and photon-number fluctuations calculated from (41) in comparison with the result from the density matrix equation. One can recognize an excellent agreement. The comparison between the probability distributions from the Fokker–Planck equation and the density matrix equation, presented in figure 8, reveals that small differences between both approaches occur only in the higher-order moments. This indicates that the scaling assumption used in the derivation of the Fokker–Planck equation leads to accurate results for the lowest-order moments.

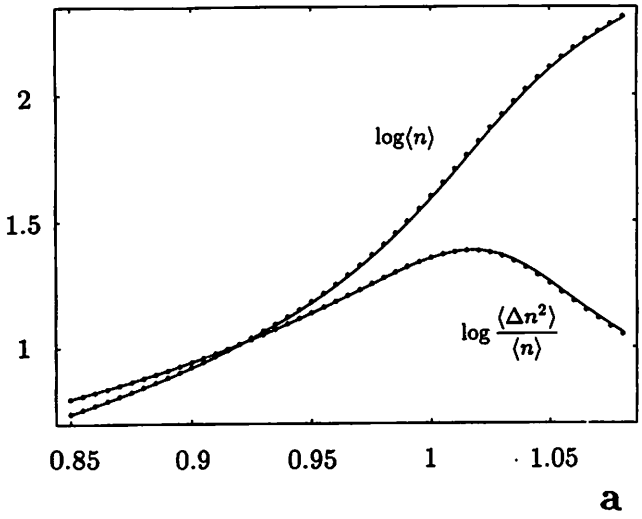

Figure 7. Mean number of photons and photon-number fluctuations from the Fokker–Planck equation (41) (full curve) and from the reduced density matrix equation (dotted curve).

Finally, it should be noted that neglecting terms of the order of n/n_s compared to unity in (37), we obtain a Gaussian solution

$$P(n) = \mathcal{N}' \exp\left\{-\frac{1}{2}\left(\frac{n - n_0}{\sigma}\right)^2\right\} \tag{43}$$

where

$$n_0 = \frac{a - 1}{cd_1} \tag{44}$$

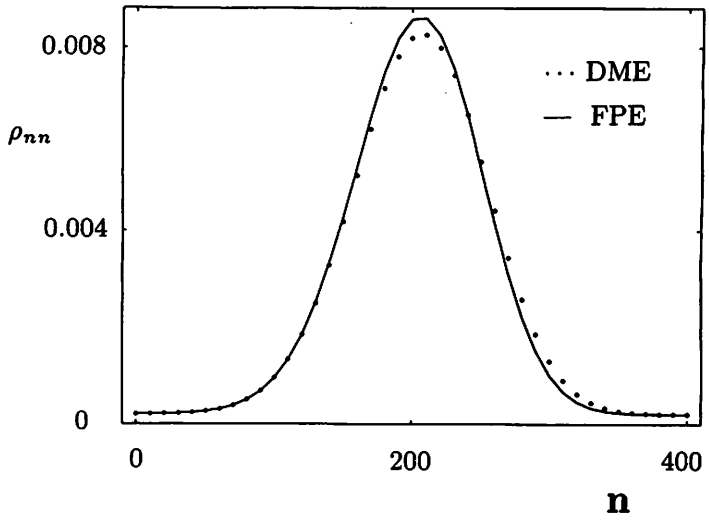

Figure 8. Comparison between steady-state photon-number distributions from the density matrix equation (dotted curve) and the Fokker–Planck equation (41) (full curve) for a relative gain of $a = 1.08$ and parameters as in figure 3.

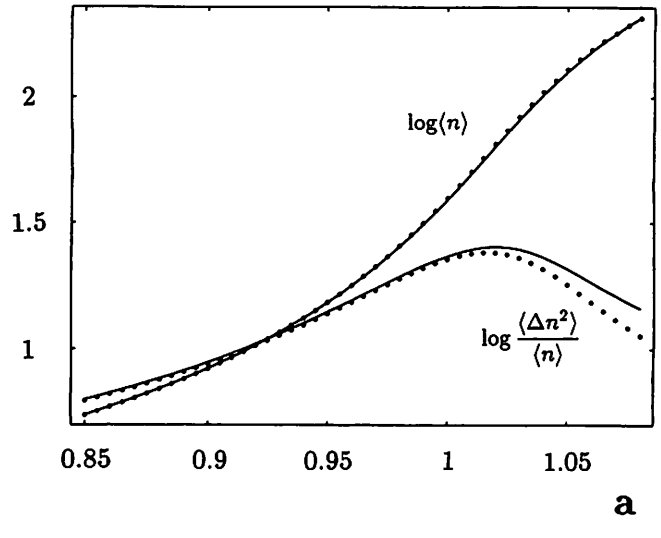

Figure 9. Mean number of photons and photon-number fluctuations from Gaussian solution of the approximate Fokker–Planck equation (full curve) and from the reduced density matrix equation (dotted curve).

$$\sigma^2 = n_0 \frac{a}{a-1}. \tag{45}$$

Figures 9 and 10 show a comparison with the results from the density matrix equation. One can see that this additional approximation slightly overestimates the fluctuations above threshold.

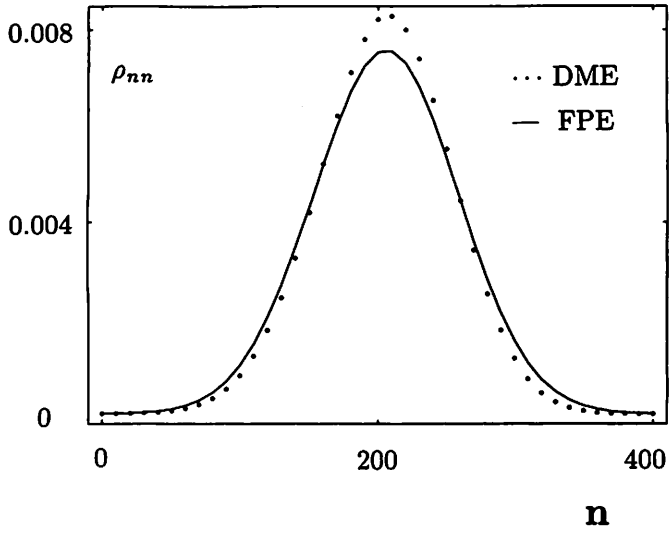

Figure 10. Comparison between steady-state photon-number distributions from the density matrix equation (dotted curve) and Gaussian distribution (43) (full curve) for a relative gain of $a = 1.08$ and parameters as in figure 3.

4. Summary

In this paper we have derived a master equation for the reduced field density operator of a three-level laser in the adiabatic limit, which holds for an arbitrary number of atoms. This equation was obtained from an effective time-evolution operator in which the degrees of freedom of the atoms are eliminated and which has the form of a time-evolution in a dissipative medium with all-order optical nonlinearities. For a near-threshold operation the infinite number of nonlinearities in the exponent of the time-evolution operator can be truncated and only the two leading terms are taken into account. The resulting perturbative density matrix equation in the Fock representation was solved for the diagonal elements in steady state. For the case of a small pump rate into the upper lasing level, the Scully–Lamb master equation was recovered. We compared our results with solutions of the total density matrix equation for a single-atom laser obtained from a damping basis approach and found good agreement. Using standard scaling arguments with the saturation photon number as a large parameter, a Fokker–Planck equation for the Glauber-P distribution was derived. Excellent agreement was found between the steady-state solutions of the density matrix- and Fokker–Planck-equation for the lowest-order moments. In contrast to most of the standard approaches, no assumption about the number of atoms has been made and the results are valid for a single-atom laser as well as a laser with very many atoms†.

The limitations of the present approach are twofold. First, in order to derive a reduced density matrix equation, we have to restrict ourselves to the adiabatic limit of fast decaying atomic coherences. Secondly, in order to truncate the infinite number of terms in the effective time-evolution operator, a small parameter had to be introduced for which we choose the atomic saturation. Therefore, our analysis is limited to a near-threshold operation of the laser. In order to derive a density matrix equation, which also holds for a laser operation well above threshold, different expansion parameters must be used. A potential

† See, however, [17] for a Fokker–Planck approach which uses the saturation photon number as a scaling parameter instead of the number of atoms.

way of doing this is to separate a semiclassical field amplitude and to include the interaction of the atoms with this field component in the calculation of the cumulants. In order to describe laser systems in which the adiabatic assumption is not fulfilled, and hence a derivation of a reduced density matrix equation is not possible, one has to resort to different methods as for example Green function techniques. Both extensions of the present work will be the subject of future studies.

Acknowledgment

We would like to thank Berthold-Georg Englert for stimulating discussions and for introducing us to the damping-basis method.

References

[1] Haken H 1970 *Handbuch der Physik* vol XXV/2c (Berlin: Springer); 1984 *Laser Theory* (Berlin: Springer)
Haken H, Risken H and Weidlich W 1967 *Z. Phys.* **206** 233, 335
[2] Lax M 1966 *Phys. Rev.* **145** 110; 1968 *Phys. Rev.* **172** 350, 362
Lax M and Louisell W H 1967 *IEEE J. Quantum Electron.* **QE-3** 47
Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
[3] Scully M O and Lamb W Jr 1966 *Phys. Rev. Lett.* **16** 853
Sargent M III, Scully M O and Lamb W E Jr 1974 *Laser Physics* (Reading, MA: Addison-Wesley)
[4] Mandel P 1974 *Physica* **77** 174
[5] Lugiato L A 1975 *Physica* A **82** 1
Casagrande F and Lugiato L A 1976 *Phys. Rev.* A **14** 778; 1976 *Phys. Rev.* A **15** 429
[6] Filipowicz P, Javanainen J and Meystre P 1986 *Phys. Rev.* A **34** 4547
[7] Yamamoto Y and Slusher R E 1993 *Physics Today* (6) 66
[8] Smith A M and Gardiner C W 1988 *Phys. Rev.* A **38** 4073; 1990 *Phys. Rev.* A **41** 2730
[9] Fleischhauer M 1994 *Phys. Rev.* A **50** 2773
[10] Kadanoff L P and Baym G 1962 *Quantum Statistical Mechanics* (New York: Benjamin)
[11] Keldysh L V 1964 *Zh. Eksp. Teor. Fiz.* **47** 1515
[12] Gardiner C W 1985 *Handbook of Stochastic Methods* (Berlin: Springer) 2nd edn
[13] Gardiner C W 1991 *Quantum Noise* (Berlin: Springer)
[14] Spohn H 1980 *Rev. Mod. Phys.* **52** 569
[15] Briegel H J and Englert B G 1993 *Phys. Rev.* A **47** 3311
[16] Lugiato L A, Casagrande F and Pizzuto L 1982 *Phys. Rev.* A **26** 3438
Casagrande F, Eschenazi E and Lugiato L A 1984 *Phys. Rev.* A **29** 239