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Quantum-theory of photodetection without the rotating wave approximation

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Abstract. A perturbative description of photodetection without the rotating wave approximation is presented. With the help of Green function techniques on an extended time contour renormalized expressions for the single- and multitime counting probabilities are derived, which explicitly reflect the causal propagation of signals. In the case of classical light fields the moments of the counting statistics can be expressed in terms of field correlation functions in agreement with the result of Bykov and Tatarskii. For non-classical radiation fields such as squeezed states the Bykov–Tatarskii formula is shown to be incorrect.

1. Introduction

The quantum theory of photodetection is one of the fundamental concepts in quantum optics and has extensively been studied in the early years of the field [1–5]. The two main approximations of the standard approaches by Mandel and Glauber [1, 2] are the perturbative treatment of the detector–field interaction and the rotating-wave approximation (RWA). An extension to efficient photodetection, where the back-action of the detector on the field cannot be neglected and a non-perturbative approach is required has been done for a single-mode field in [4–8] and for a multimode field in [9–11]. The aim of this paper is to develop an approach without the use of the RWA.

In RWA, the terms d^+E^+ and d^-E^- (where the superscript denotes positive or negative frequency parts) in the detector-field interaction Hamiltonian [12]

$$H_{\rm DF} = -\sum_j \boldsymbol{d}_j(t)\cdot\boldsymbol{E}(\boldsymbol{r}_j, t) \tag{1}$$

are neglected. Here $\boldsymbol{d}_j$ is the dipole operator of the jth detector atom and $\boldsymbol{E}$ is the operator of the transverse electric field†. This approximation is valid as long as the measurement time and the pulse length of the detected field are long compared to a typical optical cycle. It breaks down, however, when ultrashort light pulses are considered. Furthermore, as pointed out by Bykov and Tatarskii [13], the expression for the detection probability obtained with the RWA is unsatisfactory from a fundamental point of view, since it leads to (small) violations of causality.

In an approach without the RWA some problems arise, which prevent a straightforward generalization of the standard treatment. The correct choice of the initial state and the

† In fact the transverse electric field should be replaced by the electric displacement operator $\boldsymbol{D}/\epsilon_0$ [12]. This distinction is, however, of no relevance here.

projection operator onto an 'excited' state of the detector become crucial. In contrast to the case with RWA the initial state cannot be the free ground state, since the excitation probability from this state is not zero, even if the field is initially in the vacuum state

$$\frac{\mathrm{d}}{\mathrm{d}t} w_{\mathrm{ext}} \sim |d|^2 \langle 0|E(t)E(t)|0\rangle \sim \int \mathrm{d}^3\boldsymbol{k}\, k \to \infty. \tag{2}$$

One rather has to take into account the fact that the interaction with the vacuum field is present at any time and hence the initial state has to be a stationary state of the detector-field interaction. A similar problem arises with the projection operator. A projection onto a bare excited state of a detector atom does not correspond to an actual detection of a photon, since this state has a finite overlap with the interacting ground state.

An interesting way to overcome these problems has been introduced by Drummond [14]. In this approach a Hamiltonian is introduced by means of an appropriate canonical transformation, which is a combination of the $\boldsymbol{p}\cdot\boldsymbol{A}$ and $\boldsymbol{d}\cdot\boldsymbol{E}$ interactions and has no rotating terms for the interaction of the field with a harmonic oscillator of given frequency. However, since the canonical transformation of the field explicitly contains the frequency of the detector oscillator, it is not clear whether this approach can be generalized to broad-band detectors with short response times.

Recently Milonni *et al* [15] have shown, that by applying the RWA to the formal solution of the source-field interaction problem as opposed to the interaction Hamiltonian, causality can be regained. In this approach the field operator E is replaced by

$$E^{\pm}(t) = E^{\pm}_{\mathrm{free}}(t) - \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}\tau \sum_j D^{\mathrm{ret}}(0, t; \boldsymbol{r}_j, \tau) s_j^{\pm}(\tau) \tag{3}$$

where E^{free} is the free-field part, s_j is the source-dipole operator and D^{ret} is the retarded Green function of the electric field. This approach does not, however, remove the problem of vacuum contributions when higher moments are considered. For example, in $\langle E^-(t_1)E^-(t_2)E^+(t_2)E^+(t_1)\rangle$, where $E^{\pm}$ are the frequency parts as defined in (3), the free-field parts of $E^{\pm}(t_2)$ do not vanish as they in general do not commute with the source parts of $E^{\pm}(t_1)$.

The problem of renormalization, i.e. the removal of vacuum divergences will be solved in this paper by using Green function techniques [16] on an extended time contour. The initial state of the detector system and the projection operator will be related to the known eigenstates of the non-interacting systems at times $t = \pm\infty$. Since the interaction of the detector with the radiation field is always present, the finite detection time is modelled by modifying the field propagator that describes the signal propagation from the radiation sources rather than switching the detector–field interaction on and off. In the lowest order of perturbation the probability of exciting a set of detector atoms is expressed as a matrix element of the time-evolution operator on a modified Schwinger–Keldysh time contour [17].

It is shown that the moments of the single- and multitime photon counting distributions are given by retarded (i.e. causal) correlations of source dipoles. If the radiation sources are classical (deterministic or stochastic) these moments can be expressed in terms of electric field operators. In this case the detection formula suggested by Bykov and Tatarskii [13] is recovered. It is shown, however, that for non-classical radiation fields the Bykov–Tatarskii formula may lead to unphysical results such as negative probabilities.

This paper is organized as follows. In section 2 we describe the detector model and derive an expression for the excitation probability of a given set of detector atoms when exposed to the source field at all times. In section 3 we introduce an extended time-contour which removes the vacuum divergences and renormalizes the approach. In section 4 we derive the probability distribution of m counts within a time interval $(t, t + \Delta t)$ and in section 5 the joint probability distributions of counting $m_1, \ldots, m_n$ photons in the non-overlapping time intervals $(t_1, t_1 + \Delta t), \ldots, (t_n, t_n + \Delta t)$. In section 6 it is shown that the counting probability can be expressed in terms of input field operators if the radiation sources are classical. In this case the result of Bykov and Tatarskii is recovered [13]. Finally the difference between RWA and non-RWA formulae is illustrated for the simple example of an ultrashort Gaussian pulse. Section 7 summarizes the results.

2. Detector model

The point-like photodetector is assumed to be a collection of many-level atoms located at the origin with a single ground state $|g\rangle_l$ and a broad band of excited states $|e_i\rangle_l$. The index i specifies the level and l the atom. The dipole operator of the detector atom is denoted by $\boldsymbol{d}_l$. We assume that the probability of excitation of a detector atom is small, such that a perturbative treatment is justified. Note that even though we are not interested in this aspect here, this assumption does not exclude efficient photodetection. In such a case the propagation of the field inside the detector and its depletion due to absorption have to be taken into account [11, 18]. In this paper we also take into account explicitly radiation sources. The source dipoles located at position $\boldsymbol{r}_j$ are described by the dipole operators $\boldsymbol{s}_j$. The source-field and detector-field interaction operators are given in dipole approximation by

$$H_{SF} = -\sum_j \boldsymbol{s}_j(t) \cdot \boldsymbol{E}(\boldsymbol{r}_j, t) \tag{4}$$

$$H_{DF} = -\sum_l \boldsymbol{d}_l(t) \cdot \boldsymbol{E}(0, t). \tag{5}$$

As mentioned before, the finite detection time cannot be described by switching on and off the detector-field interaction (5). We will rather assume that the propagation of photons from the source to the detector will be blocked by a shutter before and after the time interval of detection. In [19] it was shown that the action of such a shutter can be described by its influence on the retarded field propagators.

$$D^{\rm ret}_{\mu\nu}(0, t; \boldsymbol{r}_l, t') \to \Theta(t - t_1)\Theta(t_2 - t) D^{\rm ret}_{\mu\nu}(0, t; \boldsymbol{r}_l, t') \tag{6}$$

$$D^{\rm ret}_{\mu\nu}(0, t; 0, t') \to D^{\rm ret}_{\mu\nu}(0, t; 0, t') \tag{7}$$

where (t_1, t_2) is the detection time interval and Θ denotes the Heaviside step function. In all the following discussions the presence of the shutter is ignored and will be taken into account only at the end by making the replacement (6).

To find the appropriate initial state, we assume that all interactions are adiabatically switched on and off for $t \to \pm\infty$. Hence we may assume that the state vector for $t \to -\infty$ is given by a product of the bare ground state of the detector atoms $|g\rangle_l$, the field vacuum $|\{0\}\rangle$ and some state of the sources $|\psi_s\rangle$

$$|\psi(t = -\infty)\rangle = |\phi_0\rangle = \prod_l |g_l\rangle |\{0\}\rangle |\psi_s\rangle. \tag{8}$$

It may be noted, that this choice will finally give rise to the retarded as opposed to advanced propagation of signals. In lowest order of the detector-field coupling a detector atom can only absorb a photon from the field. A subsequent spontaneous emission is a higher-order process and is not accounted for in the perturbative approach. Therefore the excitation of the detector can be probed by a projection onto the bare excited states as $t \to +\infty$

where again all interactions are assumed to be switched off adiabatically. In reality the detector atom is coupled to some reservoir (e.g. electric field ionization) which prevents the re-emission of photons [10]. Its description would, however, unneccesarily complicate the present discussion.

In order to obtain the probability distribution for the detection of photons we first derive the coincidence probability of finding a specific set of n atoms labelled by $l_1, \ldots, l_n$ excited. In the interaction picture, this probability can be written as

$$p^{(n)}_{l_1,\ldots,l_n} = \langle\phi_0|S^{-1}(\infty,-\infty)\hat{P}_{l_1,\ldots,l_n}S(\infty,-\infty)|\phi_0\rangle \tag{9}$$

where the projection operator $\hat{P}_{l_1,\ldots,l_n}$ is given by

$$\hat{P}_{l_1,\ldots,l_n} = \prod_{l=l_1}^{l_n}\sum_i |e_i\rangle_{ll}\langle e_i|. \tag{10}$$

$$S(\infty,-\infty) = T\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}\tau\,(H_{\mathrm{DF}}(\tau)+H_{\mathrm{SF}}(\tau))\right\} \tag{11}$$

is the time-evolution operator and T denotes time ordering. Note, that $\hat{P}_{l_1,\ldots,l_n}$ acts as a unity operator with respect to all other atoms. Thus, other atoms may also be excited.

Expanding S in equation (9) into a power series and applying the Wick theorem [16] we find in lowest order of the detector-field coupling for the single-atom probability

$$\begin{aligned}
p^{(1)}_l = \frac{1}{\hbar^2}\int\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}t'\sum_i\langle g|d_\alpha(t)|e_i\rangle_{ll}\langle e_i|d_\beta(t')|g\rangle\Big\{D^{-+}_{\alpha\beta}(0,t;0,t')\\
+\frac{1}{\hbar^2}\int\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{d}\tau'\sum_{jk}[D^{--}_{\alpha\mu}(0,t;\boldsymbol{r}_j,\tau)D^{++}_{\beta\nu}(0,t;\boldsymbol{r}_k,\tau')\langle s^j_\mu(\tau)s^k_\nu(\tau')\rangle_H\\
-D^{-+}_{\alpha\mu}(0,t;\boldsymbol{r}_j,\tau)D^{++}_{\beta\nu}(0,t;\boldsymbol{r}_k,\tau')\langle T[s^j_\mu(\tau)s^k_\nu(\tau')]\rangle_H\\
-D^{--}_{\alpha\mu}(0,t;\boldsymbol{r}_j,\tau)D^{+-}_{\beta\nu}(0,t;\boldsymbol{r}_k,\tau')\langle T^{-1}[s^j_\mu(\tau)s^k_\nu(\tau')]\rangle_H\\
+D^{-+}_{\alpha\mu}(0,t;\boldsymbol{r}_j,\tau)D^{+-}_{\beta\nu}(0,t;\boldsymbol{r}_k,\tau')\langle s^k_\nu(\tau')s^j_\mu(\tau)\rangle_H]\Big\}
\end{aligned} \tag{12}$$

where α, β, μ, ν are vector indices and

$$D^{++}_{\mu\nu}(1;2) = \langle 0|T[E_\mu(1)E_\nu(2)]|0\rangle \tag{13}$$

$$D^{-+}_{\mu\nu}(1;2) = \langle 0|E_\mu(1)E_\nu(2)|0\rangle \tag{14}$$

$$D^{+-}_{\mu\nu}(1;2) = \langle 0|E_\nu(2)E_\mu(1)|0\rangle \tag{15}$$

$$D^{--}_{\mu\nu}(1;2) = \langle 0|T^{-1}[E_\mu(1)E_\nu(2)]|0\rangle \tag{16}$$

are the Green functions of the free electric field [16]. The index 'H' in equation (12) denotes correlation functions of source operators in the Heisenberg picture, i.e. including the interaction with the radiation field. In a similar way the many-atom probabilities can be derived.

We recognize two things from equation (12). First there is a contribution that does not depend on the radiation sources. This (divergent) term is a pure vacuum contribution. Secondly, the contribution from the radiation sources are not retarded since only the combinations

$$\begin{aligned}
D^{\mathrm{ret}}_{\mu\nu}(1,2) &= D^{++}_{\mu\nu}(1,2) - D^{+-}_{\mu\nu}(1,2)\\
&= D^{-+}_{\mu\nu}(1,2) - D^{--}_{\mu\nu}(1,2)
\end{aligned} \tag{17}$$

give retarded Green functions. In the next section it is shown, that a formal manipulation of the initial expression (9) removes the vacuum terms and leads to purely retarded contributions from the sources.

3. Green functions on an extended time-contour: Renormalization and causality

The perturbative expansion of the time-evolution operator in powers of the detector-field coupling did lead to divergent vacuum contributions. The origin of these terms is obviously the perturbation expansion itself. The interaction with the vacuum field cannot be treated perturbatively, only the coupling to the source part of the field can. In this section we introduce a modified perturbation expansion which eliminates the vacuum contributions and also leads to purely retarded source contributions.

According to the Gell-Mann–Low theorem [20], the non-interacting ground state of the detector-field system for $t \to -\infty$ goes over into itself as $t \to +\infty$ apart from a phase factor, if the interaction is adiabatically switched off in the limit $|t| \to \infty$. Hence we have

$$S_1(\infty, -\infty)|\phi_0\rangle = \mathrm{e}^{\mathrm{i}\phi}|\phi_0\rangle \tag{18}$$

where

$$S_1(\infty, -\infty) = T \exp\left\{ - \frac{\mathrm{i}}{\hbar} \int_{-\infty}^{\infty} \mathrm{d}\tau \; H_{\mathrm{DF}}(\tau) \right\}. \tag{19}$$

Thus we may replace $|\phi_0\rangle$ in equation (9) by $|\tilde{\phi}_0\rangle = S_1^{-1}(\infty, -\infty)|\phi_0\rangle$. Physically $|\tilde{\phi}_0\rangle$ corresponds to an initial state of the detector-field subsystem which evolves into the non-interacting ground state as $t \to +\infty$ if no radiation sources are present. It is clear at hand, that there are no vacuum contributions in any order of perturbation in this case, since the non-interacting ground state is orthogonal to the excited state of the detector. Using $|\tilde{\phi}_0\rangle$ as the initial state we may write the excitation probability as

$$p^{(n)}_{l_1,\ldots,l_n} = \langle\phi_0|S_1(\infty, -\infty)S^{-1}(\infty, -\infty) \times \hat{P}_{l_1,\ldots,l_n} S(\infty, -\infty)S_1^{-1}(\infty, -\infty)|\phi_0\rangle. \tag{20}$$

To simplify the notation we introduce a time contour D similar to the Schwinger–Keldysh [17] contour used in non-equilibrium quantum statistics as indicated in figure 1. The contour has 4 branches denoted by (a)–(d). The corresponding time evolution operator S_D is identical to S or S^{-1} on (b) and (c) and to S_1^{-1} or S_1 on (a) and (d) respectively. With these definitions we find

$$\begin{aligned} p_l^{(1)} = \frac{1}{\hbar^2} \int_{c,d} \mathrm{d}\check{t} \int_{a,b} \mathrm{d}\check{t}' \sum_i \langle g|d_\alpha(t)|e_i\rangle\langle e_i|d_\beta(t')|g\rangle \Big\{ D_{\alpha\beta}(0, \check{t}; 0, \check{t}') \\ + \frac{1}{\hbar^2} \iint_{b,c} \mathrm{d}\check{\tau}\, \mathrm{d}\check{\tau}' \sum_{jk} D_{\alpha\mu}(0, \check{t}; \boldsymbol{r}_j, \check{\tau}) D_{\beta\nu}(0, \check{t}'; \boldsymbol{r}_k, \check{\tau}') \langle T_D[s^j_\mu(\check{\tau}) s^k_\nu(\check{\tau}')]\rangle_H \Big\} \end{aligned} \tag{21}$$

where $\check{t}$ and $\check{\tau}$ denote times on the contour, T_D is the time-ordering operator on D. For simplicity we here discuss explicitly only the single-atom probability. The generalization to the many-atom case is straightforward but somewhat involved. We will come back to the general case at a later point.

Let us first discuss the vacuum term in (21). Rewriting the contour integrals into ordinary time integrals, one realizes that the contributions from the different branches compensate each other

$$\frac{1}{\hbar^2} \int_{c,d} \mathrm{d}\check{t} \int_{a,b} \mathrm{d}\check{t}' \sum_j \langle g|d_\alpha(t)|e_j\rangle\langle e_j|d_\beta(t')|g\rangle D_{\alpha\beta}(0, \check{t}; 0, \check{t}')$$

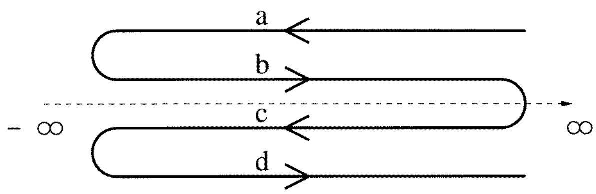

Figure 1. Extended time contour D. Time ordering on D is indicated by arrows. The time evolution on D contains on branches (a) and (d) only the detector–field interaction. In (b) and (c) it contains the detector–field and source–field coupling.

$$
\begin{aligned}
&= \frac{1}{\hbar^2} \int\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t' \sum_j \langle g|d_\alpha(t)|e_j\rangle\langle e_j|d_\beta(t')|g\rangle \\
&\quad \times [\langle 0|E_\mu(t)E_\nu(t')|0\rangle - \langle 0|E_\mu(t)E_\nu(t')|0\rangle \\
&\quad -\langle 0|E_\mu(t)E_\nu(t')|0\rangle + \langle 0|E_\mu(t)E_\nu(t')|0\rangle] = 0. \qquad (22)
\end{aligned}
$$

We now discuss the source-field term in equation (21). We first note that the integral contains two equal terms of opposite sign which cancel each other if $\check{\tau} \in (c)$ or $\check{\tau}' \in (b)$. This means we can restrict the integration to $\check{\tau} \in (b)$ and $\check{\tau}' \in (c)$. If we again rewrite the contour integrals in terms of ordinary time integrals, we find the following non-vanishing contributions:

$$
\begin{aligned}
p_l^{(1)} = \int\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t' \int\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{d}\tau'\, f^l_{\alpha\beta}(t,t') \frac{1}{\hbar^2} \sum_{jk} [D^{--}_{\alpha\mu}(0,t;\boldsymbol{r}_j,\tau) - D^{-+}_{\alpha\mu}(0,t;\boldsymbol{r}_j,\tau)] \\
\times [D^{++}_{\alpha\mu}(0,t';\boldsymbol{r}_k,\tau') - D^{-+}_{\alpha\mu}(0,t';\boldsymbol{r}_k,\tau')] \langle s^j_\mu(\tau) s^k_\nu(\tau')\rangle_H \qquad (23)
\end{aligned}
$$

where we have introduced the detector response function

$$
f^l_{\alpha\beta}(t,t') = \frac{1}{\hbar^2} \sum_i \langle g|d_\alpha(t)|e_i\rangle_{ll} \langle e_i|d_\beta(t')|g\rangle. \qquad (24)
$$

For the case of identical detector atoms the index l may be dropped. If the spectrum of excited states is dense one can replace $f_{\alpha\beta}(t,t')$ within the RWA by $g_{\alpha\beta}\delta(t-t')$. However, when the rotating terms become relevant, such a replacement is not possible, since the summation in equation (24) extends only over positive frequencies $\omega_i = (E_{e_i} - E_g)/\hbar$.

According to equations (17) the expressions in the brackets in equation (23) are the retarded Green functions of the electric field. We thus conclude from equation (23), that the excitation probability of a detector atom does not contain any vacuum contributions and is related to source expressions via retarded (i.e. rigorously causal) Green functions.

4. Single-time counting probability

In order to calculate the single-time counting probability distribution, we assume that all detector atoms interact with the source field during the same time interval $(t, t+\Delta t)$. Physically this can be realized by means of a shutter in front of the detector. Taking into account the action of this shutter we eventually arrive at

$$
\begin{aligned}
p_l^{(1)}(t,\Delta t) = -\frac{1}{\hbar^2} \int\int_t^{t+\Delta t} \mathrm{d}t'\, \mathrm{d}t''\, f^l_{\alpha\beta}(t',t'') \int\int_{-\infty}^{\infty} \mathrm{d}\tau'\, \mathrm{d}\tau'' \\
\times \sum_{jk} D^{\mathrm{ret}}_{\alpha\mu}(0,t';\boldsymbol{r}_j,\tau') D^{\mathrm{ret}}_{\beta\nu}(0,t'';\boldsymbol{r}_k,\tau'') \langle s^j_\mu(\tau') s^k_\nu(\tau'')\rangle_H \qquad (25)
\end{aligned}
$$

where Δt is the detection time interval. Note that D^{ret} contains a factor i, such that the whole expression (25) is positive.

Similarly one finds for the probability of exciting atoms $l_1, \ldots, l_n$

$$p^{(n)}_{l_1,\ldots,l_n}(t, \Delta t) = \left(-\frac{1}{\hbar^2}\right)^n \iint_t^{t+\Delta t} \mathrm{d}t_1' \, \mathrm{d}t_1'' \iint_{-\infty}^{\infty} \mathrm{d}\tau_1' \, \mathrm{d}\tau_1'' \ldots \iint_t^{t+\Delta t} \mathrm{d}t_n' \, \mathrm{d}t_n'' \iint_{-\infty}^{\infty} \mathrm{d}\tau_n' \, \mathrm{d}\tau_n''$$
$$\times f^{l_1}_{\alpha_1\beta_1}(t_1', t_1'') \ldots f^{l_n}_{\alpha_n\beta_n}(t_n', t_n'') \sum_{\{j\},\{k\}} D^{\mathrm{ret}}_{\alpha_1,\mu_1}(0, t_1'; \boldsymbol{r}_{j_1}, \tau_1')$$
$$\times D^{\mathrm{ret}}_{\beta_1,\nu_1}(0, t_1''; \boldsymbol{r}_{k_1}, \tau_1'') \ldots D^{\mathrm{ret}}_{\alpha_n,\mu_n}(0, t_n'; \boldsymbol{r}_{j_n}, \tau_n') D^{\mathrm{ret}}_{\beta_n,\nu_n}(0, t_n''; \boldsymbol{r}_{k_n}, \tau_n'')$$
$$\times \langle T^{-1}[s^{j_1}_{\mu_1}(\tau_1') \ldots s^{j_n}_{\mu_n}(\tau_n')] T[s^{k_1}_{\nu_1}(\tau_1'') \ldots s^{k_n}_{\nu_n}(\tau_n'')]\rangle. \tag{26}$$

or

$$p^{(n)}_{l_1,\ldots,l_n}(t, \Delta t) = \left\langle \prod_{l=l_1}^{l_n} \hat{\gamma}_l(t, \Delta t) \right\rangle_T \tag{27}$$

where

$$\hat{\gamma}_l(t, \Delta t) = -\frac{1}{\hbar^2} \iint_t^{t+\Delta t} \mathrm{d}t' \, \mathrm{d}t'' \iint_{-\infty}^{\infty} \mathrm{d}\tau' \, \mathrm{d}\tau'' f^l_{\alpha\beta}(t', t'')$$
$$\times \sum_{j,k} D^{\mathrm{ret}}_{\alpha\mu}(0, t'; \boldsymbol{r}_j, \tau') D^{\mathrm{ret}}_{\beta\nu}(0, t''; \boldsymbol{r}_k, \tau'') s^j_\mu(\tau') s^k_\nu(\tau'') \tag{28}$$

and T stands for apex time-ordering of the source dipole operators as in equation (26).

Following [2] we can then derive the exclusive probability of exciting exactly m atoms and therefore the single-time probability distribution of photocounts from (26) using the Bernoulli scheme. Assuming a large number of detector atoms, such that $N!/(N-m)!m! \approx N^m/m!$, we eventually arrive at the generalized Poisson distribution

$$P_m(t, \Delta t) = \frac{1}{m!} \langle \hat{\Gamma}(t, \Delta t)^m \mathrm{e}^{-\hat{\Gamma}(t,\Delta t)} \rangle_T \tag{29}$$

where $\hat{\Gamma} = N\hat{\gamma}$. The fact that $\hat{\Gamma}$ scales linearly with the number of atoms N is an artefact of the approach which neglects the depletion of the field due to the detection process. When this effect is taken into account, N is to be replaced by $(1 - \mathrm{e}^{-\eta N})$ [11], where η characterizes the absorption of the field inside the detector. The factorial moments of the photon counting statistics read explicitly

$$\langle m(m-1) \ldots (m-n+1) \rangle = \langle \hat{\Gamma}^n \rangle_T. \tag{30}$$

It should be noted that even though there is no normal ordering involved in the generalized Poisson distribution without RWA, the moments are still *time ordered* with respect to the source-dipole operators.

The generalized Poisson distribution, equations (29), is the first main result of this paper. It has been obtained without applying the RWA. The result is very similar to the Glauber–Mandel distribution, except that here no normal ordering is involved. There are no vacuum contributions (as it should be) and the source terms are retarded which shows explicitly the causal propagation of signals from the source to the detector.

5. Joint probability for detection in non-overlapping time intervals

In order to obtain information about correlations of the measured field for different times the joint probability $P_{m_1,\ldots,m_k}(t_1, \ldots, t_k; \Delta t)$ to detect $m_1, \ldots, m_k$ photons in the non-overlapping time intervals $(t_1, t_1 + \Delta t), \ldots, (t_k, t_k + \Delta t)$ is needed besides the single-time counting distribution given in equation (29). With the help of $P_{m_1,\ldots,m_k}$ properties such as bunching or antibunching of photocounts or the waiting-time statistics important for the interpretation of quantum jumps [21] can be discussed. To derive an expression for the joint probability in the framework of our discussion we have to extend the model of a single detector to a collection of detectors and a beam switch. The beam switch is used to direct the source field to different detectors in the non-overlapping time intervals. For the sake of simplicity we will restrict the discussion to the two-time joint probability. The generalization is then straightforward.

To calculate the joint probability of exciting atoms $l_1, \ldots, l_n$ of detector 1 *and* atoms $k_1, \ldots, k_m$ of detector 2, we introduce the projection operator onto the excited states of the corresponding detector atoms

$$\hat{P}_{l_1,\ldots,l_n;k_1\ldots,k_m} = \prod_{l=l_1}^{l_n} \sum_i |e_i\rangle_{ll}\langle e_i| \otimes \prod_{k=k_1}^{k_m} \sum_i |e_i\rangle_{kk}\langle e_i|. \tag{31}$$

With this we may write the joint excitation probability similar to equation (20)

$$p^{(n,m)}_{l_1,\ldots,l_n;k_1,\ldots,k_m} = \langle\phi_0|S_1(\infty,-\infty)S^{-1}(\infty,-\infty)\hat{P}_{l_1,\ldots,l_n;k_1,\ldots,k_m}S(\infty,-\infty)S_1^{-1}(\infty,-\infty)|\phi_0\rangle. \tag{32}$$

We now proceed along the same lines as in sections 2–4. The vacuum contributions again cancel by construction and only retarded source correlations remain. Taking into account the different opening times (or equivalently delay times) for the shutters in front of the two detectors, we eventually arrive at

$$p^{(n,m)}(t_1, t_2, \Delta t) = \langle\hat{\Gamma}(t_1, \Delta t)^n \hat{\Gamma}(t_2, \Delta t)^m\rangle \tag{33}$$

where $\hat{\Gamma}$ is given in equation (28). Using again the Bernoulli scheme, we find the joint probability of exciting exactly n atoms of detector 1 in the time interval $(t_1, t_1 + \Delta t)$ and m atoms of detector 2 in the time interval $(t_2, t_2 + \Delta t)$:

$$P_{n,m}(t_1, t_2; \Delta t) = \frac{1}{n!m!}\langle(\hat{\Gamma}(t_1, \Delta t))^n(\hat{\Gamma}(t_2, \Delta t))^m \exp\{-(\hat{\Gamma}(t_1, \Delta t) + \hat{\Gamma}(t_2, \Delta t))\}\rangle_T. \tag{34}$$

6. Relation to input fields

In section 4 we derived an expression for the probability distribution of photodetection without RWA in terms of source correlation functions. We will now discuss under what conditions an expression in terms of field operators can be obtained and how it differs from the Glauber–Mandel expression.

In [13] Bykov and Tatarskii suggested that the Glauber–Mandel expression for $\hat{\Gamma}$

$$\hat{\Gamma}_{\mathrm{GM}} = g_{\alpha\beta}N\int_t^{t+\Delta t} \mathrm{d}t'\, E^-_\alpha(t')E^+_\beta(t') \tag{35}$$

in the (normal and time-ordered) Poisson distribution

$$P_m = \frac{1}{m!}\langle:\hat{\Gamma}^m \mathrm{e}^{-\hat{\Gamma}}:\rangle_T \tag{36}$$

should be replaced by

$$\hat{\Gamma}_{\mathrm{BT}} = g_{\alpha\beta}N\int_t^{t+\Delta t} \mathrm{d}t'\, E_\alpha(t')E_\beta(t') \tag{37}$$

in order to obtain causal expressions. Here, $E^\pm$ are the positive and negative frequency parts of the electric field operator. This suggestion gives the detection probabilities in terms of the input field. It does, however, have an obvious problem in that the normal ordering destroys the positivity of the distribution if $\hat{\Gamma}_{\mathrm{BT}}$ is used. Hence, there can be cases where negative probabilities occur†, which means that the Bykov–Tatarskii equation is generally incorrect. For example the mean number of photons detected from a squeezed vacuum field (where $\langle E\rangle = 0$) can be negative for very short detection times (on the order of the oscillation period). According to equations (36) and (37)

$$\langle m\rangle \sim \langle : E^2 : \rangle = \langle : \Delta E^2 : \rangle < 0. \tag{38}$$

The problem of negativity does not exist, if classical light fields are considered, for which the normal ordering has no effect. We will now show, that in this case the result obtained in section 4 coincides with the Bykov–Tatarskii expression.

As shown in [22, 19] any normal and time-ordered correlation function of the electric field in the Heisenberg picture can be expressed by time-ordered source correlation functions. For example

$$\begin{aligned}\langle : E_\alpha(1)E_\beta(2) : \rangle_H = {} & \frac{1}{\hbar^2}\int_{-\infty}^{\infty}\mathrm{d}1' \int_{-\infty}^{\infty}\mathrm{d}2'\, \Theta(1-1')\Theta(2-2') \\ & \times\{\langle 0|E_\mu^+(1')E_\alpha^-(1)|0\rangle\langle 0|E_\beta^+(2)E_\nu^-(2')|0\rangle\langle s_\mu(1')s_\nu(2')\rangle_H \\ & -\langle 0|E_\mu^+(1')E_\alpha^-(1)|0\rangle\langle 0|E_\nu^+(2')E_\beta^-(2)|0\rangle\langle T^{-1}[s_\mu(1')s_\nu(2')]\rangle_H \\ & +\langle 0|E_\alpha^+(1)E_\mu^-(1')|0\rangle\langle 0|E_\nu^+(2')E_\beta^-(2)|0\rangle\langle s_\nu(2')s_\mu(1')\rangle_H \\ & -\langle 0|E_\alpha^+(1)E_\mu^-(1')|0\rangle\langle 0|E_\beta^+(2)E_\nu^-(2')|0\rangle\langle T[s_\mu(1')s_\nu(2')]\rangle_H\}.\end{aligned} \tag{39}$$

Here $1, 1', 2, 2'$ are used as abbreviations for time and space coordinates, and ‘$: \cdots :$’ denotes normal and time ordering. This expression does not coincide with the result from section 4. If, however, the source dipole operators commute, i.e. if

$$[s_\mu(1'), s_\nu(2')] = 0 \tag{40}$$

the field correlation functions in (39) can be summed to give retarded Green functions, since

$$\begin{aligned}\Theta(1-1')[\langle 0|E^+(1)E^-(1')|0\rangle - \langle 0|E^+(1')E^-(1)|0\rangle] &= \Theta(1-1')\langle 0|[E(1), E(1')]|0\rangle \\ &= D^{\mathrm{ret}}(1, 1').\end{aligned} \tag{41}$$

The source-dipole operators commute in general only if the radiation source and therefore the generated field are classical. This also includes the case of classical stochastic fields such as thermal radiation but not non-classical radiation states. Hence, the result of section 4 can be expressed in terms of field operators and coincides with the Bykov–Tatarskii formula (37) in the case of classical light fields not, however, for non-classical light such as squeezed states.

The rotating terms in the detector-field interaction become important when either the pulse length of the field or the detection time become of the order of the oscillation period. With the advance of techniques for ultrashort-pulse generation the first situation is now experimentally accessible in the optical regime. The second case is, however, only of interest for low-frequency radiation. To illustrate the difference of the RWA and non-RWA predictions let us now discuss the detection of an ultrashort pulse with Gaussian-shaped coherent amplitude

$$E(t) = E_0 \exp\left\{-\frac{\omega^2 t^2}{2\sigma^2}\right\}\cos[\omega t]. \tag{42}$$

† The author thanks Mrs Silke Biermann for pointing out this shortage of the Bykov–Tatarskii formula.

In RWA the mean number of photocounts in a detection time long compared with the pulse length is given by

$$I_{\mathrm{RWA}} = \sum_i N \frac{|d_{gi}|^2}{\hbar^2} \iint_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, \mathrm{e}^{-\mathrm{i}\omega_i (t-t')} \langle E^-(t) E^+(t') \rangle \tag{43}$$

while without RWA one has for the coherent (i.e. classical) field

$$I = \sum_i N \frac{|d_{gi}|^2}{\hbar^2} \iint_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}t'\, \mathrm{e}^{-\mathrm{i}\omega_i (t-t')} \langle : E(t) E(t') : \rangle. \tag{44}$$

Here $\omega_i = (E_i - E_g)/\hbar$ are the (positive) transition frequencies of the detector atoms. Time ordering is of no relevance since the coupling to radiation sources is ignored. Substituting equation (42) one finds

$$\frac{I}{I_{\mathrm{RWA}}} = \frac{\sum_i |d_{gi}|^2 [\mathrm{e}^{-(\omega-\omega_i)^2\sigma^2/2\omega^2} + \mathrm{e}^{-(\omega+\omega_i)^2\sigma^2/2\omega^2}]^2}{\sum_i |d_{gi}|^2 \mathrm{e}^{-(\omega-\omega_i)^2\sigma^2/\omega^2}}. \tag{45}$$

In the case of a single excited detector state with $\omega_i = \omega_0$, this simplifies to

$$\frac{I}{I_{\mathrm{RWA}}} = \left[1 + \exp\left(-2\sigma^2 \frac{\omega_0}{\omega}\right)\right]^2. \tag{46}$$

For a resonant transition $\omega_0 = \omega$ and $\sigma = 1$ the difference between the RWA and non-RWA formulae is about 29%. On the other hand, for a broad-band detector we may set $\sum_i |d_{gi}|^2 \to \eta \int_0^\infty \mathrm{d}\omega_i$. Thus we find

$$\frac{I}{I_{\mathrm{RWA}}} = 2 \frac{1 + \mathrm{e}^{-\sigma^2}}{1 + \mathrm{erf}(\sigma)} \tag{47}$$

which for $\sigma = 1$ gives rize to a difference of about 48%. This example shows that for very short pulses with pulse length of the order of the oscillation period irrespective of the actual detector response, the RWA and non-RWA formula give substantially different results.

7. Summary

In this paper a theoretical description of photodetection without RWA is presented, which is also valid on very short timescales and correctly describes the detection of ultrashort pulses. With an appropriate choice of initial and final states, which takes into account the fact that the detector–field interaction is present at all times and using Green function techniques on an extended time contour, a renormalized and causal expression for the photon-counting distribution has been derived. It has been shown that this distribution can be expressed in terms of field correlation functions only in the case of classical (deterministic or stochastic) light sources. In this case the generalization of the Glauber detection formula suggested by Bykov and Tatarskii is recovered. It is also shown that the Bykov–Tatarskii equation is incorrect in the case of non-classical light. In the case of quantum light sources, the moments of the counting distribution can be expressed in terms of time-ordered source correlations only.

The presented approach is perturbative in that the probability of excitation of a specific detector atom is assumed to be small. This assumption is well satisfied if the detector operates in the regime of linear response were saturation effects are negligible. The perturbative approach, however, also neglects the depletion of the field due to field absorption and therefore yields incorrect expressions for the detector efficiency. In order to describe efficient photodetection the depletion of the field needs to be taken into account.

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