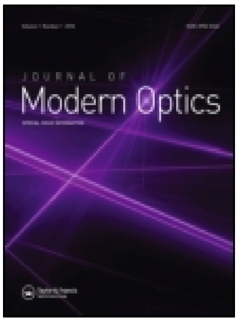


Journal of Modern Optics

ISSN: 0950-0340 (Print) 1362-3044 (Online) Journal homepage: https://www.tandfonline.com/loi/tmop20

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JOURNAL OF MODERN OPTICS, 1990, VOL. 37, NO. 6, 1075–1085

# Optical measurement accuracy in the case of non-classical light

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(*Received 4 July 1989 and accepted 10 October 1989*)

**Abstract.** The influence of optical elements on the statistical properties of the input radiation is analysed by quantum-theoretical means for a representative optical arrangement consisting of beam splitters as well as for various types of grating spectrometers. In particular non-classical light states are studied. Optimum conditions for the maintenance of the signal-to-noise ratio of sub-Poissonian input radiation are derived.

## 1. Introduction

It is the main purpose of optical measurements to gain the maximum information about the input radiation from the measured quantities that are usually output intensities behind some optical elements, where the input signal itself is assumed to carry the information on the process, on the source, or on the image to be investigated. The deviation between input and output radiation depends on spectral, temporal and stochastic properties of the input radiation and on the optical processing inside the optical arrangement. Under the conditions of high-performance optics the information limit is essentially determined by the quantum fluctuations of the radiation.

Recent experimental investigations revealed that the values of characteristic quantum fluctuation measures of so-called non-classical light, in particular of squeezed and sub-Poissonian light [1], fall considerably below the corresponding values of the 'best' classical light. The reduced variance $\langle \Delta\hat{\mathbf{E}}^2\rangle / \langle \Delta\hat{\mathbf{E}}^2\rangle_{\mathrm{vac}}$ of the electric field strength and the reduced variance $\langle \Delta\hat{n}^2\rangle / \langle \hat{n}\rangle$ of the photon-number of non-classical light can attain values near 0·2, whereas the values of quasi-ideal laser light are 1·0 (see e.g. [2]). At the first glance this fact suggests that non-classical light may yield an essential improvement of the signal-to-noise ratio (SNR) in comparison to classical light. But on the other hand it must be taken into account that, in striking contrast to the traditional description that uses spatiotemporally exactly predicted field functions, optical processing can strongly change the quantum state of light and therefore its statistical behaviour. In particular the unavoidable combination of the excited modes with the unexcited vacuum modes can diminish the advantageous signal-to-noise ratio of non-classical light. For that reason a quantum-optical procedure has to be used to study effects that act on the field fluctuations and the photon noise [3] in various optical arrangements.

Loudon [4] studied the influences of optical processing elements on the low-noise properties of squeezed light. In doing this he considered the relation between output and input radiation properties for various linear optical arrangements under the condition that the input signal states are ideal squeezed states as introduced by

Stoler [5] and Yuen [6]. Loudon found that there exist arrangements where the squeezing effect is nearly lost, while other arrangements nearly guarantee the maintenance of the squeezing effect. For the sake of generalisation, we shall consider the influence of optical processing on arbitrary input signal states. The input radiation is assumed to consist of various modes with the modal operators $\{\hat{e}_0, \hat{c}_1, \ldots, \hat{c}_\sigma\}$ whereas the modal operators of the output radiation are denoted by $\{\hat{d}_1, \ldots, \hat{d}_\sigma, \hat{e}_\sigma\}$. According to the optical arrangement considered the output operators are given functionals

$$\hat{d}_l = d_l(\hat{e}_0, \hat{c}_1, \ldots, \hat{c}_\sigma) \tag{1.1}$$

of the input operators. The input modes can represent excited signal modes or unexcited modes (non-illuminated regions). It will be shown that already the investigation of a chain of lossless identical beam splitters will enable us to gain an insight into the nature of the problem.

In spectral measurements it is the main purpose to get the maximum information about the time-dependent input spectrum $B(\omega, t)$ [7] from the measured output spectrum. This information is in particular limited by the spectral and temporal resolving power of the spectrometer, the detection noise, and the quantum fluctuations of the output radiation. The traditional treatment of the spectrometer under stationary as well as under non-stationary conditions is well known. Gase and Schubert [7, 8, 9] investigated the relation between the input spectrum and the output intensity and found conditions for the optimum temporal and spectral resolution. Moreover the influence of the noise caused by the detection process had been discussed by Schubert [10]. However the principal limit for the yield of information is given by the quantum fluctuations of the output light field, depending among other things on the statistical properties of the input signal. Hence in sections 3, 4 and 5 of this paper we shall discuss the dependence of the output radiation noise on the statistical properties of the input radiation and their alteration by the spectrometer [11] in the case of spectral resolution. In particular we will investigate optimum measurement conditions for the improvement of the SNR by use of sub-Poisson input light for various types of grating spectrometers.

## 2. Chain of lossless beam splitters

Let us consider a chain of $\sigma$ identical symmetric beam splitters. The input modes of the $j$th beam splitter are characterised by $\hat{b}_j$, $\hat{c}_j$, while the output modal operators are denoted by $\hat{d}_j$, $\hat{e}_j$. The relation between input and output operators is

$$\begin{pmatrix} \hat{e}_j \\ \hat{d}_j \end{pmatrix} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \begin{pmatrix} \hat{b}_j \\ \hat{c}_j \end{pmatrix} \tag{2.1}$$

where $t$, $r$ are the transmission and reflection coefficients that fulfil the relations:

$$|r|^2 + |t|^2 = 1; \quad tr^* + rt^* = 0. \tag{2.2}$$

The input signal mode of the chain is labelled by the modal operator $\hat{b}_1$, the mode associated with the output radiation by $\hat{e}_\sigma$. The state of the input signal radiation is characterised by the arbitrary density operator $\hat{\rho}_0$. The modes denoted by $\hat{c}_j$ are assumed to be the 'vacuum ports' of the corresponding beam splitter. The input radiation of the $j$th beam splitter characterised by the modal operator $\hat{b}_j$ results from the influence of the foregoing $(j-1)$ beam splitters on the input signal of the chain (figure 1).

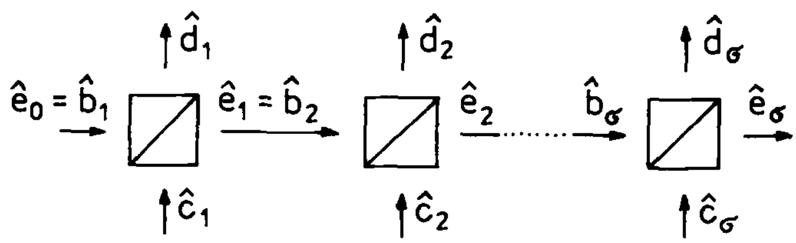


Figure 1. Chain of lossless beam splitters. $\hat{e}_0 = \hat{b}_1$ is the input signal operator of the chain; the 'vacuum' input operators are denoted by $\hat{c}_j$. $\hat{e}_\sigma$ is the output signal operator.

In the case of intensity measurements by photon counting, generally the behaviour of the signal-to-noise ratio

$$\mathrm{SNR} = \frac{\langle \hat{n} \rangle^2}{\langle \Delta \hat{n}^2 \rangle} \tag{2.3}$$

plays an important role for the estimation of the transmission of the light through an optical arrangement. The above assumptions lead to the following results at the output of the first beam splitter. The mean photon number is

$$\langle \hat{n}_1 \rangle = T \langle \hat{n}_0 \rangle, \tag{2.4}$$

where $T = |t|^2$, with the corresponding variance

$$\langle \Delta \hat{n}_1^2 \rangle = T^2 \langle \Delta \hat{n}_0^2 \rangle + T(1-T). \tag{2.5}$$

$\hat{n}_0$ is the operator of the photon number of the signal photons. $\langle \hat{A} \rangle$ means $\mathrm{Tr}\{\hat{A}\hat{\rho}_0 |0\rangle_{c_1 c_1}\langle 0|\}$, where $|0\rangle_{c_1 c_1}\langle 0|$ denotes the vacuum part of the density operator.

Thus, one arrives at the reciprocal signal-to-noise ratio

$$(\mathrm{SNR})_1^{-1} = (\mathrm{SNR})_0^{-1} + \frac{1}{\langle \hat{n}_0 \rangle} \frac{1-T}{T} \tag{2.6}$$

at the output. Since $T < 1$ in realistic cases, there follows $(\mathrm{SNR})_1 < (\mathrm{SNR})_0$. We shall now consider the reduced Poisson excess

$$P \equiv \frac{\langle \Delta \hat{n}^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}. \tag{2.7}$$

There holds

$$P_1 = P_0 T, \tag{2.8}$$

independently of the initial state. If $P_0 < 0$ (that means initial sub-Poisson light), there follows $P_1 < P_0$; this expresses a tendency to classical light because of the coupling of the unexcited vacuum mode.

From the results (2.6) and (2.8), one can easily derive the results for the chain

$$(\mathrm{SNR})_\sigma^{-1} = (\mathrm{SNR})_0^{-1} + \frac{1}{\langle \hat{n}_0 \rangle} [\exp(-\sigma \ln T) - 1] \tag{2.9}$$

$$P_\sigma = P_0 \exp(\sigma \ln T) \tag{2.10}$$

$$(\mathrm{SNR})_\sigma^{-1} = (\mathrm{SNR})_0^{-1} + \frac{1}{\langle \hat{n}_0 \rangle} \left( \frac{P_0}{P_\sigma} - 1 \right) \tag{2.11}$$

Independently of the input radiation state the SNR diminishes with growing $\sigma$. Inspection of (2.10) shows that an initially existing sub-Poisson character of the light also diminishes with growing $\sigma$; it is nearly maintained, if $\sigma|\ln T| \ll 1$.

## 3. The relation between the input and output radiation noise of grating spectrometers

In the following parts we shall study grating spectrometers. In doing this we use a simple but realistic model. The geometrical arrangement, which is symmetrical in the $x$ direction, is shown in figure 2. The spectrometer consists of an entrance slit (S) with two identical lossless lenses. The grating (G) is assumed to be an ideal reflecting screen with M periodical structures of a periodic distance g. A lens behind the grating focuses the diffraction image onto the detector (D) in the exit plane. The input field, which we identify with the field in the absence of the spectrometer, is assumed to be decomposed into plane waves according to

$$\hat{F}(\mathbf{r}, t) = \mathrm{i}\sum_{\mathbf{k}, s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}(\hat{a}_{\mathbf{k}s}\exp(\mathrm{i}\mathbf{k}\mathbf{r})\exp(\mathrm{i}\omega_k t) - \mathrm{H.c.})\mathbf{e}_{\mathbf{k}s}, \tag{3.1}$$

where $\hat{a}_{\mathbf{k}s}$, $\hat{a}^{+}_{\mathbf{k}s}$ are the annihilation and creation operators of plane waves with wave-number $\mathbf{k}$ and polarisation index $s$.

The geometrical arrangement of the entrance slit is chosen in such a way, that in the traditional description the geometrically optical approximation holds. That means, all incident plane waves with propagation directions outside a certain angle $\theta$ will be focused outside the entrance slit and will therefore be eliminated. The other waves will be transmitted apart from a conversion of $k_y$ into $-k_y$, which can be ignored for our consideration. The angle $\theta$ is an important parameter of the device, because it affects the resolution of the spectrometer. Figure 3 shows the dependence of the effective exit linewidth of a monochromatic input radiation with the wavenumber $\mathbf{k}$ on the parameter $s = Mkg\theta/2$ (van Cittert [12]). For $s \leqslant \pi$ the effective linewidth is nearly constant and only determined by the grating itself. This domain is called the high-resolution domain. In the quantum case we can model the action of the entrance slit in the following way: all modes outside the angle $\theta$ will be replaced by vacuum modes and the other signal modes remain unchanged.

The action of the grating is described by the well known Kirchhoff theory of diffraction. In doing this the field $\hat{\mathbf{E}}$ behind the grating can be calculated in terms of

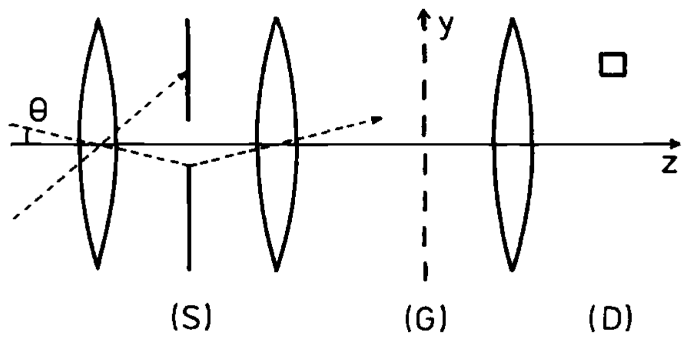


Figure 2. Geometrical arrangement of the grating spectrometer, with the entrance slit (S), the grating (G), and the detector (D).

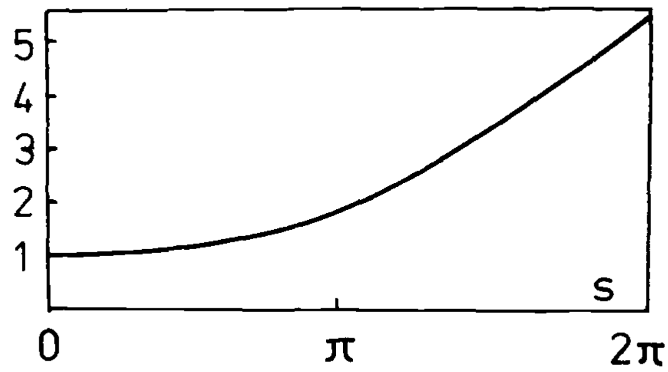


Figure 3. Dependence of the effective exit linewidth of a monochromatic input radiation on the entrance slit parameter $s=Mkg\theta/2$.

the incident field $\hat{\mathbf{F}}$. For the sake of simplicity we neglect the polarisation properties of the field and consider it as a scalar field.

$$\hat{E}(\mathbf{r},t)=\frac{1}{4\pi}\iint_{\text{apertures}} d\bar{x}\,d\bar{y}\left[\hat{F}(\bar{\mathbf{r}},\bar{t})\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\rho}\right)-\frac{1}{c\rho}\frac{\partial\rho}{\partial\bar{z}}\dot{\hat{F}}(\bar{\mathbf{r}},\bar{t})-\frac{1}{\rho}\frac{\partial}{\partial\bar{z}}\hat{F}(\bar{\mathbf{r}},\bar{t})\right]_{\bar{t}=t-\rho/c}$$

$$+\frac{1}{4\pi}\iint_{\substack{\text{non-illuminated}\\ \text{regions}}} d\bar{x}\,d\bar{y}\left[\hat{F}(\bar{\mathbf{r}},\bar{t})\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\rho}\right)-\frac{1}{c\rho}\frac{\partial\rho}{\partial\bar{z}}\dot{\hat{F}}(\bar{\mathbf{r}},\bar{t})-\frac{1}{\rho}\frac{\partial}{\partial\bar{z}}\hat{F}(\bar{\mathbf{r}},\bar{t})\right]_{\substack{\bar{t}=t-\rho/c\\ k_z\to -k_z}}. \tag{3.2}$$

Here $\rho$ is the distance between the point $\mathbf{r}$ and the point $\bar{\mathbf{r}}$ on the screen. The diffraction integral in (3.2) has to be extended over the apertures as well as over the non-illuminated regions, because the vacuum modes have to be included. Insertion of the negative frequency part of the input field into (3.2) yields the corresponding frequency component of the output field.

$$\hat{E}^{(-)}(\mathbf{r},t)=\mathrm{i}\sum_{\substack{k_x\\ k_z>0,\,|k_y/k|\le\theta}}\left(\frac{\hbar\omega_{\mathbf{k}}}{2\varepsilon_0 V}\right)^{1/2} g_{\mathbf{k}}(\mathbf{r})\hat{a}_{\mathbf{k}}\exp(-\mathrm{i}\omega_{\mathbf{k}}t)+\text{vacuum terms}, \tag{3.3}$$

where

$$g_{\mathbf{k}}(\mathbf{r})=\frac{1}{4\pi}\iint_{\text{apertures}} d\bar{x}\,d\bar{y}\exp(\mathrm{i}\mathbf{k}\bar{\mathbf{r}})\exp\left[\mathrm{i}\omega_{\mathbf{k}}\frac{\rho}{c}\right]\left[\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\rho}\right)+\frac{1}{c\rho}\frac{\partial\rho}{\partial\bar{z}}\mathrm{i}\omega_{\mathbf{k}}-\frac{k_z}{\rho}\right] \tag{3.4}$$

are the 'mode functions' of the diffracted waves, which can be calculated using standard approximations and techniques. In the usual case of absorption detectors all measurable quantities may be expressed by normally ordered products of annihilation and creation operators. In this description, the vacuum terms of (3.3) do not contribute to the quantities and therefore we need not calculate them explicitly.

According to Mandel's theory of photodetection [13], the mean number and the variance of the photoelectrons registered by a detector based on the external photoeffect in an electromagnetic field $\hat{E}$ are given by

$$\langle \hat{N}(t)\rangle = \eta \int_{V_D} \mathrm{d}V \langle \hat{A}^+(\mathbf{r},t)\hat{A}(\mathbf{r},t)\rangle, \tag{3.5}$$

$$\langle \Delta\hat{N}^2(t)\rangle = \langle \hat{N}(t)\rangle - \langle \hat{N}(t)\rangle^2 + \eta^2 \int_{V_D} \mathrm{d}V \int_{V_D} \mathrm{d}V' \langle \hat{A}^+(\mathbf{r},t)\hat{A}^+(\mathbf{r}',t)\hat{A}(\mathbf{r}',t)\hat{A}(\mathbf{r},t)\rangle. \tag{3.6}$$

$V_D$ is the detector area perpendicular to the incident radiation multiplied by the product $cT$ with $T$ being the measuring time. $\eta$ is a positive number with values less than 1, that corresponds to the detector efficiency. $\hat{A}(\mathbf{r},t)$ is the so called detection operator of the electromagnetic field. Under quasi-monochromatic conditions in all normal ordered expectation values $\hat{A}(\mathbf{r},t)$ can be replaced by $\hat{E}^{(-)}(\mathbf{r},t)$ according to

$$\hat{A}(\mathbf{r},t) \to -\mathrm{i}\left(\frac{2\varepsilon_0}{\hbar\omega_0}\right)^{1/2} \hat{E}^{(-)}(\mathbf{r},t), \tag{3.7}$$

where $\omega_0$ is the mid-frequency of the small spectral region considered. For a detector in the exit plane of a spectrometer the quasi-monochromatic condition is naturally fulfilled and we may use (3.7). With (3.3) from (3.5) and (3.6) follows the output SNR:

$$\begin{aligned} \mathrm{SNR} &\equiv \frac{\langle \hat{N}(t)\rangle^2}{\langle \Delta\hat{N}^2(t)\rangle} \\ &= \langle \hat{N}(t)\rangle \end{aligned} \tag{3.8}$$

$$\times \left[ 1 - \langle \hat{N}(t)\rangle + \frac{\eta \int_{V_D} \mathrm{d}V \int_{V_D} \mathrm{d}V' \sum'_{\mathbf{k}_1\cdots\mathbf{k}_4} g^*_{\mathbf{k}_1}(\mathbf{r}) g^*_{\mathbf{k}_2}(\mathbf{r}') g_{\mathbf{k}_3}(\mathbf{r}') g_{\mathbf{k}_4}(\mathbf{r}) \langle \hat{a}^+_{\mathbf{k}_1}(t)\hat{a}^+_{\mathbf{k}_1}(t)\hat{a}_{\mathbf{k}_3}(t)\hat{a}_{\mathbf{k}_4}(t)\rangle}{\int_{V_D} \mathrm{d}V \sum'_{\mathbf{k}_1,\mathbf{k}_2} g^*_{\mathbf{k}_1}(\mathbf{r}) g_{\mathbf{k}_2}(\mathbf{r}) \langle \hat{a}^+_{\mathbf{k}_1}(t)\hat{a}_{\mathbf{k}_2}(t)\rangle} \right],$$

where $\Sigma'$ means summation over all $\mathbf{k}$-values with $k_z > 0$ and $|k_y/k| \leqslant \theta$ and $\hat{a}_\mathbf{k}(t)$ being the abbreviation for $\hat{a}_\mathbf{k} \exp(-\mathrm{i}\omega_\mathbf{k} t)$. In this form (3.8) does not allow a simple physical interpretation, but in the next part of the paper we will introduce a statistical measure of the input field which simplifies this expression. With the help of it, the alteration of the input radiation statistics, caused by the device, can easily be described.

## 4. A measure for the spectral photon-number statistics of the input field

The photon-number statistics of a one-mode field is frequently described by the reduced Poisson excess $P$, as defined in equation (2.7). Because $\langle \Delta\hat{n}^2\rangle \geqslant 0$, $P \geqslant -1$ holds. To get an insight into the physical meaning of this quantity it is useful to express $P$ with the help of the Glauber–Sudarshan representation $\Phi(\alpha)$ [14]:

$$P = \frac{\int \mathrm{d}^2\alpha \Phi(\alpha) \left[ \alpha^*\alpha - \int \mathrm{d}^2\beta \Phi(\beta)\beta^*\beta \right]^2}{\int \mathrm{d}^2\alpha \Phi(\alpha)\alpha^*\alpha} \tag{4.1}$$

For light states with a classical analogue $\Phi(\alpha)$ is positive definite. Hence for these states follows from (4.1) $P \geqslant 0$. $P < 0$ defines sub-Poisson light, which is therefore a type of radiation without classical analogue.

A photon-number operator $\hat{n}_{VT}$, that belongs to a measurement of a multimode field within a finite volume $V_0$ which's sides are large compared to the optical wavelength, had been introduced by Mandel [13].

$$\hat{n}_{VT} \equiv \int_{V_0} \mathrm{d}V \hat{A}^+(\mathbf{r}', t)\hat{A}(\mathbf{r}, t), \tag{4.2}$$

where

$$\hat{A}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \hat{a}_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r}) \exp(-\mathrm{i}ckt) \tag{4.3}$$

corresponds to the detection operator of a field that is decomposed into plane waves. Mandel showed that $\hat{n}_{VT}$ fulfils the relation

$$\langle : \hat{n}_{VT}^2 : \rangle \equiv \langle \hat{n}_{VT}^2 \rangle - \langle \hat{n}_{VT} \rangle \tag{4.4}$$

in a good approximation. Here ': :' denotes the normal ordering belonging to the annihilation and creation operators. Hence the Poisson excess of $\hat{n}_{Vt}$ can be expressed with the multi-mode Glauber–Sudarshan representation in analogy to (4.2):

$$\begin{aligned} P_{VT} &\equiv \frac{\langle \Delta \hat{n}_{VT}^2 \rangle - \langle \hat{n}_{VT} \rangle}{\langle \hat{n}_{VT} \rangle} \rangle = \frac{\langle \hat{n}_{VT}^2 \rangle - \langle \hat{n}_{VT} \rangle^2 - \langle \hat{n}_{VT} \rangle}{\langle \hat{n}_{VT} \rangle} \\ &= \frac{\langle : (\hat{n}_{VT} - \langle \hat{n}_{VT} \rangle)^2 : \rangle}{\langle \hat{n}_{VT} \rangle} \\ &= \frac{\int \mathrm{d}^2\alpha_{\mathbf{k}}\, \Phi(\{\alpha_{\mathbf{k}}\}) \left[ \sum_{\mathbf{kk}'} \lambda_{\mathbf{kk}'} \alpha_{\mathbf{k}}^* \alpha_{\mathbf{k}'} - \int \mathrm{d}^2\beta_{\mathbf{k}}\, \Phi(\{\beta_{\mathbf{k}}\}) \sum_{\mathbf{kk}'} \lambda_{\mathbf{kk}'} \beta_{\mathbf{k}}^* \beta_{\mathbf{k}'} \right]^2}{\int \mathrm{d}^2\alpha_{\mathbf{k}}\, \Phi(\{\alpha_{\mathbf{k}}\}) \sum_{\mathbf{kk}'} \lambda_{\mathbf{kk}'} \alpha_{\mathbf{k}}^* \alpha_{\mathbf{k}}'} \end{aligned} \tag{4.5}$$

with

$$\lambda_{\mathbf{kk}'} = \int_{V_0} \mathrm{d}V \exp[\mathrm{i}(\mathbf{k} - \mathbf{k}')\mathbf{r}] \tag{4.6}$$

and therefore allows a similar interpretation. A multi-mode Fock state leads to $P_{VT} = -1$ and a global Glauber state to $P_{VT} = 0$. In spite of the fact $P$ belongs to a multi-mode field, it is not a useful measure for our purpose because according to (4.3) it includes all modes of the electromagnetic field. But a detector in the exit plane of a spectrometer responses only to a small spectral region. Hence we have to aim at a more instructive measure. The most simple description of the spectrometer action on the input statistics would be the introduction of a spectral transmission factor $\gamma$, so that the mean photoelectron number $\langle \hat{N}(t) \rangle$

$$\langle \hat{N}(t) \rangle = \eta\gamma \langle \hat{n}_{\mathrm{sp}}(t) \rangle \tag{4.7}$$

is proportional to the number of input photons $\hat{n}_{\mathrm{sp}}$ in the spectral region under consideration. Then the interesting statistical measure would be $P_{\mathrm{sp}} = (\langle \Delta \hat{n}_{\mathrm{sp}}^2 \rangle - \langle \hat{n}_{\mathrm{sp}} \rangle)^2 / \langle \hat{n}_{\mathrm{sp}} \rangle$. $P_{\mathrm{sp}}$ could be regarded as a generalisation of $P$ like

$P_{VT}$, if $\hat{n}_{sp}$ fulfills the relation (4.4). Now $P_{sp}$ can be expressed with the Glauber–Sudarshan representation in the same way as $P$:

$$P_{sp}=\frac{\int d^2\alpha_k\,\Phi(\{\alpha_k\})\left[\sum_{kk'}'\xi_{kk'}\alpha_k^*\alpha_{k'}-\int d^2\beta_k\,\Phi(\{\beta_k\})\sum_{kk'}'\xi_{kk'}\beta_k^*\beta_{k'}\right]^2}{\int d^2\alpha_k\,\Phi(\{\alpha_k\})\sum_{kk'}'\xi_{kk'}\alpha_k^*\alpha_{k'1}}, \tag{4.8}$$

with

$$\xi_{kk'}=\int_{V_D} dV g_k^*(\mathbf{r})g_{k'}(\mathbf{r}). \tag{4.9}$$

This means an input state with classical analogue would always lead to $P_{sp}\geqslant 0$, and sub-Poisson input radiation to $-1\leqslant P_{sp}<0$. Now we want to restrict ourselves to the case of spectral resolution. Then $\langle \hat{a}_k^+\hat{a}_{k'}\rangle$ is approximately constant within the small spectral region under consideration and (4.4) can be fulfilled introducing a 'photon-operator' according to (4.7)

$$\hat{n}_{sp}\equiv\frac{1}{\gamma}\int_{V_D} dV\sum_{kk'}'g_k^*(\mathbf{r})g_{k'}(\mathbf{r})\hat{a}_k^+(t)\hat{a}_{k'}(t) \tag{4.10}$$

with $\gamma$ being

$$\gamma=\frac{\frac{1}{V}\int_{V_D} dV'\sum_{k'k''}'g_{k''}^*(\mathbf{r}')\int_{V_D} dV\sum_{k}'g_k^*(\mathbf{r})g_k(\mathbf{r}')g_{k'}(\mathbf{r})}{\int_{V_D} dV\sum_{kk'}'g_{k'}^*(\mathbf{r})g_k(\mathbf{r})}. \tag{4.11}$$

With the help of $P_{sp}$, $\gamma$ and the detector efficiency $\eta$ now the output SNR can easily be expressed ($f=-P_{sp}$):

$$\mathrm{SNR}=\frac{\eta\gamma\langle\hat{n}_{sp}\rangle}{1-\eta\gamma f}. \tag{4.12}$$

From (4.12) can be seen, that sub-Poisson input light ($0<f\leqslant 1$) leads to an essential increasing of the SNR in comparison to classical input radiation with $f\leqslant 0$, if very efficient detectors ($\eta\rightarrow 1$) are used and the device losses are small, that means $\gamma$ tends toward 1. In the next section we shall investigate the conditions under which the latter case can really be reached.

## 5. Calculation of the spectral transmission factor $\gamma$ for various types of gratings in dependence of the measuring parameters

The functions $g_k(\mathbf{r})$ had been calculated in Fraunhofer approximation for three types of spectrometers in the usual way, namely for a transmission grating (figure 4 (*a*)), a plane reflection grating (figure 4 (*b*)), and an echelette-type grating (figure 4 (*c*)). The calculation showed that $\gamma$ can be decomposed into two factors according to

$$\gamma=\gamma_0\kappa(\Delta\nu_{Res}\,T,s), \tag{5.1}$$

both having a maximum value of 1. $\gamma_0$ depends only on the type of the spectrometer

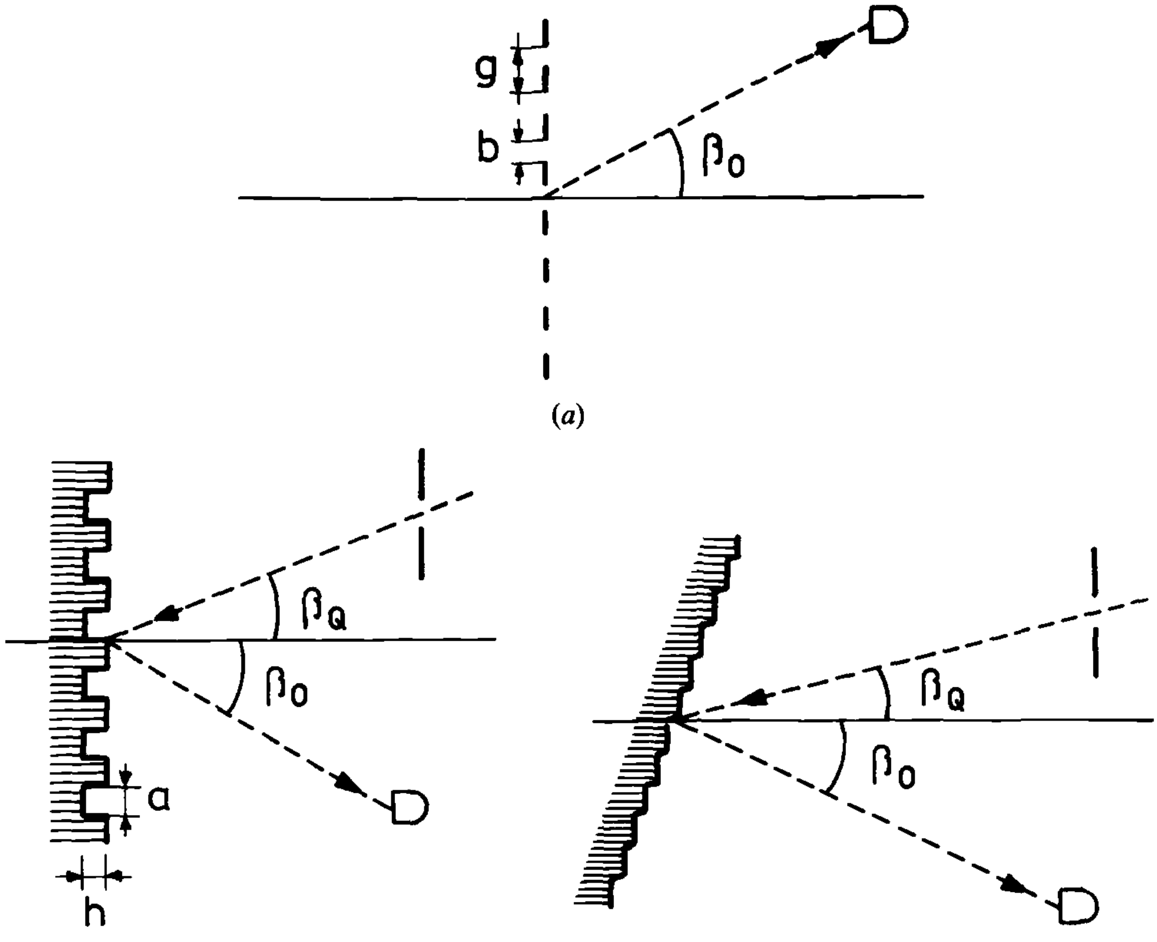


Figure 4. (*a*) transmission grating; (*b*) plane reflection grating; (*c*) echelette grating.

and the geometrical arrangement. It is given by

$$\gamma_0=\begin{cases}\underbrace{\dfrac{(\cos\beta_0+1)^2}{4\cos\beta_0}\left[\dfrac{b}{g}\right]^2\sin c^2\left(\dfrac{b}{g}m\pi\right)}_{\text{transmission grating}} & \lesssim 0\cdot 1\\ \underbrace{\dfrac{(\cos\beta_0+\cos\beta_Q)^2}{4\cos\beta_0}\sin^2(k_0 h\cos\beta_Q)\sin c^2\left[\dfrac{m\pi}{2}\right]}_{\text{plane reflection grating}} & \lesssim 0\cdot 4\\ \underbrace{\dfrac{(\cos\beta_0+\cos\beta_Q)^2}{4\cos\beta_0}\sin c^2\left[(\cos\beta_0+\cos\beta_Q)\dfrac{hk_0}{2}-m\pi\right]}_{\text{echelette-type grating}} & \lesssim 1\cdot 0\end{cases}\tag{5.2}$$

$m$ means here the order of diffraction. $k_0$ is the medium wavenumber of the small spectral region. From (5.2) follows, as can be expected, that only the echelette-type grating may yield a spectral transmission factor that attains 1. $\kappa$ is a function of the resolution bandwidth $\Delta\nu_{\mathrm{Res}}=k_0c/Mm$ times the measuring time $T$ and the entrance slit parameter $s$. Figure 5 shows the dependence of $\kappa$ on these parameters. For measuring times large compared to the inverse of the resolution bandwidth $\Delta\nu_{\mathrm{Res}}$, $\gamma$ approaches values of 0·7–0·8 even in the high-resolution domain ($s\leqslant\pi$). Hence in

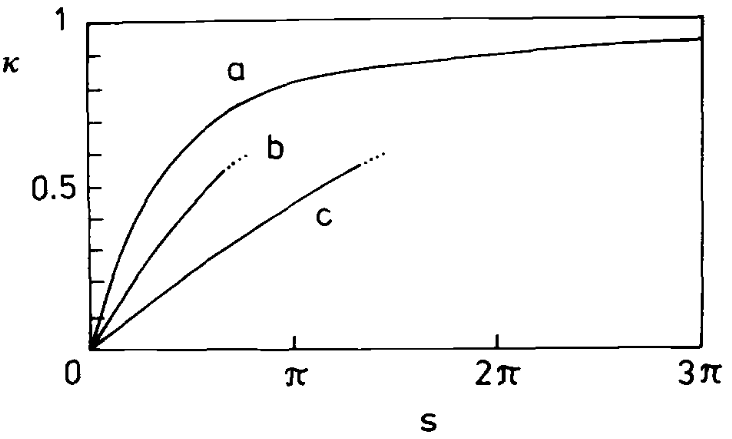


Figure 5. Dependence on the function $\kappa$ on the entrance slit parameter and the measuring time. (*a*) $\Delta\nu_{Res}T \gg 2\pi$, (*b*) $\Delta\nu_{Res}T = \pi$, (*c*) $\Delta\nu_{Res}T = \frac{1}{2}\pi$. The whole curve could be calculated only in the limit of long measuring times $T \gg 2\pi/\Delta\nu_{Res}$.

this case the use of sub-Poisson input radiation yields an essential improvement of the SNR compared with classical radiation like laser light.

## 6. Conclusions

The main purpose of this paper has been to clarify the influence of a combination of optical elements on the statistical properties of the input radiation. We aimed at the conditions of optimum information, expressed by the maximum signal-to-noise ratio. This quantity depends on the statistical behaviour of the radiation, which seems to be especially advantageous in the case of non-classical light.

We have analysed by quantum theoretical means the output quantities in terms of the input quantities for both a representative optical arrangement consisting of beam splitters and various types of grating spectrometers. Extending the calculations performed by Loudon [4], we studied in section 2 a chain of lossless beam splitters under the condition of arbitrary states of the input radiation. It was shown that also in this more general case the output properties can be described by relatively simple formulae, which reveal the effective influence of the coupling of unexcited vacuum modes to the signal modes. What concerns the spectral investigations, the influence of the device and measurement parameters (entrance slit width, resolving power, measuring time and grating type) on the statistical properties of the radiation have been studied. Introducing a special measure of the statistical properties of the input field in a small spectral region, we have been able to express the output signal-to-noise ratio in a relatively simple way. The resulting formula shows that in the case of echelette gratings the effective transmission factor can attain values that allow the approximate conservation of the advantageous input signal-to-noise ratio of non-classical light.

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