Topological Growing of Laughlin States in Synthetic Gauge Fields

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(Received 12 June 2014; published 7 October 2014)

We suggest a scheme for the preparation of highly correlated Laughlin states in the presence of synthetic
gauge fields, realizing an analogue of the fractional quantum Hall effect in photonic or atomic systems of
interacting bosons. It is based on the idea of growing such states by adding weakly interacting composite
fermions along with magnetic flux quanta one by one. The topologically protected Thouless pump
("Laughlin’s argument") is used to create two localized flux quanta and the resulting hole excitation is
subsequently filled by a single boson, which, together with one of the flux quanta, forms a composite
fermion. Using our protocol, filling 1/2 Laughlin states can be grown with particle number \( N \) increasing
linearly in time and strongly suppressed number fluctuations. To demonstrate the feasibility of our scheme,
we consider two-dimensional lattices subject to effective magnetic fields and strong on-site interactions.
We present numerical simulations of small lattice systems and also discuss the influence of losses.

DOI: 10.1103/PhysRevLett.113.155301 PACS numbers: 67.85.-d, 03.67.Lx, 42.50.Pq, 73.43.-f

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In electronic systems, the preparation of topological
states of matter relies on quick thermalization and cooling
below the many-body gap. While this is already hard to
achieve in cold-atom systems (partly due to the small
required temperatures), cooling is even less of an option in
photonic systems due to the absence of effective thermal-
zeization mechanisms. On the other hand, lasers with narrow
linewidths allow for a completely different avenue towards
preparation of extremely pure quantum states. For instance,
it was suggested to use the coherence properties of lasers to
directly excite two (and more) photon Laughlin (LN) states
in nonlinear cavity arrays [24], where the laser plays the
role of a coherent pump. However, this approach has the
inherent problem of an extremely small multiphoton
transition amplitude. While this might be acceptable for
small systems of \( N = 2, 3 \) photons, it makes the prepara-
tion of true many-body states with \( N \gg 2 \) practically
impossible. Moreover, the prepared states in this case
contain superpositions of different photon-numbers rather
than being Fock states.

In this Letter, we suggest an alternative scheme for the
preparation of topologically ordered states of strongly
interacting bosons, specifically for the 1/2 LN state, and
we discuss systems allowing for an implementation of the
scheme with state-of-the-art technology. It consists of
growing such states and makes direct use of the Thouless
pump [25] connected to the many-body topological invari-
ant. In the case of quantum Hall physics, the latter is
realized by local flux insertion in the spirit of Laughlin’s
argument for the quantization of the Hall conductivity \( \sigma_H \)
[26]: Introducing magnetic flux \( \phi/2\pi = 2 \) (in units of the
flux quantum) in the center of the system produces a
quantized outwards Hall current \( \sim \sigma_H \delta \phi \), leaving behind a
hole, see Fig. 1(a).

In the next step, the so-created hole can be replenished
by a single boson. In view of the composite fermion (CF)
picture [27,28] of the fractional quantum Hall effect, this
refilling step can be interpreted as the addition of a single
CF (composed of a bare boson and one flux quantum) into a
free orbital of the CF Landau level (LL), using up the
remaining flux quantum. To refill the hole deterministically
by a single boson, we consider a coherent pump in the
center of the system. Excitations by more than one particle
are prohibited by the many-body gap, and the coherent
coupling can not decrease the total particle number because
the central cavity is empty initially. Thus, our final state has
sub-Poissonian boson number statistics. A complementary
scheme, where holes resulting from boson losses are
dynamically refilled in the entire system using single
photon pumps, has recently been suggested for photonic
systems [29]. Our protocol, in contrast, does not rely on an explicit single photon source.

A key advantage of our scheme, compared to [24,30,31], is the ability to grow LN states with a size increasing linearly in time. To reach $N$ particles with given fidelity $1-\epsilon$, the protocol has to be carried out sufficiently slowly to avoid errors in the repumping protocol. For $\epsilon \ll 1$, the total required time scales like

$$T \sim \frac{N^{3/2}}{\Delta_{\text{LN}}^{3/2}},$$

where $\Delta_{\text{LN}}$ is the bulk many-body gap. In contrast to previously proposed schemes [24,30,31], $T$ only grows algebraically with $N$.

**Model.**—We consider a 2D lattice with complex hopping elements (amplitude $J$) realizing an effective magnetic field, supplemented by Hubbard-type on-site interactions (strength $U$). This model is illustrated in Fig. 1 and can be described by the following Hamiltonian:

$$\hat{H}_{\text{int}} + \hat{H}_0 = \frac{J}{2} \sum_{m,n} \hat{a}_m^\dagger \hat{a}_{m+1, n} + \hat{a}_{m, n+1}^\dagger \hat{a}_{m+1, n} + \text{H.c.},$$

where we used the Landau gauge and set $\hbar = 1$. Following Jaksch and Zoller’s proposal for the creation of synthetic gauge fields [32], there have been numerous suggestions regarding how this Hamiltonian can be implemented in photonic [18,23,24,33,34], circuit-QED [35–37], or atomic [31,38] systems, and in the last case, this goal has already been achieved [15,16].

Local flux insertion can most easily be realized by changing the hopping elements from site $(m \geq 0, n = 0)$ to $(m, 1)$ by a factor $e^{i\phi}$, see Fig. 1(b). These links are, thus, described by

$$\hat{H}_\phi = -J \sum_{m \geq 0} e^{-i\phi} \hat{a}_{m, 1}^\dagger \hat{a}_{m, 0} + \text{H.c.},$$

modifying the total magnetic flux through the central plaquette to $\alpha - \phi/2\pi$. $\hat{H}_\phi$ is motivated by recent experiments with photons [18,23], where the hopping phases can locally and temporarily be manipulated [39]. Finally to replenish the system with bosons, we place a weak coherent pump ($\Omega \ll 4\pi a J$) in the center,

$$\hat{H}_\Omega = \Omega e^{-i\omega t} \hat{a}_{0,0}^\dagger + \text{H.c.}$$

In the following, we present the details of our scheme, neglecting local boson losses (rate $\gamma$) for the moment. We include losses again, afterwards, in the discussion of the performance of our scheme.

**Protocol—continuum.**—We begin by discussing the continuum case when the magnetic flux per plaquette $\alpha / 2 \pi$ is small, allowing us to make use of angular momentum $L_z$ as a conserved quantum number. The continuum can be described by LLs, which are eigenstates of $\hat{H}_\Omega$ in the limit $\alpha \to 0$ with energies $E_n = (n + 1/2)\omega$, $(n = 0, 1, 2, \ldots)$ and $\omega = 4\pi a J$ denoting the cyclotron frequency, see, e.g., [28]. The magnetic length is defined as $a_B = a/\sqrt{2\pi\alpha}$, where $a$ denotes the lattice constant. In symmetric gauge, the single particle states of the lowest LL (LLL) are labeled by their angular momentum quantum number $l = 0, 1, 2, \ldots$ [28], and we define boson creation operators of these orbitals as $\hat{b}_l^\dagger$. Now, we discuss the preparation of filling $\nu = N/N_A = 1/2$ LN states, but the generalization to other fillings is straightforward.

To create the first excitation from vacuum $|0\rangle$, we switch on the coherent pump (3) with frequency $\omega = \omega_c/2$, which, due the blockade [40] (caused by strong boson-boson interactions), only allows a single particle to enter the system. Since we drive locally in the center, no angular momentum is transferred, and we, thus, arrive at the state $|\Psi_i\rangle = \hat{b}_0^\dagger |0\rangle$. This argument is true when excitations of higher LLs are neglected, allowing us to project the coherent pump (3) into the LLL. $\hat{H}_\Omega \approx \hat{b}_0^\dagger e^{-i\omega t} \Omega^{(1)} + \text{H.c.}$ with $\Omega^{(1)} = \Omega \sqrt{a}$. To this end, we require a weak pump, $\Omega \ll \omega_c$, and we also sufficiently feeble for the blockade to work, i.e., $\Omega^{(1)} \ll \Delta_{\text{LN}}$.

In the continuum, the gap can be estimated from $\Delta_{\text{LN}} \approx \min(V_0, \omega_c)$, where $V_0 = Ua/2$ is given by Haldane’s zeroth-order pseudopotential [41]. To prepare $|\Psi_1\rangle$ from $|0\rangle$ as described, the coherent pump has to be switched on for a time $T_\pi = \pi/2\Omega^{(1)}$ (corresponding to a $\pi$ pulse in the effective two-level system defined by $|0\rangle$ and $|\Psi_1\rangle$), which works when losses are negligible, $\gamma \ll \Omega^{(1)}$ [42].

Next, we adiabatically introduce two units of magnetic flux into the center of the system. Thereby, the initial state $|\Psi_1\rangle = \hat{b}_0^\dagger |0\rangle$ attains two units of angular momentum, and we end up in $|\Psi_2\rangle = \hat{b}_1^\dagger |0\rangle$ [43]. This state has a ring structure with a hole in its center, which—repeating the first step of our protocol—can be replenished by an additional
particle using the coherent pump. Because the latter only couples to the center of the system, it can not reduce the total particle number. The combined insertion of magnetic flux and a boson can be understood as addition of a single CF, with one flux-quantum binding to the boson to form a CF in the reduced magnetic field corresponding to the remaining flux quantum. Crucially, in contrast to the first step, the new state is not the simple product state $|\hat{b}_0^\dagger \Psi_2^\dagger\rangle$. Instead, the blockade mechanism allows us to pump only into the $N = 2$ LN state $|LN, 2\rangle$, which is the only zero-energy state with the correct total angular momentum $L_z = 2$, while all other states are detuned from the pumping frequency by the gap $\Delta_{LN}$. As a consequence, the corresponding Rabi frequency is reduced by a Franck-Condon factor (FCF), $\Omega_\text{eff}(2)/\Omega = (LN, 2|\hat{b}_0^\dagger \Psi_2^\dagger)/\sqrt{\alpha}$.

Having established our protocol for two bosons, the extension to $N$-particle LN states $|LN, N\rangle$ is straightforward. In this case, local flux insertion is used to create the state $|2\text{qh}, N - 1\rangle$ with two quasiholes (qh), which are subsequently refilled by the coherent pump to prepare $|LN, N\rangle$. The corresponding transition amplitude $\Omega_\text{eff}^{(N)}$ is reduced by a many-body FCF,

$$\Omega_\text{eff}^{(N)}/\Omega = \sqrt{\alpha(LN, N|\hat{b}_0^\dagger |2\text{qh}, N - 1\rangle).} \quad (4)$$

Using exact diagonalization (ED) of small systems ($N = 1, \ldots, 9$), we find that $\Omega_\text{eff}^{(N)}$ is nearly constant as a function of $N$, and we extrapolate $\Omega_\text{eff}^{(\infty)} \approx 0.70 \sqrt{\alpha}$. Thus, our pump works equally for large and small boson numbers.

A natural explanation regarding why highly correlated many-body states can be grown in the relatively simple fashion described above is provided by the composite fermion picture: LN states are separable (Slater determinant) states of noninteracting CFs filling the CF-LLL [27]. Thus, introducing CFs one-by-one into the orbitals of this LLL, LN states can easily be grown.

Protocol—lattice.—To ensure a sizable cyclotron gap $\omega_c$, a not too small flux per plaquette $\alpha$ is desirable, where lattice effects become important. We will now study this regime, which is also of great experimental relevance [15,16,23]. The spectrum of the Hamiltonian $\hat{H}_0(\alpha)$ is the famous Hofstadter butterfly [44], consisting of a self-similar structure of magnetic sub-bands. When interactions are taken into account, LN-type states can still be identified at filling $\nu = 1/2$ [31,38].

The basic ideas directly carry over from the continuum to the lattice case. Because the many-body Chern number is strictly quantized, Laughlin’s argument shows that a hole excitation can still be created by local flux insertion. However, due to the formation of magnetic sub-bands, such a quasihole becomes dispersive and will propagate away from the center. This leaves us only a restricted time to refill the defect, and leads to a reduced efficiency of repumping. To circumvent this problem, we introduce a trap for quasiholes. A static, repulsive potential of the form

$$\hat{H}_\text{pot} = \sum_{m,n} \frac{\hbar}{2\pi l_B^2} e^{-\left((m^2+n^2)a^2/2\hbar^2\right)} \hat{a}_{m,n}^\dagger \hat{a}_{m,n}, \quad (5)$$

is sufficient for a gapped ground state at every point in the protocol. An alternative would be to include carefully chosen long-range hoppings leading to a completely flat band [45].

In the following, we use ED to simulate our protocol for small systems. To get rid of boundary effects, which can be pronounced in small systems, we consider a spherical geometry [46] and take into account lattice effects by using a buckyball-type lattice. The hopping elements on all links have amplitude $J$, and their phases were chosen such that the flux per plaquette is $\alpha$. Because the total flux $\Phi_\text{tot}$ is integer quantized, it holds $\alpha = \Phi_\text{tot}/N_p$ with $N_p = 32$ being the number of plaquettes. We checked numerically (using ED) that, for the values of $\alpha \lesssim 0.2$ used in this Letter, there are gapped LN-type ground states, provided that the condition $\Phi_\text{tot} = 2(N - 1)$ for $\nu = 1/2$ LN states on a sphere is fulfilled. We find gaps of the order $\Delta_{LN} \approx 0.1J$, as predicted for a square lattice [31,38]. To describe the effect of local flux insertion $\Phi_\text{tot} \to \Phi_\text{tot} + \Phi/2\pi$, we slightly increase $\alpha \to \alpha + \Phi/(2\pi N_p)$ everywhere, except on the central plaquette where $\alpha \to \alpha - (1 - 1/N_p)\Phi/2\pi$ changes by $-\Phi/2\pi$ in thermodynamic limit (i.e., for $N_p \to \infty$).

Starting from an incompressible LN-type ground state, we checked numerically that the correct number of low-lying quasihole states is obtained and that they can be gapped out by the potential Eq. (5) [47].

In Fig. 2, we present a numerical simulation of our full protocol on the $C_{60}$ buckyball lattice. We start from vacuum and $\Phi_\text{tot} = 0$ flux quanta. Then, the coherent pump Eq. (3) is switched on for a time $T_\Omega = 6\pi/\Omega$ (with $\Omega = 0.05J$) and one boson is inserted with an overlap close to one to the target $N = 1$ ground state. The driving frequency $\omega$ is

![FIG. 2 (color online). Simulation of the full protocol on a C\textsubscript{60} buckyball lattice. We start from vacuum and N\textsubscript{p} = 0 flux quanta. Then, the coherent pump Eq. (3) is switched on for a time T\_\Omega = 6\pi/\Omega (with \(\Omega = 0.05J\)) and one boson is inserted with an overlap close to one to the target N = 1 ground state. The driving frequency \(\omega\) is chosen long-range hoppings leading to a completely flat band [45].

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chosen to be resonant on the transition from the $N = 0$ to the $N = 1$ ground state. After introducing two more flux quanta in a time $2 \times 20\pi/J$, of the order $2\pi/\Delta_{\text{LN}} \approx 60/J$, the whole protocol is repeated, and we, finally, arrive close to a three particle LN-type ground state. We find that the overlaps of the prepared states to the targeted $N$ particle ground states $|\text{gs}_N\rangle$ are close to one after all steps, and the overlaps conditioned on having the correct particle number $N$ (occurring with probability $P_N$) are even larger. At the end of the protocol, the $N = 3$ boson ground state at $N_g = 4$ is prepared with high fidelity, which carries the signatures of a LN-type state. Importantly, the particle number fluctuations after a completed cycle are strongly suppressed $\langle (\hat{N}^2) - \langle \hat{N} \rangle^2 \rangle / \langle \hat{N} \rangle \ll 1$.

In our simulations, we neglected edge effects and bulk losses. The latter result in a finite boson lifetime, such that, in noninteracting CFs on a lattice and show that our protocol still works when edge-effects are taken into account. The growing scheme, the mean density losses. The latter result in a finite boson lifetime, such that, in noninteracting CFs on a lattice and show that our protocol still works when edge-effects are taken into account.

We also expect that our scheme can be adapted for potentials, mimicking the effect of long-range interactions opens the possibility to simulate exotic Haldane pseudo-Abelian topological order, can also be grown using our technique. Moreover, preparing bosons in higher LLs of weakly interacting CFs, and our protocol is based on the idea of growing noncorrelated states of the latter. We demonstrated that this can be achieved by first creating artificial gauge fields. LN states can be understood in terms of weakly interacting CFs, and our protocol is based on the idea of growing noncorrelated states of the latter. We demonstrated that this can be achieved by first creating LN quasihole excitations which are subsequently refilled with bosons. Importantly, our protocol only requires a preparation time scaling slightly faster than linear with system size.

Our scheme is not restricted to the preparation of LN states of bosons. For example, we expect that the $\nu = 1$ bosonic Moore-Read Pfaffian [5,60,61] supporting non-Abelian topological order, can also be grown using our technique. Moreover, preparing bosons in higher LLs opens the possibility to simulate exotic Haldane pseudo-potentials, mimicking the effect of long-range interactions without the need to implement these in the first place. We also expect that our scheme can be adapted for the preparation of fractional quantum Hall states of fermions.

The authors thank N. Yao, M. Lukin, M. Höning, and N. Lauck for helpful discussions. Support was provided by the NSF-funded Physics Frontier Center at the JQI and by

Experimental realization.—Our protocol can be implemented in photonic cavity arrays [18,23,24,33,34], where the main experimental challenges are the required large interactions $U \gtrsim J$ and small losses $\gamma \ll \Delta_{\text{LN}}/N^{5/2}$. Strong nonlinearities can be realized, e.g., by placing single atoms into the cavities [33] or coupling them to quantum dots [23] or Rydberg gases [23,49,50]. Most promising are circuit-QED systems, where loss rates $\gamma = (0.1 \text{ ms})^{-1}$ have been achieved [51] (and $\gamma = 1 \text{ ns}^{-1}$ seems feasible). The strong coupling regime can be reached and single-photon nonlinearities $U = 100 \text{ MHz}$ are realistic [37]. For the case when $U = J$ and for $\gamma = 0.1$, the LN gap can be estimated to $\Delta_{\text{LN}} \approx 0.05U = 5 \text{ MHz}$ [38], which corresponds to $\Delta_{\text{LN}}/\gamma \approx 3 \times 10^3$. For an infidelity of $\epsilon = 0.1$, this yields $N_{\text{max}} = 7.4$ in a continuum system ($N_{\text{max}} = 3.4$ for $\Delta = 10$ as in our simulation). To observe interesting many-body physics on a qualitative level, $\epsilon = 0.5$ should be sufficient, which results in $N_{\text{max}} \approx 20$ in continuum. To reach even larger photon numbers, an array of multiple flux and photon pumps could be envisioned.

Alternatively, our scheme could be realized in ultracold atomic systems [15,16], where large interactions $U$ and negligible decay $\gamma$ are readily available [52]. In this case, an idea for realizing local flux insertion would be to use optical Raman beams with nonzero angular momentum [53], or as an alternative, quasiholes could be introduced by placing a focused laser-beam close to the edge of the system and increasing its intensity adiabatically [54]. Independent of the system, means for detecting LN-type ground states are required, and several approaches were discussed regarding how this can potentially be achieved [24,31,55–59].

Summary and outlook.—We proposed a scheme for the preparation of highly correlated LN states of bosons in artificial gauge fields. LN states can be understood in terms of weakly interacting CFs, and our protocol is based on the idea of growing noncorrelated states of the latter. We demonstrated that this can be achieved by first creating LN quasihole excitations which are subsequently refilled with bosons. Importantly, our protocol only requires a preparation time scaling slightly faster than linear with system size.

Our scheme is not restricted to the preparation of LN states of bosons. For example, we expect that the $\nu = 1$ bosonic Moore-Read Pfaffian [5,60,61] supporting non-Abelian topological order, can also be grown using our technique. Moreover, preparing bosons in higher LLs opens the possibility to simulate exotic Haldane pseudo-potentials, mimicking the effect of long-range interactions without the need to implement these in the first place. We also expect that our scheme can be adapted for the preparation of fractional quantum Hall states of fermions.

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ARO MURI Grant No. W911NF0910406. F.G. received support through the Excellence Initiative (DFG/GSC 266) and he gratefully acknowledges financial support from the “Marion Köser Stiftung.” Financial support by the DFG within the SFB/TR 49 is also acknowledged.

[42] Note that the coherent drive $\Omega = \kappa / \sqrt{N_0}$ can be related to the single-boson coupling strength $\kappa$ of the central site to a single-mode reservoir with average boson number $N_0$. To neglect additional spontaneous emission into the reservoir, we also require $\kappa \ll \gamma$, and thus, $N_0 \gg 1$.
[43] Here, for simplicity, we assumed that the Hamiltonian (2) for flux insertion is replaced by one with a symmetric gauge choice preserving rotational symmetry, see, e.g., A. R. Kolovsky, F. Grusdt, and M. Fleischhauer, Phys. Rev. A 89, 033607 (2014).