A dynamical equation for a maser with non-poissonian injection statistics

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A derivation of the coarse grained dynamical equation for a maser with periodic injection of atoms suggested by Briegel and Englert [Phys.Rev.A. **52**, 2361 (1995)] from the microscopic masterequation is presented.

The description of the dynamics of a maser with non-poissonian injection statistics of atoms has been the subject of several discussions in the literature [1]. While a microscopic description of the field evolution by a sequence of kicks from the atoms and subsequent periods of free decay adeqately models the maser dynamics [2], several approaches to derive a quasi-continous masterequation on a coarse grained time-scale failed for non-poissonian pumping [3]. Briegel and Englert suggested a macroscopic masterequation which overcomes these problems [4].

$$\frac{\partial}{\partial t}\bar{\rho}(t) = \mathcal{L}\bar{\rho}(t) + \mathcal{K}\frac{\mathcal{L}}{1-e^{-\mathcal{L}T}}\bar{\rho}(t). \tag{1}$$

The aim of this short note is to derive this equation from the microscopic masterequation.

We here consider the following microscopic model of the maser: Some pumping mechanism periodically injects excited two-level atoms into a resonator. The number of atoms entering the cavity at the periodic times $t_j = jT$ $(j = 0, \pm 1, \ldots)$ can fluctuate. The probability for a k-atom event is denoted by p_k. If we assume, that the transit time of the atom(s) is short compared to the time scale of interest, the effect of the atoms on the field can be described by a sequence of quasi instantaneous kicks

$$\rho(jT+0) = (1+\mathcal{K})\rho(jT-0) \tag{2}$$

where

$$\mathcal{K} = \sum_{k=1}^{\infty} p_k \mathcal{M}_k \tag{3}$$

is the operator describing the average effect of the injected atoms on the field. $\mathcal{M}_k$ accounts for the change in the field resulting from a k-atom event and is not further specified here. It depends on the actual interaction process and the passage time. Denoting the Liouvillian that describes the coupling to the cavity reservoir by $\mathcal{L}$ the microscopic masterequation for the field evolution in the interaction picture reads:

$$\frac{\partial}{\partial t}\rho(t) = \mathcal{L}\rho(t) + \lim_{\epsilon\to+0}\mathcal{K}\sum_j \delta(t-jT)\rho(t-\epsilon). \tag{4}$$

In order to derive an equation of motion on a coarse grained time scale we introduce a time averaged density operator:

$$\bar{\rho}(t) = \int_{-T_0/2}^{T_0/2} d\tau \rho(t-\tau) f(\tau), \tag{5}$$

where $f(\tau)$ is a properly normalized, slowly varying function of time. The averaging interval T_0 is assumed to be larger, or of the order of, the injection period T. It is clear at hand, that we can not define a coarse grained density operator in the immediate vicinity of the initial time $t = 0$. This is a generic feature of any coarse-graining approximation to an initial value problem and the definition of $\bar{\rho}(t)$ makes only sense for times larger than the averaging interval $T_0/2$.

In this sense we may rewrite Eq.(5) in a form convenient for a Laplace-transformation

$$\bar{\rho}(t) = \int_0^t d\tau \rho(t-\tau) f(\tau). \tag{6}$$

We now use this equation as a definition of a coarse-grained density operator, noting that it has the correct properties of a density operator only for time $t \geq T_0$. The Laplace-transform of $\bar{\rho}$ is then simply obtained from that of ρ via $\bar{\rho}(s) = \rho(s)f(s)$, where, for notational simplicity, we used the same symbols for the functions in Laplace-space.

We proceed by transforming the microscopic masterequation (4). The multiplication with the "filterfunction" $f(s)$ will allow some approximations which eventually yield the desired macroscopic masterequation.

$$\begin{aligned}(s-\mathcal{L})\rho(s) &= \rho(t=0) \\ &+ \lim_{\epsilon\to+0}\int_0^\infty dt \sum_j \delta(t-jT)e^{-st}\mathcal{K}\rho(t-\epsilon).\end{aligned} \tag{7}$$

The sum of delta-functions in Eq.(7) is equivalent to a sum of exponentials

$$\sum_{j=-\infty}^{\infty}\delta(t-jT) = \frac{1}{T}\sum_{\nu=-\infty}^{\infty} e^{2\pi i\nu t/T} \tag{8}$$

which yields

$$\begin{aligned}(s-\mathcal{L})\rho(s) &= \rho(t=0) \\ &+ \lim_{\epsilon\to+0}\frac{\mathcal{K}}{T}\sum_\nu \int_0^\infty dt\ e^{-(s-2\pi i\nu/T)t}\rho(t-\epsilon).\end{aligned} \tag{9}$$

In Eq.(9) we can immediately identify the Laplace-transform of ρ with a shifted argument:

$$(s-\mathcal{L})\rho(s) = \rho(t=0) + \lim_{\epsilon\to+0}\frac{\mathcal{K}}{T}\sum_{\nu} e^{-(s-2\pi i\nu/T)\epsilon}\rho(s-\frac{2\pi i\nu}{T}). \tag{10}$$

Since the r.h.s. of Eq.(10) is invariant under the transformation $s\to s+2\pi i n/T$ with $n=0,\pm1,\ldots$, the l.h.s is invariant as well. From this we infer

$$\rho(s+\frac{2\pi i\nu}{T}) = \left[s+\frac{2\pi i\nu}{T}-\mathcal{L}\right]^{-1}(s-\mathcal{L})\ \rho(s). \tag{11}$$

Inserting this result into Eq.(10) yields

$$\begin{aligned}(s-\mathcal{L})\rho(s)-\rho(t=0) &= \lim_{\epsilon\to+0}\frac{\mathcal{K}}{T}\sum_{\nu=-\infty}^{\infty}\frac{s-\mathcal{L}}{s+\frac{2\pi i\nu}{T}-\mathcal{L}}\ \rho(s)\ e^{-(s+2\pi i\nu/T)\epsilon} \\ &= \lim_{\epsilon\to+0}\frac{\mathcal{K}}{T}\sum_{\nu=-\infty}^{\infty}\left[1-\frac{i\frac{2\pi\nu}{T}}{s+\frac{2\pi i\nu}{T}-\mathcal{L}}\right]\ \rho(s)\ e^{-(s+2\pi i\nu/T)\epsilon}.\end{aligned} \tag{12}$$

So far no approximations are made. We now multiply Eq.(12) with the filterfunction $f(s)$, which rapidely decreases for $|s|>T_0^{-1}$. If we take an averaging intervall long compared to the period of injection T, we may therefore neglect s in the denominator of Eq.(12) as compared to $2\pi i\nu/T$. Note that the second term in the brackets vanishes for $\nu=0$.

If we furthermore rewrite the sum over ν as a sum of integrals with the help of the Poisson summation formula [5], we obtain:

$$(s-\mathcal{L})\bar\rho(s) = \rho(t=0)f(s)+\lim_{\epsilon\to+0}\frac{\mathcal{K}}{T}\times \times\int_{-\infty}^{\infty}d\nu\sum_{l=-\infty}^{\infty}e^{-i2\pi\nu(\epsilon/T+l)}\left[1-\frac{\nu}{\nu+i\frac{\mathcal{L}T}{2\pi}}\right]\bar\rho(s). \tag{13}$$

The ν integration of the first term gives deltafunctions $\delta(\epsilon/T+l)$ which vanish for $l=0,\pm1,\ldots$. The ν integration of the second term can be carried out by residual integration. Noting, that the eigenvalues of $\mathcal{L}$ are zero or negative, we find that only terms with negative l values contribute. We thus have

$$\begin{aligned}(s-\mathcal{L})\bar\rho(s)-\rho(t=0)f(s) &= -\mathcal{K}\sum_{l=-\infty}^{-1}\mathcal{L}e^{-\mathcal{L}Tl}\bar\rho(s) \\ &= -\mathcal{K}\sum_{l=1}^{\infty}\mathcal{L}e^{\mathcal{L}Tl}\bar\rho(s) \\ &= \mathcal{K}\frac{\mathcal{L}}{1-e^{-\mathcal{L}T}}\ \bar\rho(s).\end{aligned} \tag{14}$$

A transformation into the time domain yields

$$\frac{\partial}{\partial t}\bar\rho(t) = \rho(t=0)f(t)-\bar\rho(t=0) +\mathcal{L}\bar\rho(t)+\mathcal{K}\frac{\mathcal{L}}{1-e^{-\mathcal{L}T}}\ \bar\rho(s). \tag{15}$$

Noting, that according to Eq.(6) $\bar\rho(t=0)=0$ and that for $t>T_0$ $f(t)\equiv0$, we obtain the macroscopic masterequation of Briegel and Englert [4]:

$$\frac{\partial}{\partial t}\bar\rho(t) = \mathcal{L}\bar\rho(t)+\mathcal{K}\frac{\mathcal{L}}{1-e^{-\mathcal{L}T}}\ \bar\rho(t). \tag{16}$$

Since Eq.(6) defines a correct coarse-grained density operator only for $t\geq T_0$, this equation is only true for these times. In order to find the correct initial value of $\bar\rho(t)$ one has to solve the microscopic equation for some small time intervall and calculate $\bar\rho(T_0)$ according to Eq.(5). If the dynamics is however sufficiently slow one may to a good approximation apply the macroscopic equation also in the initial time period and identify the macroscopic initial value with the microscopic one.

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